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The transverse structure of Lie flows of codimension 3

By

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1. Introduction

This paper deals with the problem of the realization of a given Lie algebra as transverse algebra to a Lie foliation on a compact manifold.

Lie foliations have been studied by several authors ([E.H.S], [E.N], [F], [H.M], [M], [Ma], etc.). The importance of this study was increased by the fact that they arise naturally in Molino's classification of Riemannian foliations [M].

To each Lie foliation are associated two Lie algebras, the Lie algebra $\mathcal{G}$ of the Lie group on which the foliation is modeled and the structural Lie algebra $\mathcal{H}$. The latter algebra is the Lie algebra of the Lie foliation $\mathcal{F}$ restricted to the closure of any one of its leaves. In particular, it is a subalgebra of $\mathcal{G}$. We remark that although $\mathcal{H}$ is canonically associated to $\mathcal{F}$, $\mathcal{G}$ is not.

Thus two interesting problems are naturally posed: the *realization problem* and the *change problem*.

The *realization problem* is to know which pairs of Lie algebras $(\mathcal{G}, \mathcal{H})$, with $\mathcal{H}$ subalgebra of $\mathcal{G}$, can arise as transverse and structural Lie algebras, respectively, of a Lie foliation $\mathcal{F}$ on a compact oriented manifold M.

This problem is closely related to the following Haefliger's problem [Ha]: given a Lie subgroup Γ of a Lie group G, is there a Lie G-foliation on a compact manifold M with holonomy group Γ? E. Ghys [Gh] and G. Meigniez [Mg] also studied this problem and they gave necessary conditions for a pair (G, Γ) to be realizable.

Our formulation of the realization problem is a little different: We shall say that the pair $(\mathcal{G}, q)$ is *realizable* if there is a compact oriented manifold endowed with a Lie foliation transversely modeled on $\mathcal{G}$ and with structural Lie algebra of dimension q. We also say that $\mathcal{G}$ is realizable as *transverse* to a Lie foliation.

This formulation of the *realization problem* has been considered in [Ll], [H], [G, R] and [H.Ll.R] making a very detailed study of Lie flows of codimension 3 (cf. §8). But a complete classification was not obtained because of the following open question:

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Let $\mathcal{G}_8^h$ be the family of Lie algebras for which there is a basis e_1, e_2, e_3 such that

$$[e_1, e_2]=0, \quad [e_1, e_3]=e_2, \quad [e_2, e_3]=-e_1+he_2, \quad h\in(0,2).$$

For which h is there a Lie $\mathcal{G}_8^h$-flow on a compact manifold with basic dimension 2?

We solve this problem here, showing that there is no algebra of the family $\mathcal{G}_8^h$, $h\neq 0$, realizable as transverse to a Lie flow of basic dimension 2 (Theorem 5.1).

The *change problem* is to know if a given Lie $\mathcal{G}$-foliation can be at the same time a Lie $\mathcal{G}'$-foliation, where $\mathcal{G}$ and $\mathcal{G}'$ are two non isomorphic Lie algebras. The only *a priori* restriction is that the structural Lie algebra $\mathcal{H}$ must be a Lie subalgebra of $\mathcal{G}$ and $\mathcal{G}'$.

A first example of this situation was given by P. Molino [G.R]:
Let $\theta^0, \theta^1, \theta^2, \theta^3$ denote the canonical coordinates in $T^3\times T^1$. The vector filed $X=\partial/\partial\theta^0+\alpha\partial/\partial\theta^1+\beta\partial/\partial\theta^2$, with α,β rationally independent, admits $\partial/\partial\theta^0$, $\partial/\partial\theta^1$, $\partial/\partial\theta^3$ as an abelian parallelism. But

$$e_1=\cos\theta^3\frac{\partial}{\partial\theta^0}+\sin\theta^3\frac{\partial}{\partial\theta^1}$$
$$e_2=-\sin\theta^3\frac{\partial}{\partial\theta^0}+\cos\theta^3\frac{\partial}{\partial\theta^1}$$
$$e_3=-\frac{\partial}{\partial\theta^3}$$

is a new parallelism with $[e_1, e_2]=0$, $[e_1, e_3]=e_2$, $[e_2, e_3]=-e_1$, i.e. the flow is also transversely modeled on $\mathcal{G}_8^0$.

A systematic study of the *change problem* was first made in [H]. The case of Lie flows of codimension 3 and basic dimension 1 was made in [H.Ll.R] (cf. §8).

In this paper we complete the classification, in relation with the change problem, of Lie flows of codimension 3. The cases of codimension 1 and 2 are easy (cf. §3). We expect that this study becomes usuful in order to attack the general case.

The main results of this paper are the following.

Theorem 5.1. (1) *The Lie group G_8^h admits, for countable many values of h, a closed Lie subgroup H which is the closure of a finitely generated subgroup and such that the homogeneous space G/H is a compact manifold of dimension* 2.
(2) *For these h the pair* $(\mathcal{G}_8^h, 1)$ *is realizable as transverse to a Lie foliation.*
(3) *The pair* $(\mathcal{G}_8^h, 1)$ *is not realizable as transverse to a Lie flow for any* $h\neq 0$.

Theorem 6.1. *Let $\mathcal{F}$ be a codimension* 3 *Lie flow of basic dimension* 2 *on*

a compact oriented manifold M. Then
(1) *$\mathcal{F}$ can be modeled exactly on one or exactly on two Lie algebras. This second case arises if and only if $\mathcal{F}$ is modeled on $\mathcal{G}_1$ and $\mathcal{G}_8^0$ or on $\mathcal{G}_4$* and *$\mathcal{G}_5$.*
(2) *$\mathcal{F}$ is modeled on $\mathcal{G}_1$ if and only if it is modeled on $\mathcal{G}_8^0$.*
(3) *If $\mathcal{F}$ is modeled on $\mathcal{G}_5$ then it is modeled on $\mathcal{G}_4$.*
(4) *There are $\mathcal{G}_4$ Lie flows which are not $\mathcal{G}_5$ Lie flows.*

Theorem 7.1. *Let $\mathcal{F}$ be a codimension* 3 *Lie foliation on a compact oriented manifold M with compact leaves. Then*
(1) *$\mathcal{F}$ can be modeled exactly on one or exactly on two Lie algebras.*
(2) *$\mathcal{F}$ can be modeled on two Lie algebras if and only if it is modeled on $\mathcal{G}_1$. In this case the pair is $(\mathcal{G}_1, \mathcal{G}_8^0)$.*
(3) *There are Lie $\mathcal{G}_8^0$-foliations that can not be modeled on $\mathcal{G}_1$.*

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2. Preliminaries

Let $\mathcal{F}$ be a smooth foliation of codimension n on a differentiable manifold M given by an integrable subbundle $L \subset TM$. We denote by $T\mathcal{F}$ the Lie algebra of the vector fields tangents to the foliation, i.e. the sections of L. A vector field $Y \in \mathfrak{X}(M)$ is said to be *$\mathcal{F}$-foliated* (or simply *foliated*) if $[X, Y] \in T\mathcal{F}$ for all $X \in T\mathcal{F}$. The Lie algebra of foliated vector fields is denoted by $\mathcal{L}(M, \mathcal{F})$. Clearly, $T\mathcal{F}$ is an ideal of $\mathcal{L}(M, \mathcal{F})$ and the elements of $\mathfrak{X}(M/\mathcal{F}) = \mathcal{L}(M, \mathcal{F})/T\mathcal{F}$ are called *transverse* (or *basic*) *vector fields.*

If there is a family $\{X_1, \ldots, X_n\}$ of foliated vector fields on M such that the corresponding family $\{\bar{X}_1, \ldots, \bar{X}_n\}$ of basic vector fields has rank n everywhere the foliation is called transversely parallelizable and $\{\bar{X}_1, \ldots, \bar{X}_n\}$ is a transverse parallelism. If the vector subspace $\mathcal{G}$ of $\mathfrak{X}(M/\mathcal{F})$ generated by $\{\bar{X}_1, \ldots, \bar{X}_n\}$ is a Lie subalgebra, the foliation is called *Lie $\mathcal{G}$-foliation* and we say that $\mathcal{F}$ is *transversely modeled* on the Lie algebra $\mathcal{G}$.

We shall use the following structure theorems:

Theorem 2.1. ([M]). *Let $\mathcal{F}$ be a transversely parallelizable foliation on a compact manifold M, of codimension n. Then*
a) *There is a Lie algebra $\mathcal{H}$ of dimension $q \leq n$.*
b) *There is a locally trivial fibration π: $M \rightarrow W$ with compact fibre F and*

$$\dim W = n - q = m.$$

c) *There is a dense Lie $\mathcal{H}$-foliation on F such that:*
  i) *The fibres of π are the closures of the leaves of $\mathcal{F}$.*

ii) The foliation induced by $\mathcal{F}$ on each fibre of π: $M \to W$ is isomorphic to ohe Lie$\mathcal{H}$-foliation on $\mathcal{F}$.

$\mathcal{H}$ is called the *structural Lie algebra* of $(M, \mathcal{F})$, π the *basic fibration* and W the *basic manifold*. The foliation given by the fibres of π is denoted by $\bar{\mathcal{F}}$.

Note that the *basic dimension* (i.e. the dimension of W) is

$$\dim W = \operatorname{codim} \bar{\mathcal{F}} = \operatorname{codim} \mathcal{F} - \dim \mathcal{H}$$

Theorem 2.2 ([F]). *$\mathcal{F}$ is a Lie $\mathcal{G}$-foliation on a compact connected manifold M if and only if there exists a homomorphism h: $\pi_1(M) \to G$, where G is the connected and simply connected Lie group with its Lie algebre $\mathcal{G}$, a covering map p: $\widetilde{M} \to M$, and a locally trivial fibration D: $\widetilde{M} \to G$ such that*

i) D: $\widetilde{M} \to G$ is equivariant under the group **Aut** (p).

ii) The fibres of D are the leaves of the lift foliation $\widetilde{\mathcal{F}} = p^*\mathcal{F}$ of $\mathcal{F}$.

Condition i) means that

$$D(\gamma \cdot x) = h(\gamma) \cdot D(x) \quad \forall x \in \widetilde{M} \quad \forall \gamma \in \pi_1(M).$$

We also say that $\mathcal{F}$ is a Lie G-foliation. The subgroup $\Gamma = \operatorname{Im} h$ is called the *holonomy group* of the foliation.

For a Lie $\mathcal{G}$-foliation the structural Lie algebra $\mathcal{H}$ is always a subalgebra of $\mathcal{G}$.

The basic cohomology $H^*(M/\mathcal{F})$ of a foliation $\mathcal{F}$ on a manifold M is the cohomology of the complex of basic forms, i.e. the subcomplex $\Omega^*(M/\mathcal{F}) \subset \Omega(M)$ of the De Rham complex given by the forms α satisfying $i_X\alpha = 0$ and $L_X\alpha = 0$ for any vector field $X \in T\mathcal{F}$.

For a Riemannian foliation on a compact manifold M it is well known [E.H.S] that $H^n(M/\mathcal{F}) = 0$ or $\mathbf{R}$, where n is the codimension of the foliation. If $H^n(M/\mathcal{F}) \cong \mathbf{R}$ the foliation $\mathcal{F}$ is called *unimodular*.

We have the result

Theorem 2.3 ([LI.R.2]). *Let $\mathcal{F}$ be a Lie $\mathcal{G}$-foliation of codimension n on a compact oriented manifold M.*

i) If $\mathcal{F}$ is unimodular then $H^n(\mathcal{G}) \cong \mathbf{R}$ and $H^p(\mathcal{G}) \subset H^p(M/\mathcal{F})$.

ii) If $H^n(\mathcal{G}) \cong \mathbf{R}$ and the structural Lie algebra is an ideal of $\mathcal{G}$ then $H^n(M/\mathcal{F}) \cong \mathbf{R}$.

We shall also use the following results on unimodularity.

Theorem 2.4 ([LI.R.1]). *Let $\mathcal{F}$ be a $sl(2, \mathbf{R})$ Lie flow of codimension 3 and of basic dimension 2 on a compact oriented manifold. Then $\mathcal{F}$ is not unimodular.*

Theorem 2.5 ([Ll.R.2]). *Let $\mathcal{F}$ be a Lie $\mathcal{G}$-foliation with $\mathcal{G}$ a nilpotent Lie algebra on a compact oriented manifold. Then $\mathcal{F}$ is unimodular.*

Now we recall the definition of *commuting sheaf* associated to a foliation [M]. Let U be an open subset of M and let $\bar{Z}_U \in \mathcal{X}(U/\mathcal{F})$ be a local transverse field. We will say that $\bar{Z}_U$ is a *local commuting transverse field* if, for all $\bar{X} \in \mathcal{X}(M/\mathcal{F})$, the restriction of $\bar{X}$ to U commutes with $\bar{Z}_U$. The set of these local commuting transverse fields forms a subalgebra $C(U)$ of $\mathcal{X}(U/\mathcal{F})$. These subalgebras, together the natural restrictions, can be considered as a presheaf of algebras. The *commuting sheaf* is then the sheaf associated to this presheaf.

For instance, in the case of *dense* Lie foliations, where the transverse Lie algebra can be identified with the Lie algebra of left invariant vector fields on G, the commuting sheaf is nothing but the germs of the right invariant ones.

Theorem 2.6 ([M.S]). *Let $\mathcal{F}$ be a riemannian flow on a compact oriented manifotd M. Then $\mathcal{F}$ is unimodular if and only if the commuting sheaf is globaly trivial.*

We shall use the following classification of the 3 dimensional Lie algebras:

- $\mathcal{G}_1$ (Abelian):
$$[e_1, e_2] = [e_1, e_3] = [e_2, e_3] = 0$$
- $\mathcal{G}_2$ (Heisenberg):
$$[e_1, e_2] = [e_1, e_3] = 0,\ [e_2, e_3] = e_1$$
- $\mathcal{G}_3$ ($so(3)$):
$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2$$
- $\mathcal{G}_4$ ($\mathrm{sl}(2)$):
$$[e_1, e_2] = e_3, \quad [e_2, e_3] = -e_1, \quad [e_3, e_1] = e_2$$
- $\mathcal{G}_5$ (Affine):
$$[e_1, e_2] = e_1, \quad [e_1, e_3] = [e_2, e_3] = 0$$
- $\mathcal{G}_6$:
$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_1 \quad [e_2, e_3] = e_1 + e_2$$
- The family $\mathcal{G}_7^k$:
$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_1 \quad [e_2, e_3] = ke_2 \quad k \neq 0$$

The algebras $\mathcal{G}_7^k$ and $\mathcal{G}_7^{k'}$ are isomorphic if and only if $k = k'$ or $k = \frac{1}{k'}$. From now on we consider that the family is parametrized by $h \in [-1, 0) \cup (0, 1]$.

- The family $\mathcal{G}_8^h$:
$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_2, \quad [e_2, e_3] = -e_1 + he_2 \qquad h^2 < 4$$

The algebras $\mathcal{G}_8^h$ and $\mathcal{G}_8^{h'}$ are isomorphic if and only if $h=h'$ or $h=-h'$. From now on we consider thet family is parametrized by $h\in[0, 2)$. Notice that for $h^2\geq 4$ we obtain an algebra isomorphic to $\mathcal{G}_6$.

The Lie algebras $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$ are unimodular. The Lie algebras $\mathcal{G}_5$, $\mathcal{G}_6$ are not unimodular. The only unimodular Lie algebra in the family $\mathcal{G}_7$ is $\mathcal{G}_7^{-1}$ and the only unimodular Lie algebra in the family $\mathcal{G}_8$ is $\mathcal{G}_8^0$.

The connected simply connected Lie groups corresponding to $\mathcal{G}_5$, $\mathcal{G}_7^k$, $\mathcal{G}_8^h$ are given by

$$G_5=\left\{\begin{pmatrix} e^t & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix};\ x, y, t\in\mathbf{R}\right\}$$

$$G_7^k=\left\{\begin{pmatrix} e^{-t} & 0 & x \\ 0 & e^{-kt} & y \\ 0 & 0 & 1 \end{pmatrix};\ x, y, t\in\mathbf{R}\right\}$$

$$G_8^h=\left\{\begin{pmatrix} c(t)\cos(\varphi+t) & -c(t)\sin t & x \\ c(t)\sin t & c(t)\cos(\varphi-t) & y \\ 0 & 0 & 1 \end{pmatrix};\ x, y, t\in\mathbf{R}\right\}$$

where $c(t)=\dfrac{2e^{\beta t}}{\alpha}$, $\alpha=\sqrt{4-h^2}$ and $\beta=\tan\varphi=\dfrac{h}{\alpha}$.

They can also be described as $\mathbf{R}^3=\mathbf{R}^2\times\mathbf{R}$ with the product

$$(p, t)\cdot(p', t')=(p+e^{-At}p', t+t')$$

where

$$A=\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for } G_5$$

$$A=\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} \quad \text{for } G_7^k$$

$$A=\begin{pmatrix} 0 & 1 \\ -1 & h \end{pmatrix} \quad \text{for } G_8^h$$

In §7 we shall use the following result concerning semisimple Lie groups.

Theorem 2.7 ([G]). *Let Γ be a lattice in a semisimple Lie group G that has no compact semisimple factors, let $W=G/\Gamma$, and let G' be a Lie group acting transitively and locally effectively on W. Then G' is locally isomorphic to G.*

We shall also use the following results obtained in [H.Ll.R.]:

Proposition 2.8. *Every no dense Lie abelian foliation of codimension* 3 *on a compact manifold M is also a Lie $\mathcal{G}_8^{h=0}$-foliation.*

The converse is not true. Nevertheless it is true for basic dimension 2 (and also for Lie *flows* of basic dimension 1):

Proposition 2.9. *Let $\mathcal{F}$ be a Lie $\mathcal{G}_8^{h=0}$-foliation on a compact manifold M with basic dimension* 2, *then $\mathcal{F}$ is also a Lie abelian foliation.*

Proposition 2.10. *Let $\mathcal{F}$ be a Lie foliation on a compact manifold transversely modeled on two nilpotent Lie algebras $\mathcal{G}$ and $\mathcal{H}$. Then $\mathcal{G}$ and $\mathcal{H}$ are isomorphiec.*

3. Codimensions 1 and 2

The realization and the change problems are very easy in these two codimensions. *Codimension* 1. There is only one Lie algebra of dimension 1, so that the change problem has no sense in this case. The linear flows on T^2 (rational slope for basic dimension 1 and irrational slope for basic dimension 0) are examples of such foliations.

Codimension 2. There are two Lie algebras of dimension 2: the abelian and the affine Lie algebras. Examples of the abelian case in basic dimension 0, 1 or 2 are given by linear flows of suitable slope on T^3. A dense (basic dimension 0) affine Lie foliation of codimension 2 was given by A. Haefliger in [Gh]. This situation is not possible for Lie flows [C].

The flow on the hyperbolic torus T_A^3 induced by one of the eigenvectors of $A \in SL\ (2,\ \mathbf{Z})$, $\mathrm{tr}(A) > 2$, is one example of an affine Lie flow of basic dimension 2 and basic dimension 1 [C].

But it is not possible to have a codimension 2 affine Lie flow of basic dimension 2. This is because in this case the leaves are compact and the folitation is in fact a locally trivial bundle on a compact manifold. In particular, since the basic cohomology coincides with the cohomology of the base space of the bundle, the foliation is unimodular. But this is not possible because the affine Lie algebra is not unimodular (Theorem 2.3).

On the other hand, since every Lie foliation transversely modeled on the abelian Lie algebra is unimodular (Theorem 2.3) it is clear that it is not possible to *change* the transverse algebra of a given codimension 2 Lie foliation.

4. Review of some general facts

First of all we note that the language of Lie algebras can be translated into the language of Lie groups.

The pair of Lie algebras $(\mathcal{G}, \mathcal{H})$ *is realizable if and only if the pair of Lie groups* (G, Γ) *is realizable, where* G *is the connected* 1*-connected Lie group corresponding to* $\mathcal{G}$ *and* Γ *is a subgroup of* G *such that the Lie algebra of the connected component of the identity of* $\overline{\Gamma}$ *is* $\mathcal{H}$.

But we can have subgroups Γ and Γ' of G, with the same Lie algebra $\mathcal{H}$, and such that (G, Γ) is realizable and (G, Γ') is not [Gh].

The non-isomorphic Lie algebras $\mathcal{G}$ *and* $\mathcal{G}'$ *are transverse to the same Lie foliation of and only if on the connected* 1*-connected Lie group* $(G, \cdot)$ *corresponding to* $\mathcal{G}$ *there exists another operation of group,* $*$, *such that the Lie algcbra of* $(G, *)$ *is* $\mathcal{G}'$ *and there exist a subgroup* Γ' *of* $(G, *)$ *and an isomorphism* Φ *between the holonomy group* Γ *of the given foliation and* Γ' *such that* $\gamma g = \Phi(\gamma) * g$ *for each* $\gamma \in \Gamma$ *and* $g \in G$.

This last condition implies, in particular, that Φ can be extended to a unique continuous isomorphism $\tilde{\Phi}$ between $\overline{\Gamma}$ and $\overline{\Phi(\Gamma)}$. Since the holonomy group Γ of a *dense* Lie foliation is dense in G, $\tilde{\Phi}$ is an isomorphism between $(G, \cdot)$ and $(G, *)$, i.e. a dense Lie foliation can be modeled only on one Lie algebra.

Another consequence of this interpretation is that a Lie foliation transversely modeled on a Lie algebra such that the corresponding connected 1-connected Lie group is *compact*, can be modeled only on this Lie algebra. This follows from the fact [F] that the Lie group G is diffeomorphic to the manifold $\widetilde{M}/\widetilde{\mathcal{F}}$, which depends only on the foliation, and that compact Lie groups with the same homotopy type are locally isomorphic [S].

We remark that in the case of Lie foliations on *simply connected* manifolds the holonomy group is trivial. Hence, in this case, a Lie G-foliation is also a Lie foliation with respect to every structure of Lie group on the manifold G. There are not topological obstructions to the *change* problem. If the manifold is not only simply connected, but compact, then the foliation is a locally trivial bundle over G. In particular the foliation can not be dense and the connected 1-connected group G corresponding to $\mathcal{G}$ is compact. Hence there are no Lie foliations of codimension 1 or 2 on compact simply connected manifolds.

5. On the dimension of the foliation

The dimension of the foliation plays an important role in the realization and the change problems. To see this we shall construct examples of pairs $(\mathcal{G}, q)$ such that they are not realizable as Lie flows but that they are realizable as Lie foliations of dimension greater than one (Theorem 5.1). Also, we shall give examples of pairs $(\mathcal{G}, q)$ such that one can change the transverse Lie algebra $\mathcal{G}$ on any of their realizations as Lie flows and examples of foliations on the same pair $(\mathcal{G}, q)$ on which it is not possible to change the transverse Lie algebra $\mathcal{G}$ (Example 5.1).

It follows directly from [F] that a necessary condition for a pair $(\mathcal{G}, q)$ to be realizable is that the connected simply connected Lie group G corresponding to $\mathcal{G}$ admits a closed Lie subgroup H of dimension q. Moreover this H must be the closure of a finitely generated sugroup Γ of G. These conditions are not easy to check up in general and moreover they are not sufficient (Theorem 5.1). More specific necessary conditions are given in [Gh]. We begin with the following

Lemma 5.1. *The basic manifold of a Lie $\mathcal{G}_8^h$-foliation of basic dimension 2 on a compact manifold M is diffeomorphie to T^2 or K^2.*

Proof. For each point $x \in M$ there exists a foliated vector field Z_U in a neighbourhood U of x, such that Z_U is tangent to $\overline{\mathcal{F}}$, is not tangent to $\mathcal{F}$, and commutes (modulo $T\mathcal{F}$) with every global foliated vector field, that is, we consider a local section of the commuting sheaf [M]. Moreover if Z_V is another vector field in a neighbourhood V of x with the same property then $Z_U = \alpha Z_V$ (modulo $T\mathcal{F}$) where α is a locally constant function.

We can assume that the vector field Z_U is

$$Z_U = a_U Y_1 + b_U Y_2 + c_U Y_3$$

where a_U, b_U, c_U are basic functions on U and $\{\overline{Y}_1, \overline{Y}_2, \overline{Y}_3\}$ is a parallelism corresponding to the basis of $\mathcal{G}_8^h$ considered in section 2. Since $[Y_i, Z_U] \in T\mathcal{F}$ we obtain the equations:

$$Y_1(c_U) = 0 \quad Y_2(c_U) = 0 \quad Y_3(c_U) = 0$$

We deduce from these equations that c_U is constant on U.

Since $Z_U = \alpha Z_V$ (modulo $T\mathcal{F}$), with α a locally constant function, if $c_U = 0$ then $c_V = 0$. Then there are only two possibilities:

i) for any point $y \in M$ and any neighbourhood U of y we have $c_U = 0$ or

ii) for any point $y \in M$ and any neighbourhood U of y we have $c_U \neq 0$. Let us prove that ii) is not possible:

In this case it is easy to see that Y_1, Y_2 are not tangents to $\overline{\mathcal{F}}$ at any point. Then we have $T(M) = T\overline{\mathcal{F}} \oplus \langle Y_1, Y_2 \rangle$ and we define Y_i^N as the component of Y_i in $\langle Y_1, Y_2 \rangle$. Then Y_3^N is a combination of Y_1^N and Y_2^N at each point, i.e. there are basic functions f, g such that

$$(Y_3^N)_p = f(p)\,(Y_1^N)_p + g(p)\,(Y_2^N)_p \quad \forall p \in M.$$

In this case we obtain

$$Y_2^N = [Y_1, Y_3]^N = Y_1(f)\, Y_1^N + Y_1(g)\, Y_2^N$$

then $Y_1(q)$ is the constant 1, but this is not possible because g is a continuous function on a compact manifold.

Thus $c_U=0$ in each neighbourhood U and this means that Y_3 is not tangent to $\overline{\mathscr{F}}$ at any point. Hence the projection of the foliated vector field Y_3 on the basic manifold W is a non-vanishing vector field, i.e. $W=T^2$ or K^2.

Theorem 5.1. (1) *The Lie group G_8^h admits, for countable many values of h, a closed Lie subgroup H which is the closure of a finitely generated subgroup and such that the homogeneous space G/H is a compact manifold of dimension* 2.

(2) *For these h the pair $(\mathscr{G}_8^h, 1)$ is realizable as transverse to a Lie foliation.*

(3) *The pair $(\mathscr{G}_8^h, 1)$ is not realizable as transverse to a Lie flow for any $h\neq 0$.*

Proof of (1). Let $\Gamma=\langle(1,0,0),(\xi,0,0),(0,1,0),(0,0,\pi)\rangle$, $\xi\notin\mathbf{Q}$, be a subgroup of G_8^h and let $H=\overline{\Gamma}$.

We shall first prove that dim $H=1$, for at least countable many values of h. Note that every element of Γ can be written as a product of elements of type

$$(a, m, k\pi) \quad a\in\mathbf{R},\ m,\ k\in\mathbf{Z}$$

But

$$(a, m, k\pi)\cdot(a', m', k'\pi)=\left(c(k\pi)\begin{pmatrix}\cos(\phi+k\pi) & -\sin k\pi\\ \sin k\pi & \cos(\phi-k\pi)\end{pmatrix}\begin{pmatrix}a'\\ m'\end{pmatrix}\right.$$

$$\left.+\begin{pmatrix}a\\ m\end{pmatrix},\ (k+k')\pi\right)=$$

$$=(a+e^{\frac{kh\pi}{\sqrt{4-h^2}}}\cdot(\pm a'),\ e^{\frac{kh\pi}{\sqrt{4-h^2}}}\cdot(\pm m')+m,\ (k+k')\pi)$$

Now, as $\frac{1}{\pi}\log n>0$, $\forall n\in\mathbf{Z}$, $n>1$, there is $h\in[0,2)$ such that $\frac{h}{\sqrt{4-h^2}}=\frac{1}{\pi}\log n$

Thus $(a, m, k\pi)\cdot(a', m', k'\pi)=(--, \pm n^k m'+m, (k+k')\pi$ and $H=(\mathbf{R}\times\mathbf{Z})\ltimes\pi\mathbf{Z}$., i.e. dim $H=1$.

Moreover the $G_8^h/H\cong T^2$, i.e. it is a compact manifold of dimension two.

Proof of (2). Since Γ is a polycyclic, finitely generated subgroup of G_8^h with G_8^h/H compact, Γ is realizable [Mg].

More precisely Megniez theorem states that if a finitely generated subgroup Γ of a conected simply connected solvable Lie group G contains a uniform polycyclic subgroup, then Γ is the holonomy group of a Lie G-foliation on a compact manifold M.

In particular the pair $(\mathscr{G}_8^h, 1)$ is realizable.

But the Meigniez construction gives rise to a 2-dimensional Lie $\mathscr{G}_8^h$-foliation. In fact this realization can be made in the following way:

Let $S=(\mathbf{R}^2\times\mathbf{R}^2)\ltimes\mathbf{R}$ be the Lie group given by
$(a,b,c,d,e)\cdot(a',b',c',d',e')=((a,b)+A(e)(a',b'),\ (c,d)+A(e)(c',d'),\ e+e')$
where

$$A(e)=c(e)\begin{pmatrix}\cos(\phi+e) & -\sin e\\ \sin e & \cos(\phi-e)\end{pmatrix}$$

Now we consider the submersion

$$\begin{array}{ccc}\phi:S & \to & \mathbf{R}^2\ltimes\mathbf{R}=G_8^h\\ (a,b,c,d,e) & \to & (a+\xi c,\ b+d,\ e)\end{array}$$

It is easy to see that $\phi(\gamma x)=\phi(\gamma)\cdot\phi(x)$, $\forall\gamma\in\mathbf{Z}^4\ltimes\pi\mathbf{Z}$, $\forall x\in S$ and hence we have a 2-dimensional Lie G_8^h-foliation on $S/\mathbf{Z}^4\ltimes\pi\mathbf{Z})$.

Before proving part (3) we remark that it is the answer to one of the open questions stated in [GR] and it closes the realization problem for Lie flows of codimension 3 and basic dimension 2.

Proof of (3). Assume that there is a $\mathcal{G}_8^h$, $h\neq 0$, Lie flow of basic dimension 2 on a compact oriented manifold M. By replacing, if necessary, the basic manifold W with its double cover we can assume that the basic fibration is a T^2 bundle over T^2. Using now the classification of such bundles given by Sakamoto-Fukuhara [S.F] we obtain that M is diffeomorphic to $(T^2\times\mathbf{R}\times\mathbf{R})/\sim$, the quotient space of $(T^2\times\mathbf{R}\times\mathbf{R})$ by the equivalence relation "$\sim$" generated by

$$(\pi(s,t),x,y)\sim(\pi(s,t),x+1,y)$$

and

$$(\pi(s,t),x,y)\sim(\pi(A(s,t)+x(m,n)),x,y+1)$$

where π: $\mathbf{R}^2\to T^2$ is the canonical projection, $m,n\in\mathbf{Z}$ and $A\in SL(2,\mathbf{Z})$.

This description enable us to construct a well defined and injective map j: $T^2\to M$, that we shall use later, by

$$j(\pi(a,b))=p(\pi(0,a),b,0)$$

where p: $T^2\times\mathbf{R}^2\to M$ is the canonical projection.

On the other hand the homotopy sequence of the basic fibration induces the exact sequence of fundamental groups

$$0\to\pi_1(T^2)\to\pi_1(M)\to\pi_1(T^2)\to 0$$

Since $\pi_1(T_2)$ is a free abelian group we have

$$\pi_1(M)=\mathbf{Z}^2\ltimes\mathbf{Z}^2$$

In particular, and because of the injectivity of the holonomy representation $h: \pi_1(M) \to G_8^h$ of a *non-compact Lie flow*, we have two subgroups $h(\mathbf{Z}^2 \ltimes 0)$ and $h(0 \ltimes Z^2)$ of G_8^h isomorphic to $\boldsymbol{Z} \oplus \boldsymbol{Z}$. But it can be seen that if S is a subgroup of G_8^h isomorphic to $\boldsymbol{Z} \oplus \boldsymbol{Z}$ then $S \subset \mathbf{R}^2 \times \{0\}$ or $S \subset \{a^m b^n;\ ab = ba,\ m, n \in \mathbf{Z}\}$ for $a, b \notin \mathbf{R}^2 \times \{0\}$. In this case if $a = (a_1, a_2, \xi)$ and $b = (b_1, b_2, \eta)$, ξ and η are rationally independent. Hence we have four possibilities:

(1) $h(\mathbf{Z}^2 \ltimes 0)$ and $h(0 \ltimes Z^2)$ are both contained in $\mathbf{R}^2 \times \{0\}$. Then the holonomy group $\Gamma = h(\pi_1(M))$, generated by $h(\mathbf{Z}^2 \ltimes 0)$ and $h(0 \times \boldsymbol{Z}^2)$, is contained in $\mathbf{R}^2 \times \{0\}$, which contradicts $G_8^h / \bar{\Gamma} = T^2$.

(2)

$$h(\mathbf{Z}^2 \ltimes 0) = \{a^m b^n;\ ab = ba,\ m, n \in \mathbf{Z}\} \quad a, b \notin \mathbf{R}^2 \times \{0\}$$

and

$$h(0 \ltimes Z^2) = \{c^m d^n;\ cd = dc,\ m, n \in \mathbf{Z}\} \quad c, d \notin \mathbf{R}^2 \times \{0\}$$

As $h(\mathbf{Z}^2 \ltimes 0)$ is a normal subgroup of Γ we have $cac^{-1} \in h(\mathbf{Z}^2 \ltimes 0)$, i.e $cac^{-1} = a^m b^n$. This implies $m\xi + n\eta = \xi$, and since ξ and η are rationally independent we have $m = 1$ and $n = 0$, and hence $ca = ac$. Analogously $ad = da$, $bc = cb$, and $bd = db$. Thus $\pi_1(M)$ is abelian and the matrix A in the Sakamoto-Fukuhara classification is the identity [S.F]. But this implies that the flow is isometric [A.M] which is impossible because $\mathcal{G}_8^h$, $h \neq 0$, is not unimodular.

(3) $h(\mathbf{Z}^2 \ltimes 0) \subset \mathbf{R}^2 \times \{0\}$ and $h(0 \ltimes Z^2) = \{a^m b^n;\ ab = ba,\ m, n \in \mathbf{Z}\}$ $a, b \notin \mathbf{R}^2 \times \{0\}$ or vice versa. By the construction of the map j we have $j_*[\gamma] \in \mathbf{Z}^2 \ltimes 0$ and $j_*[\delta] \in 0 \ltimes \mathbf{Z}^2$, where $[\gamma]$, $[\delta]$ are the generators of $\pi_1(T^2)$. Then $h(j_*[\gamma]) = (x, y, 0)$, $(x, y) \neq (0, 0)$, and $h(j_*[\delta]) = (p, q, t)$, $t \neq 0$ but this is not possible because $(x, y, 0) \cdot (p, q, t) \neq (p, q, t) \cdot (x, y, 0)$

We end this section showing, by means of an example, that the dimension of the foliation plays an important role in the change problem.

First we recall that the Molino's example in §1 is at the same time a realization of the pair $(\mathcal{G}_8^0, 2)$ and of the pair $(\mathcal{G}_1, 2)$.

In fact we have proved in [H.Ll.R] that any realization of the pair $(\mathcal{G}_8^0, 2)$ as transverse to a Lie flow is, at the same time, an abelian flow.

But there are realizations of the same pair $(\mathcal{G}_8^0, 2)$ as transverse to a Lie *foliation* which are not Lie foliations for any other Lie algebra. To see this we consider the following example:

Example 5.1. Let Γ be the subgroup of G_8^0 given by

$$\Gamma=\{(1,0,0),\ (\xi,0,0),\ (0,1,0),\ (0,\eta,0),\ (0,0,\pi)\}$$

where ξ and $\eta \notin \mathbf{Q}$.

Since Γ is a polycyclic finitely generated subgroup of G_8^0 the pair (G_8^0, Γ) is realizable [Mg], as transverse to a 2-dimensional foliation. Note that the connected component of the identity of $\bar{\Gamma}$ is $\mathbf{R}^2 \times \{0\}$. Moreover, as the structural Lie algebra $\mathscr{H}$ is an ideal of $\mathscr{G}^0{}_8$ and $\mathscr{G}^0{}_8$ is unimodular, the foliation is unimodular (Theorem 2.3), and hence it only can be modeled on umimodular Lie algebras, i.e. on $\mathscr{G}_1, \mathscr{G}_2, \mathscr{G}_3, \mathscr{G}_4, \mathscr{G}_7^{-1}$. But

(1) It can not be modeled on $\mathscr{G}_1$ because the holonomy group Γ is not abelian.

(2) It can not be modeled neither on $\mathscr{G}_3$ nor on $\mathscr{G}_4$ because they do not have abelian subalgebras of dimension 2.

(3) It can not be modeled neither on $\mathscr{G}_3$ nor on $\mathscr{G}_7^{-1}$. To see this assume that it is modeled on $\mathscr{G}_2$ (resp. $\mathscr{G}_7^{-1}$). That means that on the same underlying manifold $\mathbf{R}^3$ we have two structures of Lie group, the corresponding to G_8^0 and the corresponding to G_2 (resp. G_7^{-1}). Moreover we have $\gamma \cdot g=\Phi(\gamma) * g$ for all g in the underlying manifold $\mathbf{R}^3$ where $\cdot$ and $*$ are the respective Lie multiplications (§4). Then the subgroup $H=\{(1,0,0), (0,1,0), (0,0,\pi)\}$ of Γ is a discrete uniform subgroup of G_8^0 and $\Phi(H)$ is a discrete uniform subgroup of G_2 (resp. G_7^{-1}) isomorphic to H. But this is a contradiction to the classification of the uniform subgrups of these Lie groups (§7).

6. Lie flows of codimension 3 and basic dimension 2.

The realization and the change problem in basic dimension 1 was first considered in [G.R] and completely solved in [H.Ll.R].

In basic dimension 2, the realization problem was also considered in the above two papers but the case $\mathscr{G}_8^h$, $h \neq 0$ remained open. This case has been solved here in Theorem 5.1.

So it only remains to study the change problem.

We begin with the following

Lemma 6.1. *The basic manifold of a Lie $\mathscr{G}_5$-flow of basic dimension 2 is diffeomorphic to the torus T^2.*

Proof Let Z_U be a local section of the commuting sheaf as in Lemma 5.1. Recall that if Z_V is another section of this sheaf then $Z_U=\alpha Z_V$ (modulo $T\mathscr{F}$) where α is a locally constant function.

We can assume that the vector field Z_U is

$$Z_U = a_U Y_1 + b_U Y_2 + c_U Y_3$$

where a_U, b_U, c_U are basic functions on U and $\{\overline{Y}_1, \overline{Y}_2, \overline{Y}_3\}$ is a parallelism corresponding to the basis of $\mathcal{G}_5$ considered in section 2. Since $[Y_i, Z_U] \in T\mathcal{F}$ we obtain the equations:

$$\begin{array}{lll} Y_1(a_U) = -b_U & Y_2(a_U) = a_U & Y_3(a_U) = 0 \\ Y_1(b_U) = 0 & Y_2(b_U) = 0 & Y_3(b_U) = 0 \\ Y_1(c_U) = 0 & Y_2(c_U) = 0 & Y_3(c_U) = 0 \end{array}$$

Thus b_U and c_U are locally constant. As $b_U = \alpha_{UV} b_V$ and $c_U = \alpha_{UV} C_V$ we have $b_U = 0$ everywhere or $b_U \neq 0$ for each U and the same for c_U.

But $b_U \neq 0$ everywhere implies that $\frac{Z_U}{b_U}$ is a global section of the commuting sheaf which is impossible because this flow is not unimodular [M.S]. Hence $b_U = 0$ everywhere.

As the same is true for c_U we have $Z_U = a_U Y_1$. In particular Y_2 and Y_3 are never tangent to $\overline{\mathcal{F}}$ and the basic manifold is diffeomorphic to T^2.

Theorem 6.1. *Let $\mathcal{F}$ be a codimension* 3 *Lie flow of basic dimension* 2 *on a compact oriented manifold M. Then*

(1) *$\mathcal{F}$ can be modeled exactly on one or exactly on two Lie algebras. This second case arise if and only if $\mathcal{F}$ is modeled on $\mathcal{G}_1$ and $\mathcal{G}_8^0$ or on $\mathcal{G}_4$ and $\mathcal{G}_5$*

(2) *$\mathcal{F}$ is modeled on $\mathcal{G}_1$ if and only if it is modeled on $\mathcal{G}_8^0$.*

(3) *If $\mathcal{F}$ is modeled on $\mathcal{G}_5$ then it is modeled on $\mathcal{G}_4$*

(4) *There are Lie $\mathcal{G}_4$-flows which are not Lie $\mathcal{G}_5$-flows.*

Proof. First we recall that (2) is proved in [H.Ll.R].

Proof of (1). It follows from the results on realization in [G.R] and [H.Ll.R] and Theorem 5.1 that the only Lie algebras that can appear as transverse Lie algebras to this Lie flow are

$$\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_5, \mathcal{G}_8^0$$

Also if $\mathcal{F}$ is modeled on $\mathcal{G}_3$ it is not possible to change the algebra because $\mathcal{G}_3$ is the only Lie algebra of dimension 3 with compact connected 1-connected associated Lie group.

Assume that $\mathcal{F}$ is modeled on $\mathcal{G}_1$. Then the flow is unimodular [Ll.R.2] and hence it can not be modeled on $\mathcal{G}_5$, because the affine algebra is not unimodular, and it can not be modeled on $\mathcal{G}_4$, because the Lie flows of basic dimension 2 on $\mathcal{G}_4$ are not unimodular [Ll.R.1]. Finally it can not modeled on $\mathcal{G}_2$ because they are both nilpotent [H.Ll.R].

Assume that $\mathcal{F}$ is modeled on $\mathcal{G}_2$. Since the flow is unimodular [Ll.R.2], we have as before that it can not be modeled neither on $\mathcal{G}_4$ nor on $\mathcal{G}_5$.

Proof of (3). If $\mathcal{F}$ is modeled on $\mathcal{G}_5$ the basic fibration is a T^2 bundle over T^2 (Lemma 6.1). Using now the classification of T^2 bundle over T^2 given by Sakamoto-Fukuhara [S.F] we have that there are a matrix $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z})$ and numbers $m, n \in \mathbf{Z}$ such that M is diffeomorphic to the quotient of $\mathbf{R}^4$ by the natural action of $\mathbf{Z}^3 \ltimes \mathbf{Z}$. This semidirect product is given by

$$(s, d)(s', d') = (s+B^d(s'), d+d') s, s' \in \mathbf{Z}^3, d, d' \in \mathbf{Z}$$

and

$$B=\begin{pmatrix} a & b & m \\ c & d & n \\ 0 & 0 & 1 \end{pmatrix}$$

That corresponds exactly to the action on the universal covering $\widetilde{M}$ of the fundamental group of M.

Set $G_4=\widetilde{SL(2, \mathbf{R})}$ and $G_5=Aff^+(\mathbf{R}) \times \mathbf{R}$. Let D: $\widetilde{M} \to G_5$ and h: $\mathbf{Z}^3 \ltimes \mathbf{Z} \to G_5$ be the developing map and the holonomy morphism associated to the given Lie flow $\mathcal{F}$. Let $\Gamma = h(\mathbf{Z}^3 \ltimes \mathbf{Z})$ be the holonomy group. The vector field Y_1 considered in the proof of Lemma 6.1 is the projection on M of a vector field Y_1 on $\overline{M}$ such that $D_*(\overline{Y}_1) = e_1$ where e_1 is the left invariant vector field on G_5 given by

$$e_1 = y\frac{\partial}{\partial x}$$

where the coordinates x, y correspond to the notation

$$Aff^+(\mathbf{R}) = \{\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}; y>0\}$$

Since Y_1 is tangent to the closure of the leaves and $M/\overline{\mathcal{F}} \cong G_5/\overline{\Gamma}$ it follows that $\overline{\Gamma} = \mathbf{R} \times H$ where H is a subgroup $\{x=0\}$ in G_5. Since $G_5/\overline{\Gamma}$ has dimension 2, H must be a discrete subgroup and hence $\Gamma \subset Aff^+(\mathbf{R}) \times \epsilon\mathbf{Z}$ for a given $\epsilon \in \mathbf{R}$. We may assume $\epsilon = 1$. Next we define

$$\begin{array}{cccc} \Phi: & G_5 & \longrightarrow & G_4 \\ & (a, b, t) & \mapsto & \tilde{\alpha}(a, b) \cdot \tilde{\phi}(t) \end{array}$$

where $\tilde{\phi}$ is a lifting of

$$\begin{array}{cccc} \phi: & \mathbf{R} & \longrightarrow & SL(2, \mathbf{R}) \\ & t & \mapsto & \begin{pmatrix} \cos 2\pi t & -\sin 2\pi t \\ \sin 2\pi t & \cos 2\pi t \end{pmatrix} \end{array}$$

and $\tilde{\alpha}$ is a lifting of

$$\begin{array}{rcll} \alpha: Aff^{+}(\mathbf{R}) & \rightarrow & SL(2, \mathbf{R}) & \\ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} & \mapsto & \frac{1}{\sqrt{a}}\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} & a>0 \end{array}$$

to the universal covering G_4 of $SL(2, \mathbf{R})$.

Since $\tilde{\phi}(n)$, $n \in \mathbf{Z}$, is in the center of G_4 [P], Φ is a morphism when restricted to Γ. Then if we define $\widetilde{D}=\Phi \circ D$ and $\widetilde{h}=\Phi \circ h$ we have

$$\begin{aligned} \widetilde{D}(\gamma \cdot \tilde{x}) &= \Phi(h(\gamma) \cdot D(x)) = \Phi((a, b, n) \cdot (x, y, z)) = \Phi(a+bx, by, n+z) = \\ &= \tilde{\alpha}(a+bx, by) \cdot \tilde{\phi}(n+z) = \tilde{\alpha}(a, b) \cdot \tilde{\phi}(n) \cdot \tilde{\alpha}(x, y) \cdot \tilde{\phi}(z) = \\ &= \widetilde{h}(\gamma) \cdot \widetilde{D}(x) \end{aligned}$$

and hence $\mathscr{F}$ is also a $\mathscr{G}_4$ Lie flow.

Proof of (4). Let $D=T^2 \# T^2$ be the double torus and let T_1D be the unit tangent bundle over D. It is well known that T_1D is diffeomorphic to the quotient space $T_1\mathbf{H}/\pi_1(D)$ where the action of the fundamental group of the double torus $\pi_1(D)$ on the unit tangent bundle of the hyperbolic plane $\mathbf{H}$ is by hyperbolic isometries.

Next we consider the diagonal action of $\pi_1(D)$ on the producxt $T_1\mathbf{H} \times S^1$, where the action on the second component is the the identity.

As $T_1\mathbf{H} \times S^1$ is diffeomorphic to $\mathbf{H} \times S^1 \times S^1$ we can take coordinates t, s on $S^1 \times S^1$ and consider the vector field on $\mathbf{H} \times S^1 \times S^1$ given by $\widetilde{X}=\frac{\partial}{\partial t}+\xi\frac{\partial}{\partial s}$, $\xi \notin \mathbf{Q}$. Since the above action is by rotations on the first S^1 and the identity on the second one, the vector field $\overline{X}$ induces a vector field X on the quotient space $\mathbf{H} \times S^1 \times S^1/\pi_1(D) \cong T_1D \times S^1$.

We claim that X is a $\mathscr{G}_4$ Lie flow on the compact manifold $T_1D \times S^1$.

To see this recall that $\mathbf{H} \times S^1$ is diffeomorphic to $PSL(2, \mathbf{R})$ and hence we have on $\mathbf{H} \times S^1 \times S^1$ vector fields $(\widetilde{Y}_1, 0)$, $(\widetilde{Y}_2, 0)$, $(\widetilde{Y}_3, 0)$ induced by a basis of the Lie algebra $\mathscr{G}_4$ of $PSL(2, \mathbf{R})$. It is worth to say that the identification between $PSL(2, \mathbf{R})$ and $\mathbf{H} \times S^1$ is given by $A \rightarrow (A^{-1}(i), A_{*}u)$ where $Az=\frac{az+b}{cz+d}$ and u is a unit tangent vector in $T_{A(i)}\mathbf{H}$ with a previous fixed direction.

Since left invariant vector fields on the group $PSL(2, \mathbf{R})$ correspond to infinitestimal hyperbolic isometries, the above vector fields induce vector fields Y_1, Y_2, Y_3 on the quotient space $\mathbf{H} \times S^1 \times S^1/\pi_1(D) \cong T_1D \times S^1$.

Again because they are infinitesimal isometries we have $[Y_i, X]=0$, i.e. they are foliated with respect to X. Since they are obviously transverse to X, they are the desired $\mathscr{G}_4$ Lie parallelisme.

But X can not be modeled on $\mathscr{G}_5$ because the basic manifold is D (Lemma 6.1).

7. Compact Lie Flows of codimension 3

The results of this section are equally true both for Lie flows and Lie foliations.

First we shall recall the classification of discrete uniform subgroups of dimension 3 Lie groups [A.G.H]. This classification directly solves the realization problem: *Only the algebras* $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_7^{-1}, \mathcal{G}_8^0$ *are realizable.* We have:

(1) The only connected 1-connected Lie groups of dimension 3 with uniform discrete subgroups are those corresponding to the Lie algebras

$$\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_7^{-1}, \mathcal{G}_8^0$$

(2) In the abelian case, $\mathcal{G}_1$, the group is isomorphic to the matrix group

$$\begin{pmatrix} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & x_2 \\ 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (x_1, x_2, x_3) \in \mathbf{R}^3$$

and the uniform discrete subgroup Γ corresponds to

$$\begin{pmatrix} 1 & 0 & 0 & n_1 \\ 0 & 1 & 0 & n_2 \\ 0 & 0 & 1 & n_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (n_1, n_2, n_3) \in \mathbf{Z}^3$$

(3) In the non-abelian nilpotent case, $\mathcal{G}_2$, the group is isomorphic to the matrix group

$$\begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} \quad (x_1, x_2, x_3) \in \mathbf{R}^3$$

and the subgroup Γ corresponds to the subgroup generated by

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & \frac{1}{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for a given $k \in \mathbf{N}$. We call this subgroup $\Gamma(k)$ and we have $\Gamma(k) \cong \Gamma(k') \Leftrightarrow k = k'$.

Moreover $Z(\Gamma(k)) \,/\, [\Gamma(k), \Gamma(k)] \cap Z(\Gamma(k)) \cong \mathbf{Z}/k\mathbf{Z}$, and $Z(\Gamma(1)) \neq 0$

[A.G.H].

(4) In the case of $\mathcal{G}_7^{-1}$ the group is isomorphic to

$$\begin{pmatrix} e^{kz} & 0 & 0 & x \\ 0 & e^{-kz} & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (x, y, z) \in \mathbf{R}^3 \quad \text{for a fixed } k \in \mathbf{Z} \text{ such that } e^k + e^{-k} \neq 2$$

and Γ corresponds to the subgroup generated by

$$e_1 = \begin{pmatrix} e^k & 0 & 0 & 0 \\ 0 & e^{-k} & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad e_2 = \begin{pmatrix} 1 & 0 & 0 & u_1 \\ 0 & 1 & 0 & v_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad e_3 = \begin{pmatrix} 1 & 0 & 0 & u_2 \\ 0 & 1 & 0 & v_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with $\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \neq 0$. It is not abelian.

(5) In the case $\mathcal{G}_8^0$ the group is isomorphic to

$$\begin{pmatrix} \cos(2\pi z) & \sin(2\pi z) & 0 & x \\ -\sin(2\pi z) & \cos(2\pi z) & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (x, y, z) \in \mathbf{R}^3$$

and Γ corresponds to a subgroup generated either by

$$w_1 = \begin{pmatrix} \cos\left(\frac{2\pi n}{p}\right) & \sin\left(\frac{2\pi n}{p}\right) & 0 & 0 \\ -\sin\left(\frac{2\pi n}{p}\right) & \cos\left(\frac{2\pi n}{p}\right) & 0 & 0 \\ 0 & 0 & 1 & \frac{n}{p} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$w_2 = \begin{pmatrix} 1 & 0 & 0 & u_1 \\ 0 & 1 & 0 & v_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad w_3 = \begin{pmatrix} 1 & 0 & 0 & u_2 \\ 0 & 1 & 0 & v_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with $n \in \mathbf{Z}$, $p \in \{2, 3, 4, 6\}$ and $\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \neq 0$

or to a subgroup generated by

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & u \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & n \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with $n \in \mathbf{Z}$. It is not abelian.

We shall also use, and this is an strightforward computation, that the center of Γ_7^{-1} is trivial and that $Z(\Gamma_8^0)/([\Gamma_8^0, \Gamma_8^0] \cap Z(\Gamma_8^0)) \cong \mathbf{Z}^i$ on $i \in \{1, 3\}$ where Γ_7^{-1} and Γ_8^0 are the corresponding uniform discrete subgroups of $\mathscr{G}_7^{-1}$ and $\mathscr{G}_8^0$.

Using this we can solve the change problem:

Theorem 7.1. *Let $\mathscr{F}$ be a codimension 3 Lie foliation on a compact oriented manifold M with compact leaves. Then*

(1) *$\mathscr{F}$ can be modeted exactly on one or exactly on two Lie algebras.*

(2) *$\mathscr{F}$ can be modeted on two Lie algebras if and only if it is modeled on $\mathscr{G}_1$. In this case the pair is $(\mathscr{G}_1, \mathscr{G}_8^0)$.*

(3) *There are Lie $\mathscr{G}_8^0$-foliations that can not be modeled on $\mathscr{G}_1$.*

Proof. We study the six possibilities $\mathscr{G}_1$, $\mathscr{G}_2$, $\mathscr{G}_3$, $\mathscr{G}_4$, $\mathscr{G}_7^{-1}$, $\mathscr{G}_8^0$.

(1) $\mathscr{F}$ is a Lie $\mathscr{G}_3$-foliation.

It is not possible to change the algebra because $\mathscr{G}_3$ is the only algebra such that the corresponding connected 1-connected group is compact.

(2) $\mathscr{F}$ is a Lie $\mathscr{G}_4$-foliation.

Let Γ be the holonomy group of $\mathscr{F}$. Assume that $\mathscr{F}$ is also modeled on another Lie algebra of connected 1-connected group G' with holonomy group Γ. Then G' acts transitive and locally effective on the basic manifold $W \cong G_4/\Gamma \cong G'/\Gamma$. Moreover G_4 is semisimple and has not compact semisimple factors [A.G.H]. Thus we are in the hypothesis of Gorbacevič theorem [G] and hence G is locally isomorphic to G', i.e. it is not possible to change the algebra $\mathscr{G}_4$.

Thus it only remains to study the four solvable Lie algebras.

(3) $\mathscr{F}$ is a Lie $\mathscr{G}_2$-foliation.

Then it can not be modeled on $\mathscr{G}_1$ because they are both nilpotent [H.Ll.R]. On the other hand the above results about the center of the uniform discrete subgroups of G_2, G_7^{-1}, and G_8^0 tells us that it is not possible to have a uniform discrete subgroup (the holonomy group) which is at the same time subgroup of two of these three groups. Hence it is not possible to change the algebra $\mathscr{G}_2$.

(4) $\mathscr{F}$ is a Lie $\mathscr{G}_7^{-1}$-foliation. The same argument shows that neither in

this case it is possible to change the algebra.

(5) $\mathcal{F}$ is a Lie $\mathcal{G}_1$-foliation. It is known [H.Ll.R] that in this case $\mathcal{F}$ is also a Lie $\mathcal{G}_8^0$-foliation.

(6) $\mathcal{F}$ is a Lie $\mathcal{G}_8^0$-foliation. One of the uniform discrete subgroups of $\mathcal{G}_8^0$ described above is not abelian. So it can not be a subgroup of $\mathcal{G}_1$, and hence there are examples of Lie $\mathcal{G}_8^0$-foliations which are not Lie $\mathcal{G}_1$-*foliations*.

8. Summary

We sum up here the classification of Lie flows of codimension 3 ([G.R], [H.Ll.R] and the present paper).

Realization Problem. (1) *Basic dimension* 3. Only $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$, $\mathcal{G}_7^{-1}$, $\mathcal{G}_8^0$ are realizable.

(2) *Basic dimension* 2. Only $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$, $\mathcal{G}_5$, $\mathcal{G}_8^0$ are realizable.

(3) *Basic dimension* 1. $\mathcal{G}_1$, $\mathcal{G}_5$, $\mathcal{G}_8^0$ are realizable. $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$, $\mathcal{G}_6$, $\mathcal{G}_7^k$, $k\in\mathbf{Q}$ are not realizable.

$\mathcal{G}_7^k$ is realizable if and only if

$$k=\frac{\ln b}{\ln a},\ k\notin\mathbf{Q}$$

where a, b, $\frac{1}{ab}$ are positive real roots of a monic polynomial of degree 3 with integer coefficients.

$\mathcal{G}_8^h$, $h\neq 0$ is realizable if and only if

$$h=\frac{2\ln\lambda}{\sqrt{4\omega^2+\ln^2\lambda}}$$

where λ and ω are two real numbers, with $\lambda>1$ and $\omega\neq k\pi\,(k\in\mathbf{Z})$, such that λ, $\frac{1}{\sqrt{\lambda}}(\cos\omega\pm\mathbf{i}\sin\omega)$ are the roots of a monic polynomial of degree 3 with integer coefficients.

(4) *Basic dimension* 0. Only $\mathcal{G}_1$ is realizable.

Change Problem. *Let $\mathcal{F}$ be a codimension* 3 *Lie flow on a compact oriented manifold M. Then $\mathcal{F}$ can be modeled on one, two, or countable many Lie algebras.*

(1) *Basic dimension* 3. $\mathcal{F}$ can be modeled exactly on one or exactly on two Lie algebras. $\mathcal{F}$ is modeled on two Lie algebras if and only if it is modeled on $\mathcal{G}_1$, and the pair is $(\mathcal{G}_1, \mathcal{G}_8^0)$. But there are $\mathcal{G}_8^0$ Lie flows which are not modeled on $\mathcal{G}_1$.

(2) *Basic dimension* 2. $\mathscr{F}$ can be modeled exactly on one or exactly on two Lie algebras. $\mathscr{F}$ can be modeled on two Lie algebras in the cases $(\mathscr{G}_1, \mathscr{G}_8^0)$ or $(\mathscr{G}_4, \mathscr{G}_5)$. $\mathscr{F}$ is modeled on $\mathscr{G}_1$ if and only if it is modeled on $\mathscr{G}_8^0$. If $\mathscr{F}$ is modeled on $\mathscr{G}_5$ then it is modeled on $\mathscr{G}_4$. But there are $\mathscr{G}_4$ Lie flows which are not modeled on $\mathscr{G}_5$.

(3) *Basic dimension* 1. $\mathscr{F}$ can be modeled on one, two, or countable many Lie algebras. $\mathscr{F}$ is modeled on one Lie algebra in the case $\mathscr{G}_5$ or $\mathscr{G}_7^k$.
$\mathscr{F}$ is modeled on two Lie algebras if and only if it is modeled on $\mathscr{G}_1$ or on $\mathscr{G}_8^0$, and the pair is $(\mathscr{G}_1, \mathscr{G}_8^0)$.
$\mathscr{F}$ is modeled on countable many Lie algebras if and only if it is modeled on $\mathscr{G}_8^h$, with

$$h=\frac{2\ln\lambda}{\sqrt{4(\omega+2k\pi)^2+\ln^2\lambda}} \quad \forall k\in\mathbf{Z}$$

where λ and ω are defined as above.

(4) *Basic dimension* 0. It is not possible to change the Lie algebra.

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