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LIE FLOWS OF CODIMENSION 3

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ABSTRACT. We study the following realization problem: given a Lie algebra of dimension 3 and an integer q, $0 \leq q \leq 3$, is there a compact manifold endowed with a Lie flow transversely modeled on $\mathscr{G}$ and with structural Lie algebra of dimension q? We give here a quite complete answer to this problem but some questions remain still open (cf. §2).

0. INTRODUCTION

Among the class of foliations with a transverse structure, Lie foliations stand out. These are foliations transversely modeled on Lie groups. They have been studied by several authors, mainly by Fedida (cf. [3]). Apart from its intrinsic interest, the importance of this study is increased by the fact that they arise naturally in Molino's classification of Riemannian foliations [6].

To each Lie foliation are associated two Lie algebras, the Lie algebra $\mathscr{G}$ of the Lie group on which it is modeled and the structural Lie algebra $\mathscr{H}$. The latter algebra is the Lie algebra of the Lie foliation $\mathscr{F}$ restricted to the closure of any one of its leaves. In particular, it is a subalgebra of $\mathscr{G}$. We remark that although $\mathscr{H}$ is canonically associated to $\mathscr{F}$, $\mathscr{G}$ is not.

Thus, one natural and interesting question is to know which pairs of Lie algebras $(\mathscr{G}, \mathscr{H})$, with $\mathscr{H}$ a subalgebra of $\mathscr{G}$, can arise as transverse algebra and structural Lie algebra, respectively, of a Lie foliation $\mathscr{F}$ on a *compact* manifold M.

We shall study here a particular but interesting case; namely, given a Lie algebra of dimension 3 and an integer q, $0 \leq q \leq 3$, is there a compact manifold endowed with a Lie flow transversely modeled on $\mathscr{G}$ and with structural Lie algebra of dimension q? For simplicity's sake we shall say that the pair $(\mathscr{G}, q)$ is (or is not) realizable.

By using the classification of the 3-dimensional Lie algebras and the fact that the structural Lie algebra of a Lie flow is abelian (cf. [1]) it becomes apparent that certain pairs $(\mathscr{G}, q)$ are not realizable (for instance, $(\mathrm{sl}(2), 2)$ and $(\mathrm{so}(3), 2)$ are not realizable because $\mathrm{sl}(2)$ and $\mathrm{so}(3)$ have no abelian subalgebras of dimension two).

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Nevertheless, in some cases the obstruction for certain pairs to be realizable is rooted in the compactness of M and not based on purely algebraic reasons (for instance, the pair (*affine*, 0) is not realizable (cf. Theorem 1)).

We classify the 3-dimensional Lie algebras in 6 algebras $\mathscr{G}_1, \ldots, \mathscr{G}_6$ and two families $\mathscr{G}_7$ (parametrized by $k \in \mathbf{R}$, $k \neq 0$) and $\mathscr{G}_8$ (parametrized by $h \in \mathbf{R}$, $h^2 < 4$) (cf. §1). We obtain

Theorem 1. *If the structural Lie algebra is zero, i.e. $\mathscr{F}$ is a compact foliation, then $\mathscr{G}_1$, $\mathscr{G}_2$, $\mathscr{G}_3$, and $\mathscr{G}_4$ are realizable. $\mathscr{G}_5$ and $\mathscr{G}_6$ are not realizable. $\mathscr{G}_7$ is realizable if and only if $k = -1$, and $\mathscr{G}_8$ is realizable if and only if $h = 0$.*

Theorem 2. *If the structural Lie algebra has dimension* 1, *then $\mathscr{G}_1$, $\mathscr{G}_2$, $\mathscr{G}_3$, $\mathscr{G}_4$, and $\mathscr{G}_5$ are realizable. $\mathscr{G}_6$ and $\mathscr{G}_7$ are not realizable and $\mathscr{G}_8$ with $h = 0$ is realizable.*

We do not know any realization of $\mathscr{G}_8$ with $h \neq 0$ and 1-dimensional structural Lie algebra of dimension 1.

Finally, it is remarkable that the realization of the pair $(\mathscr{G}_7, 2)$ depends on k. In fact we have

Theorem 3. *If the structural Lie algebra has dimension* 2, *then $\mathscr{G}_1$, $\mathscr{G}_5$, and $\mathscr{G}_8$ with $h = 0$ are realizable. $\mathscr{G}_2$, $\mathscr{G}_3$, $\mathscr{G}_4$, $\mathscr{G}_6$, and $\mathscr{G}_7$ with $k \in \mathbf{Q}$ are not realizable.*

We give a realization of $\mathscr{G}_7$ with $k \notin \mathbf{Q}$. A characterization of those k for which $\mathscr{G}_7$ is realizable and the $\mathscr{G}_8$ case, are still open.

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1. Preliminary definitions and results

Let $\mathscr{F}$ be a smooth foliation of codimension n on a smooth manifold M given by an integrable subbundle $L \subset TM$. We denote by $\mathscr{L}(M, \mathscr{F})$ the Lie algebra of foliated vector fields, i.e. $X \in \mathscr{L}(M, \mathscr{F})$ if and only if $[X, Y] \in \Gamma L$ for all $Y \in \Gamma L$. Thus, the set of sections of L, ΓL, is an ideal of $\mathscr{L}(M, \mathscr{F})$. The elements of $\mathscr{L}(M, \mathscr{F})/\Gamma L(M, \mathscr{F})$ are called basic vector fields.

If there is a family $\{X_1, \ldots, X_n\}$ of foliated vector fields of M such that the corresponding family $\{\overline{X}_1, \ldots, \overline{X}_n\}$ of basic vector fields has rank n everywhere, the foliation is called transversely parallelizable and $\{\overline{X}_1, \ldots, \overline{X}_n\}$ a transverse parallelism. If the vector subspace $\mathscr{G}$ of $\mathscr{X}(M, \mathscr{F})$ generated by $\{\overline{X}_1, \ldots, \overline{X}_n\}$ is a Lie subalgebra, the foliation is called a Lie foliation.

We shall use the following structure theorems (cf. [3] and [6]):

Theorem A. *Let $\mathscr{F}$ be a transversally parallelizable foliation on a compact manifold M of codimension n. Then*:

(a) *There is a Lie algebra $\mathscr{H}$ of dimension $q \leq n$.*
(b) *There is a locally trivial fibration $\pi\colon M \to W$ with compact fibre F and* $\dim W = n - q = m$.

(c) *There is a dense Lie* $\mathscr{H}$*-foliation on* F *such that:*
  (i) *The fibres of* π *are the closures of the leaves of* $\mathscr{F}$.
  (ii) *The foliation induced by* $\mathscr{F}$ *on each fibre of* π *is isomorphic to the* $\mathscr{H}$*-foliation on* F.

$\mathscr{H}$ is called the structural Lie algebra of $(M, \mathscr{F})$, π the basic fibration, and W the basic manifold. The foliation given by the fibres of π is denoted by $\overline{\mathscr{F}}$. Note that $\operatorname{codim}\overline{\mathscr{F}} + q = \operatorname{codim}\mathscr{F}$.

Theorem B. *Let* $\mathscr{F}$ *be a* $\mathscr{G}$*-foliation on a compact manifold* M *and let* G *be the connected simply-connected Lie group with Lie algebra* $\mathscr{G}$. *Let* $p: \widetilde{M} \to M$ *be the universal covering of* M. *Then there is a locally trivial fibration* $D: \widetilde{M} \to G$ *equivariant by* $\operatorname{Aut}(p)$ *(i.e. if* $D(x) = D(y)$ *then* $D(gx) = D(gy)$ *for all* $x, y \in \widetilde{M}$ *and* $g \in \operatorname{Aut}(p)$*) such that the foliation* $\widetilde{\mathscr{F}} = p^*\mathscr{F}$ *is given by the fibres of* D.

The natural morphism $h: \pi_1(M) \to \operatorname{Diff}(G)$ *is such that* $\Gamma = \operatorname{im}(h) \subset G$, *where the inclusion* $G \subset \operatorname{Diff}(G)$ *is by left translations.*

We shall also use some cohomological properties of the foliation. Recall that the basic forms complex is given by the forms $\alpha \in \Omega^*(M)$ such that $\mathscr{L}_X\alpha = 0$ and $i_X\alpha = 0$ for all $X \in \Gamma L$. The cohomology of this complex, $H^*(M, \mathscr{F})$, is the basic cohomology of the foliated manifold $(M, \mathscr{F})$. If $H^n(M, \mathscr{F}) \neq 0$ we say that $\mathscr{F}$ is homologically orientable or unimodular. We have (cf. [5]):

Theorem C. *Let* $\mathscr{F}$ *be an unimodular Lie* $\mathscr{G}$*-foliation on a compact manifold* M. *Then the Lie algebra* $\mathscr{G}$ *is unimodular.*

Finally, we recall that the 3-dimensional Lie algebras can be classified in eight families.

$\mathscr{G}_1$ (Abelian):
$$[e_1, e_2] = [e_1, e_3] = [e_2, e_3] = 0.$$
$\mathscr{G}_2$ (Heisenberg):
$$[e_1, e_2] = [e_1, e_3] = 0, \quad [e_2, e_3] = e_1.$$
$\mathscr{G}_3$ (so(3)):
$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2.$$
$\mathscr{G}_4$ (sl(2)):
$$[e_1, e_2] = e_3, \quad [e_2, e_3] = -e_1, \quad [e_3, e_1] = e_2.$$
$\mathscr{G}_5$ (Affine):
$$[e_1, e_2] = e_1, \quad [e_1, e_3] = [e_2, e_3] = 0.$$
$\mathscr{G}_6$:
$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = e_1 + e_2.$$
$\mathscr{G}_7$:
$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = ke_2, \ k \neq 0.$$

$\mathscr{G}_8$:

$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_2, \quad [e_2, e_3] = -e_1 + he_2, \quad h^2 < 4.$$

Notice that $\mathscr{G}_1$, $\mathscr{G}_2$, $\mathscr{G}_3$, and $\mathscr{G}_4$ are unimodular, $\mathscr{G}_5$ and $\mathscr{G}_6$ are not unimodular, $\mathscr{G}_7$ is unimodular only if $k = -1$, and $\mathscr{G}_8$ only if $h = 0$.

Remark. We can think that $\mathscr{G}_7$ is parametrized by $k \in [-1, 0) \cup (0, 1]$. In fact two of these algebras are isomorphic if and only if $k \cdot k' = 1$.

2. Lie flows of codimension 3

Let $\mathscr{F}$ be a Lie flow of codimension 3 on a compact manifold M. Since the closures of the leaves of $\mathscr{F}$ are the fibres of a bundle (cf. Theorem A), there are four possible cases.

Case 1. $\operatorname{codim} \overline{\mathscr{F}} = 3$.

In this case $\mathscr{F}$ is compact and the basic bundle is $M \to M/\mathscr{F}$. Thus the basic cohomology coincides with the de Rham cohomology of the compact manifold $M/\mathscr{F}$, and hence $H^3(M/\mathscr{F}) \neq 0$. By Theorem C, if such a flow exists it is transversely modeled on a unimodular Lie algebra. So $\mathscr{G}_5$ and $\mathscr{G}_6$ are not realizables, $\mathscr{G}_7$ is realizable (a priori) only if $k = -1$, and $\mathscr{G}_8$ only if $h = 0$.

We now give examples for each one of the remainder algebras.

- $\mathscr{G}_1$: Just consider the trivial bundle $T^1 \times T^3 \to T^3$.
- $\mathscr{G}_2$: Consider the trivial bundle $T^1 \times M \to M$ where M is the homogeneous space N/Γ of the Heisenberg group

$$N = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} ; a, b, c \in \mathbf{R} \right\}$$

by the discret uniform subgroup Γ of N given by the matrices of N with integer coefficients.

- $\mathscr{G}_3$: Just consider the trivial bundle $T^1 \times S^3 \to S^3$.
- $\mathscr{G}_4$: Consider the trivial bundle $T^1 \times T_1 W \to T_1 W$ where $T_1 W$ is the unit sphere bundle of the two hole torus W. $T_1 W$ is the homogeneous space $\mathrm{PSL}(2, \mathbf{R})/\pi_1(W)$ and therefore we have the desired example.
- $\mathscr{G}_7$ (with $k = -1$): Let $A \in \mathrm{SL}(2, \mathbf{Z})$ be a matrix with eigenvalues λ, $1/\lambda$ (being $\lambda > 0$ and $\lambda \neq 1$). We can give a solvable Lie group structure on $\mathbf{R}^3 = \mathbf{R} \times \mathbf{R}^2$ by

$$(t, u) \cdot (s, v) = (t + s, A^t \cdot v + u).$$

The Lie algebra of this group is $\mathscr{G}_7$ with $k = -1$ (cf. [4]). Moreover, the points of $\mathbf{R}^3$ with integer coordinates constitute a uniform discret subgroup Γ of $\mathbf{R}^3$. The quotient is usually denoted by T^3_A. Then, one example of a Lie flow transversely modeled on $\mathscr{G}_7$, with $k = -1$, is given by the trivial bundle $T^1 \times T^3_A \to T^3_A$.

• $\mathscr{G}_8$ (with $h = 0$) (P. Molino): Let us consider the flow given by the fibres of the trivial bundle $T^1 \times T^3 \to T^3$. Let $\theta^0, \theta^1, \theta^2, \theta^3$ denote the canonical coordinates in $T^1 \times T^3$. The parallelism given by $\partial/\partial\theta^1, \partial/\partial\theta^2, \partial/\partial\theta^3$ makes the fibres of the bundle an abelian Lie foliation. But we have basic functions enough to modify this parallelism. In fact, we can take

$$\begin{aligned} e_1 &= \cos\theta^1 \cdot \partial/\partial\theta^2 + \sin\theta^1 \cdot \partial/\partial\theta^3, \\ e_2 &= -\sin\theta^1 \cdot \partial/\partial\theta^2 + \cos\theta^1 \cdot \partial/\partial\theta^3, \\ e_3 &= -\partial/\partial\theta^1, \end{aligned}$$

to obtain a new parallelism with $[e_1, e_2] = 0$, $[e_1, e_3] = e_2$, $[e_2, e_3] = -e_1$, i.e. the flow is also transversely modeled on $\mathscr{G}_8$ (with $h = 0$).

Case 2. $\operatorname{codim}\overline{\mathscr{F}} = 2$.

In this case we give examples for $\mathscr{G}_1$, $\mathscr{G}_2$, $\mathscr{G}_3$, $\mathscr{G}_4$, $\mathscr{G}_5$, and $\mathscr{G}_8$ (with $h = 0$). We also prove that $\mathscr{G}_6$ and $\mathscr{G}_7$ are not realizable.

• $\mathscr{G}_1$: One example is given by the flow $(X, 0)$ on $T^2 \times T^2$ where X is a dense linear flow on T^2.

• $\mathscr{G}_2$: Let M be the homogeneous space of the Heisenberg group considered before. The flow on $M \times T^1$ whose integral curves are given by

$$\varphi_t(p) = \left(\begin{pmatrix} 1 & a & b+t \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, t + d \right)$$

where

$$p = \left(\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, d \right) \quad \text{and} \quad d \in \mathbf{R}\backslash\mathbf{Q}$$

is transverse to M, and the closure of each leaf is T^2. Hence it is one example of a $\mathscr{G}_2$-Lie flow with $\operatorname{codim}\overline{\mathscr{F}} = 2$.

• $\mathscr{G}_3$: As $S^3 = \mathrm{SU}(3)$, an example can be constructed by suspending the representation $h\colon \pi_1(S^1) \to \mathrm{Diff}(S^3)$ given by $h(1) = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$, where $\alpha \in \mathbf{R}\backslash\mathbf{Q}$.

• $\mathscr{G}_4$ (A. ElKacimi): Let $\mathscr{F}_0$ be the transverse affine Lie flow on T^3_A (cf. [1]). Using the fact that the affine group GA can be considered, lifting the map

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \to \frac{1}{\sqrt{a}} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

of GA in $\mathrm{SL}(2, \mathbf{R})$, as a Lie subgroup of $\widetilde{\mathrm{SL}}(2, \mathbf{R})$, and using also that the unfolding diagram of $\mathscr{F}_0$ (cf. Theorem B), $D_0\colon \widetilde{T}^3_A \to GA$, $\rho_0\colon \pi_1(T^3_A) \to GA$, the desired foliation can be constructed as follows:

Let $\widetilde{M} = \widetilde{T}^3_A \times \mathbf{R}$ be the universal covering of $M = T^3_A \times S^1$ and define $D\colon \widetilde{M} \to \widetilde{\mathrm{SL}}(2, \mathbf{R})$ and $\rho\colon \pi_1(M) \to \widetilde{\mathrm{SL}}(2, \mathbf{R})$ by $D(x, t) = D_0 x \cdot \tilde{\varphi}(t)$ and

$\rho(\gamma, n) = \rho_0(\gamma)\cdot\varphi(n)$, where $\tilde{\varphi}\colon \mathbf{R}\to\widetilde{\mathrm{SL}}(2, \mathbf{R})$ is a lift of the uniparametric subgroup $\varphi\colon \mathbf{R}\to \mathrm{SL}(2, \mathbf{R})$ given by

$$\varphi = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

It turns out, using the fact that $\tilde{\varphi}(n)$ is in the center of $\widetilde{\mathrm{SL}}(2, \mathbf{R})$, that ρ is an homomorphism and D is equivariant (i.e., $D((\gamma, n)\cdot(x, t)) = \rho(\gamma, n)\cdot D(x, t))$. Thus the fibres of D induce the desired Lie foliation on M (cf. [2] for details).

• $\mathscr{G}_5$: Let X be the generator of the transversely affine Lie flow on T^3_A. As we have that $\mathscr{G}_5 = \mathscr{A} + \mathbf{R}$, where $\mathscr{A}$ is the affine Lie algebra of dimension 2, the vector field $(X, 0)$ on $T^3_A \times S^1$ is transversely modeled on $\mathscr{G}_5$ and $\operatorname{codim}\overline{\mathscr{F}} = 2$.

• $\mathscr{G}_6$ and $\mathscr{G}_7$ are not realizable: Let $\mathscr{F}$ be a $\mathscr{G}_6$ or a $\mathscr{G}_7$ Lie flow on a compact manifold M. Fix a generator X of $\mathscr{F}$ and a transverse parallelism Y_1, Y_2, Y_3 such that $[\overline{Y}_1, \overline{Y}_2] = 0$, $[\overline{Y}_1, \overline{Y}_3] = \overline{Y}_1$, $[\overline{Y}_2, \overline{Y}_3] = \overline{Y}_1 + \overline{Y}_2$ for $\mathscr{G}_6$, and $[\overline{Y}_1, \overline{Y}_2] = 0$, $[\overline{Y}_1, \overline{Y}_3] = \overline{Y}_1$, $[\overline{Y}_2, \overline{Y}_3] = kY_2$ for $\mathscr{G}_7$. Let g be a Riemannian metric on M. Then we have the orthogonal decomposition $TM = T\overline{\mathscr{F}} + T\overline{\mathscr{F}}^{\perp}$ and we shall denote by Z^t and Z^n the tangent and the orthogonal parts of a vector field Z on M.

The set $T = \{p \in M;\ Y_1^n(p) = 0\}$ is open. In fact, if $p\in T$, Y_1 is tangent to $\overline{\mathscr{F}}$ in p, therefore Y_2^n, Y_3^n are independent in p. Hence they are independent in an open neighborhood U of p and we can write $Y_1^n = \lambda Y_2^n + \mu Y_3^n$ where λ, μ are basic functions on U. Computing now $[Y_1^n, Y_2^n]$ and $[Y_1^n, Y_3^n]$, we deduce the following system of differential equations:

$$\begin{aligned} Y_2^n(\lambda) + \mu\lambda + \mu &= 0, \\ Y_2^n(\mu) + \mu^2 &= 0, \\ Y_3^n(\lambda) - \lambda^2 &= 0, \\ Y_3^n(\mu) - \mu\lambda + \mu &= 0, \end{aligned}$$

for $\mathscr{G}_6$, and

$$\begin{aligned} Y_2^n(\lambda) + k\mu &= 0, \\ Y_2^n(\mu) &= 0, \\ Y_3^n(\lambda) + (1-k)\lambda &= 0, \\ Y_3^n(\mu) + \mu &= 0, \end{aligned}$$

for $\mathscr{G}_7$, with the initial conditions $\lambda(p) = \mu(p) = 0$.

This implies that $\mu = 0$ on the integral curves of Y_3 and Y_2. Due to transverse transitivity, $\mu = 0$ on U. It follows, in a similar way, that $\lambda = 0$ on U. Thus $Y = 0$ on U and T is open.

As it is also closed and M is supposed to be connected, $T = \varnothing$ or $T = M$.

But if $T = M$, we arrive in both cases ($\mathcal{G}_6$ and $\mathcal{G}_7$) to a contradiction. In fact, if we denote by $\theta^0, \theta^1, \theta^2, \theta^3$ the dual basis of X, Y_1, Y_2, Y_3 we have $d\theta^2 = -\theta^2 \wedge \theta^3$ in $\mathcal{G}_6$ and $d\theta^2 = k\theta^2 \wedge \theta^3$ $(k \neq 0)$ in $\mathcal{G}_7$. As $\theta^2(Z) = \theta^3(Z) = d\theta^2(Z, \cdot) = d\theta^3(Z, \cdot) = 0$ for each vector field Z tangent to $\overline{\mathcal{F}}$, the 1-forms θ^2 and θ^3 are projectable on the basic manifold $W = M/\overline{\mathcal{F}}$.

So we would have an exact volume element on the compact manifold W, which is a contradiction.

Therefore $T = \varnothing$.

Next we consider the set $Q = \bigcup_{a\in\mathbf{R}} Q_a$ where $Q_a = \{p \in M; Y_2^n(p) = aY_1^n(p)\}$.

Q is open: If $p \in Q$, there is $a \in \mathbf{R}$ such that $Y_2^n(p) = aY_1^n(p)$ and hence Y_3^n and Y_1^n are independent in p. So $Y_2^n = \lambda Y_1^n + \mu Y_3^n$ is an open neighborhood U of p with $\lambda(p) = a$ and $\mu(p) = 0$. Computing now $[Y_1, Y_2^n]$, $[Y_3, Y_2^n]$ and considering their tangent and normal parts one obtains the equations:

$$\begin{aligned} Y_1(\lambda) + \mu &= 0, \\ Y_1(\mu) &= 0, \\ Y_3(\lambda) + 1 &= 0, \\ Y_3(\mu) - \mu &= 0. \end{aligned}$$

As before, this yields $\mu = 0$, i.e. $Y_2^n = \lambda Y_1^n$ on U. Thus every point $x \in U$ is in $Q_{\lambda(x)} \subset Q$ and Q is open.

Q is closed: If $p \notin Q$, for each $a \in \mathbf{R}$, $Y_2^n(p) \neq aY_1^n(p)$. In particular, $Y_2^n(p) = 0$. As we have proved that $Y_1^n \neq 0$, the vector fields Y_1, Y_2 are linearly independent on p. Hence they are independent in an open neighborhood U of p, i.e. $U \subset M\backslash Q$ and Q is closed.

As M is connected, $Q = \varnothing$ or $Q = M$.

If $Q = \varnothing$, Y_1^n and Y_2^n are linearly independent in each point. So there are differentiable functions λ and μ globally defined on M, such that $Y_3^n = \lambda Y_1^n + \mu Y_2^n$. Computing now $[Y_1, Y_3^n]$, we obtain $Y_1(\lambda) = 1$, but as M is compact this is impossible.

If $Q = M$, for each $p \in M$ there is $a(p) \in \mathbf{R}$ such that $Y_2^n(p) = a(p)Y_1^n(p)$. This gives rise to a differentiable basic function a on M with $Y_2^n = a \cdot Y_1^n$. Equivalently, $Y_2 - a \cdot Y_1$ is everywhere tangent to $\overline{\mathcal{F}}$. Since $[Y_3, Y_2 - a \cdot Y_1]$ must be in $\overline{\mathcal{F}}$ we obtain $Y_3(a) = -1$ for $\mathcal{G}_6$, which is again a contradiction, and $Y_3(a) = (1-k)a$ for $\mathcal{G}_7$. If $k \neq 1$, the only possibility is $a = 0$ and so Y_2 is everywhere tangent to $\overline{\mathcal{F}}$. As before, this yields a contradiction because $d\theta^1 = -\theta^1 \wedge \theta^3$, with θ^1 and θ^3 projectables on $W = M/\overline{\mathcal{F}}$. If $k = 1$ it follows that a is constant over the integral curves of Y_1, Y_2, Y_3, i.e. a is constant. With $\omega^0, \omega^1, \omega^2, \omega^3$ the dual basis of $X, Y_2 - aY_1, Y_1, Y_3$, we obtain $d\omega^2 = -\omega^2 \wedge \omega^3$ with ω^2, ω^3 projectables on W, again a contradiction. This proves that $\mathcal{G}_6$ and $\mathcal{G}_7$ are not realizable.

• $\mathscr{G}_8$ (with $h=0$): The same construction as before. If $\theta^0, \theta^1, \theta^2, \theta^3$ are the canonical coordinates on $T^2 \times T^2$, the vector field $X = \partial/\partial\theta^0 + \alpha\partial/\partial\theta^1$, $\alpha \in \mathbf{R}\backslash\mathbf{Q}$ is transversely abelian for the parallelism $\partial/\partial\theta^1, \partial/\partial\theta^2, \partial/\partial\theta^3$ and has $\operatorname{codim}\overline{\mathscr{F}} = 2$. We modify this parallelism by taking

$$\begin{aligned} e_1 &= \cos\theta^2 \cdot \partial/\partial\theta^1 + \sin\theta^2 \cdot \partial/\partial\theta^3, \\ e_2 &= -\sin\theta^2 \cdot \partial/\partial\theta^1 + \cos\theta^2 \cdot \partial/\partial\theta^3, \\ e_3 &= -\partial/\partial\theta^2. \end{aligned}$$

Thus X is also transversely modeled on $\mathscr{G}_8$ (with $h=0$).

Case 3. $\operatorname{codim}\overline{\mathscr{F}} = 1$.

In this case the structural Lie algebra has dimension 2. As this algebra is abelian (cf. [1]), $\mathscr{G}_3$ and $\mathscr{G}_4$ are not realizable because they do not have abelian subalgebras of dimension 2. Examples for the algebras $\mathscr{G}_1, \mathscr{G}_2, \mathscr{G}_5$, and $\mathscr{G}_8$ ($h=0$) are given. For the algebra $\mathscr{G}_7$ we prove that the only realizable cases are when $k \notin \mathbf{Q}$, an example will be given. We also prove that $\mathscr{G}_6$ is not realizable.

• $\mathscr{G}_1$: Consider the flow $(X, 0)$ on $T^3 \times T^1$ where X is a dense linear flow on T^3.

• $\mathscr{G}_2$ is not realizable: As $\mathscr{G}_2$ is unimodular and $\operatorname{codim}\overline{\mathscr{F}} = 1$, $\mathscr{F}$ is unimodular (cf. [5]) and it follows, from the results by Molino (cf. [6]), that the central transverse sheaf $\mathscr{C}$ admits a global trivialization, i.e. there are independent foliated vector fields v, w tangents to the $\mathscr{F}$ closure which commute, as transverse fields, with every global foliated vector field. In particular, $[v, e_i] = [w, e_i] = 0$. Writing

$$v = \lambda e_1 + \mu e_2 + \nu e_3, \qquad w = \alpha e_1 + \beta e_2 + \gamma e_3$$

we obtain $v = \lambda e_1$ and $w = \alpha e_1$, which is a contradiction.

• $\mathscr{G}_5$: Let X be the generator of the transversely affine Lie flow on T_A^3. The vector field $(X, \alpha\partial/\partial\theta)$ on $T_A^3 \times S^1$, with $\alpha \in \mathbf{R}\backslash\mathbf{Q}$ and θ the coordinate function on S^1, is transversely modeled on $\mathscr{G}_5 = \mathscr{A} + \mathbf{R}$ and $\operatorname{codim}\overline{\mathscr{F}} = 1$.

• $\mathscr{G}_8$ ($h=0$): The same construction as before. If $\theta^0, \theta^1, \theta^2, \theta^3$ are the canonical coordinates on $T^3 \times T^1$, the vector field $X = \partial/\partial\theta^0 + \alpha\partial/\partial\theta^1 + \beta\partial/\partial\theta^2$ with α, β rationally independent, admits

$$\begin{aligned} e_1 &= \cos\theta^3 \cdot \partial/\partial\theta^0 + \sin\theta^3 \cdot \partial/\partial\theta^1, \\ e_2 &= -\sin\theta^3 \cdot \partial/\partial\theta^0 + \cos\theta^3 \cdot \partial/\partial\theta^1, \\ e_3 &= -\partial/\partial\theta^3, \end{aligned}$$

as a transverse parallelism. But e_1, e_2, e_3 is a basis of $\mathscr{G}_8$ with $h=0$.

• Next we study the remainder algebras $\mathscr{G}_6, \mathscr{G}_7$, and $\mathscr{G}_8$ ($h \neq 0$). As the center of these algebras are trivial, the corresponding connected simply-connected groups G_6, G_7, G_8 can be obtained as $e^{t\cdot \mathrm{ad}\,\alpha}$, $\alpha \in \mathscr{G}_i$ with $i = 1, 2, 3$. We find that these groups can be thought of as $\mathbf{R}^3 = \mathbf{R}^2 \times \mathbf{R}$ with the product $(p, t)\cdot(p', t') = (p + e^{-\Lambda t}\cdot p', t + t')$ and Λ depending on the algebra.

For $\mathscr{G}_6$,

$$\Lambda = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad e^{-\Lambda t} = \begin{pmatrix} e^{-t} & -te^{-t} \\ 0 & e^{-t} \end{pmatrix}.$$

For $\mathscr{G}_7$,

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}, \qquad e^{-\Lambda t} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-kt} \end{pmatrix}.$$

For $\mathscr{G}_8$,

$$\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & h \end{pmatrix}, \qquad e^{-\Lambda t} = C(t)\cdot \begin{pmatrix} \cos(\varphi + t) & -\sin t \\ \sin t & \cos(\varphi - t) \end{pmatrix},$$

where $C(t) = \frac{2}{\alpha}e^{\beta t}$ and $\alpha = \sqrt{4 - h^2}$, $\beta = \tan\varphi = h/\alpha$, $(\sin\varphi = h/2, \cos\varphi = \alpha/2)$.

The basis we have used to define the algebras are given in this case by the following left invariant fields.

For $\mathscr{G}_6$,

$$e_1 = e^{-t}\frac{\partial}{\partial x}, \quad e_2 = -te^{-t}\frac{\partial}{\partial x} + e^{-t}\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial t}.$$

For $\mathscr{G}_7$,

$$e_1 = e^{-t}\frac{\partial}{\partial x}, \quad e_2 = e^{-kt}\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial t}.$$

For $\mathscr{G}_8$,

$$e_1 = \frac{2}{\alpha}e^{-\beta t}\left(\cos(t + \varphi)\frac{\partial}{\partial x} + \sin t\frac{\partial}{\partial y}\right),$$
$$e_2 = \frac{2}{\alpha}e^{-\beta t}\left(-\sin t\frac{\partial}{\partial x} + \cos(t - \varphi)\frac{\partial}{\partial y}\right),$$
$$e_3 = -\frac{\alpha}{2}\frac{\partial}{\partial t}.$$

Suppose now that we have a $\operatorname{codim}\overline{\mathscr{F}} = 1$ realization on a compact manifold M of one of these algebras. We shall denote the algebra by $\mathscr{G}$ and the corresponding group by G. The basic fibration is: $T^3 \to M \to T^1$ and, as $\pi_1(T^3) = \mathbf{Z}^3$, $\pi_1(T^1) = \mathbf{Z}$, and $\pi_2(T^1) = 0$, the corresponding homotopy exact sequence is $0 \to \mathbf{Z}^3 \to \pi_1(M) \to \mathbf{Z} \to 0$.

Since this exact sequence has a section, $\pi_1(M)$ is the semidirect product of $\mathbf{Z}^3$ with $\mathbf{Z}$, i.e. $\pi_1(M)$ is the product $\mathbf{Z}^3 \times \mathbf{Z}$ with the operation $(x, t)\cdot(y, x) = (x + t\cdot y, t + s)$ where $t\cdot y$ represents the natural action of $\mathbf{Z}$ on $\mathbf{Z}^3$. To be precise, if $\varphi\colon T^3 \to T^3$ is the diffeomorphism which gives the bundle, then the action is $t\cdot y = \varphi_*^t\cdot y$ where $\varphi_*\colon \pi_1(T^3) \to \pi_1(T^3)$ is the morphism induced by φ. We shall denote the group by $\mathbf{Z}^3 \times_\varphi \mathbf{Z}$.

Since $\mathscr{F}$ is a Lie foliation, we have the unfolding diagram:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{D} & G \\ \downarrow & & \\ M & & \end{array}$$

and the holonomy representation $h: \pi_1(M) \to h(\pi_1(M)) = \Gamma \subset G$ with $D(\gamma \cdot \tilde{x}) = h(\gamma) \cdot D\tilde{x}$, $\tilde{x} \in \widetilde{M}$, $\gamma \in \pi_1(M)$.

As $M/\overline{\mathscr{F}}$ is diffeomorphic to $G/\overline{\Gamma}$ (cf., for instance, [5]) we have that $\overline{\Gamma}$ is a two-dimensional closed subgroup of G.

The Lie algebra $\mathscr{H}$ of $\overline{\Gamma}_e$ (the identity component of $\overline{\Gamma}$) is named the structural Lie algebra of $\mathscr{F}$ and, in the case of flows, it is abelian (cf. [1]).

But it is easy to see that the only two-dimensional abelian subalgebra of $\mathscr{G}$ is $\langle e_1, e_2 \rangle$, thus $\mathscr{H} = \langle e_1, e_2 \rangle$. Looking at the expressions for e_1 and e_2 in $\mathscr{G}_6$, $\mathscr{G}_7$, and $\mathscr{G}_8$, we see that $\mathscr{H} = \langle \partial/\partial x, \partial/\partial y \rangle$ and hence $\overline{\Gamma} \simeq \mathbf{R}^2 \times \mathbf{Z}\varepsilon$, $\varepsilon > 0$.

Notice that $\overline{\Gamma}_e = \mathbf{R}^2 \times \{0\}$ is abelian.

Lemma. *Let A be an abelian subgroup of Γ. Then A is contained in $\mathbf{R}^2 \times \{0\}$ or there is an element $a = (a_1, a_2, a_3)$ with $a_3 \neq 0$ such that $A = \{a^n, n \in \mathbf{Z}\}$.*

Proof. If A is not in $\mathbf{R}^2 \times \{0\}$, then $A \cap (\mathbf{R}^2 \times \{0\}) = 0 \in \mathbf{R}^3$.

Otherwise, there is $(p, 0) \in A$, $p \neq 0$, and $(q, t) \in A$, $t \neq 0$. As A is abelian we have that $(p, 0)(q, t) = (q, t)(p, 0)$. Then $q + e^{-\Lambda t} \cdot p = q + p$ and this implies that $t = 0$, except for $\mathscr{G}_8$ with $h = 0$, but this case is not considered here. Therefore $A \cap (\mathbf{R}^2 \times \{0\}) = 0 \in \mathbf{R}^3$.

In particular, A has at most one element in each level $\mathbf{R}^2 \times \{m\varepsilon\}$, $m \in \mathbf{Z}$. In fact, $a_1 \cdot a_2^{-1} \in A \cap (\mathbf{R}^2 \times \{0\}) = 0$ and $a_1 = a_2$.

Let $a = (a_1, a_2, n\varepsilon)$ be the element of A in the lower level. For each $b = (b_1, b_2, m\varepsilon) \in A$, we put $m = nd + r$; then ba^{-d} is an element of A in the $r\varepsilon$ level and hence $r = 0$, i.e. $b = a^d$ and this proves the lemma.

Proposition 1. *Let the notation be as above. Then $(\mathbf{R}^2 \times \{0\}) \cap \Gamma = h(\mathbf{Z}^3)$.*

Proof. Applying the lemma we have four possibilities:

(i) $h(\mathbf{Z}^3)$ and $h(\mathbf{Z})$ are both contained in $\mathbf{R}^2 \times \{0\}$. Then Γ, generated by $h(\mathbf{Z}^3)$ and $h(\mathbf{Z})$, is contained in $\mathbf{R}^2 \times \{0\}$, which contradicts $\mathbf{R}^3/\overline{\Gamma} = S^1$.

(ii) $h(\mathbf{Z}^3)$ is contained in $\mathbf{R}^2 \times \{0\}$ and $h(\mathbf{Z}) = \{a^n, n \in \mathbf{Z}\}$ with $a \notin \mathbf{R}^2 \times \{0\}$. As $h(\mathbf{Z}^3)$ is a normal subgroup of Γ, for each $b \in h(\mathbf{Z}^3)$ we have $aba^{-1} = b'$ which is in $h(\mathbf{Z}^3)$. Hence, the elements of $(\mathbf{R}^2 \times \{0\}) \cap \Gamma$ can be written as

$$\sigma = b_1 a^{r_1} b_2 a^{r_2} b_3 a^{r_3} \cdots b_k a^{r_k}$$

with $\sum r_i = 0$ and $b_i \in h(\mathbf{Z}^3)$. That is, $\sigma = \tilde{b} \cdot a^{\sum r_i} = \tilde{b} \in h(\mathbf{Z}^3)$, i.e. $(\mathbf{R}^2 \times \{0\}) \cap \Gamma = h(\mathbf{Z}^3)$.

(iii) $h(\mathbf{Z})$ is contained in $\mathbf{R}^2 \times \{0\}$ and $h(\mathbf{Z}^3) = \{a^n, n \in \mathbf{Z}\}$ with $a \notin \mathbf{R}^2 \times \{0\}$. In this case, Γ is abelian because if we let $h(1) = b$ we have $bab^{-1} = a^k$. This implies $k = 1$ and $ab = ba$. As in (ii), this implies that $\Gamma \cap \mathbf{R}^2 \times \{0\} = h(\mathbf{Z})$ which is not dense in $\mathbf{R}^2 \times \{0\}$.

(iv) $h(\mathbf{Z}) = \{a^n, n \in \mathbf{Z}\}$ and $h(\mathbf{Z}^3) = \{b^n, n \in \mathbf{Z}\}$ with $a, b \notin \mathbf{R}^2 \times \{0\}$. As before, $aba^{-1} = b^k$ and therefore Γ is abelian. So the elements of $\mathbf{R}^2 \times \{0\}) \cap \Gamma$ can be written as $a^n \cdot b^{-n} = (a \cdot b^{-1})^n$ and this is not dense in $\mathbf{R}^2 \times \{0\}$.

Remark. Three elements $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{R}^3$ can generate a dense subgroup of $\mathbf{R}^2$. In fact, it suffices to take $\mathbf{u} = \lambda \mathbf{v} + \mu \mathbf{w}$ with λ, μ, and $\lambda/\mu \in \mathbf{R}\backslash\mathbf{Q}$. So, a priori, it is possible to have $\overline{h(\mathbf{Z}^3)} = \mathbf{R}^2 \times \{0\}$.

Proposition 2. *$\mathcal{G}_6$ is not realizable.*

Proof. If such a realization exists, the subgroup $h(\mathbf{Z}^3)$ is normal in Γ. Let $h(\mathbf{Z}^3) = \langle (p_1, 0), (p_2, 0), (p_3, 0) \rangle$ and $h(\mathbf{Z}) = \langle (p, t) \rangle$ with $t > 0$; then the normality condition can be written as $e^{-\Lambda t} \cdot p_i = \sum_{j=1}^{3} \lambda_1^j \cdot p_j$, where $\lambda_i^j \in \mathbf{Z}$.

The matrix $A = (\lambda_i^j)$ corresponds, in fact, to $\varphi_*: \mathbf{Z}^3 \to \mathbf{Z}^3$ so it is invertible, and, as we are assuming orientability, we have $\det A = 1$. Let $v_1 = (a_1, b_1, c_1)$ and $v_2 = (a_2, b_2, c_2)$, where $p_1 = (a_1, a_2)$, $p_2 = (b_1, b_2)$, and $p_3 = (c_1, c_2)$. From the above equations we have

$$Av_1 = a \cdot a_1 + a \log a \cdot v_2, \quad Av_2 = a \cdot v_2,$$

where $a = e^{-t}$.

Completing v_1, v_2 to a basis $\{v_1, v_2, v_3\}$, the matrix A can be written

$$\begin{pmatrix} a & 0 & \alpha \\ a \log a & a & \beta \\ 0 & 0 & a^{-2} \end{pmatrix}$$

and satisfies

$$2a + \frac{1}{a^2} = p, \quad a^2 + \frac{2}{a} = q,$$

with $p, q \in \mathbf{Z}$, $0 < a < 1$, and $a \in \mathbf{R}\backslash\mathbf{Q}$. But this is impossible because the equations imply that $pa^2 - 2qa + 3 = 0$ and hence $a = (q \pm \sqrt{q^2 - 3p})/p$. In particular, $\sqrt{q^2 - 3p} \in \mathbf{R}\backslash\mathbf{Q}$. Substituting a in the first equation above we conclude, after a short computation, that $p = q = 3$, which is in contradiction with $a \in \mathbf{R}\backslash\mathbf{Q}$. So $\mathcal{G}_6$ is not realizable.

Proposition 3. *The Lie algebras of the $\mathcal{G}_7$ family with $k \in \mathbf{Q}$ are not realizable.*

Proof. Proceeding as in Proposition 2, we obtain that e^{-t} and e^{-kt} are eigenvalues of A.

The characteristic polynomial of A, $x^3 - px^2 + qx - 1$, has three roots, ξ, ξ^k, and $\xi^{-(k+1)}$ with $\xi = e^{-t}$. As $t > 0$ we have $0 < \xi < 1$. This implies, from standard arguments in Galois theory (see lemma below), that $k \notin \mathbf{Q}$; i.e. the Lie algebras of $\mathscr{G}_7$ with $k \in \mathbf{Q}$ are not realizable.

The authors are grateful to P. Ara for his remarks about the following lemma.

Lemma. *Let* $f(x) = x^3 - px^2 + qx - 1$ *be a polynomial with* $p, q \in \mathbf{Z}$. *If there are* $k \in \mathbf{R}\backslash\{0\}$ *and* $\xi \in (0, 1)$ *such that the roots of* $f(x)$ *can be written as* $\xi, \xi^k, \xi^{-(k+1)}$, *then* $k \in \mathbf{R}\backslash\mathbf{Q}$.

Proof. First we observe that any rational root of this polynomial must be 1 or -1, and so it is irreducible over $\mathbf{Q}$. Hence the Galois group of $f(x)$ over $\mathbf{Q}$ is $\mathbf{Z}_3$ or the symmetric group S_3. In both cases there is an automorphism σ of the splitting field $\mathscr{K}$ of order 3, which is the identity over $\mathbf{Q}$. This automorphism permutes the roots, i.e. $\sigma(\xi) = \xi^k$, $\sigma(\xi^k) = \xi^{-k-1}$, $\sigma(\xi^{-k-1}) = \xi$, or $\sigma(\xi) = \xi^{-k-1}$, $\sigma(\xi^k) = \xi$, $\sigma(\xi^{-k-1}) = \xi^k$.

If $k = p/q$, using that $\sigma(x^{1/q}) = \pm\sigma(x)^{1/q}$, we obtain $\xi^{-k-1} = \sigma(\xi^k) = \sigma(\xi^{p/q}) = \sigma((\xi^p)^{1/q}) = (\sigma(\xi)^p)^{1/q} = \xi^{k^2}$ in the first case and $\xi = \sigma(\xi^x) = \xi^{(-k-1)k}$ in the second. This implies that $k^2 + k + 1 = 0$, which is impossible. Thus $k \notin \mathbf{Q}$ and the lemma is proved.

Example. Now we give an example of a Lie flow on a compact manifold M transversely modeled over a Lie algebra $\mathscr{G}$ of the family $\mathscr{G}_7$ with structural Lie algebra of dimension 2.

Let

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & 3 \end{pmatrix}$$

be an element of $\mathrm{SL}(3; \mathbf{R})$.

The eigenvalues are $\lambda_j = 2 + 2\cos((6\pi_j - 4\pi)/9)$ where $j = 1, 2, 3$. We have $2 + 2\cos(8\pi/9) < 1 < 2 + 2\cos(14\pi/9) < 2 + 2\cos(2\pi/9)$. If we let $\xi = \lambda_2$, there is a $k < 0$ such that $\xi^k = \lambda_3$. In this case $\lambda_1 = \xi^{-k-1}$. Here k is the quotient of logarithms of algebraic numbers. Notice that this is a necessary condition for the corresponding algebra to be realizable.

Thus we have the eigenvectors $u_j = (\lambda_j - 3, \lambda_j(\lambda_j - 3) - 1, 1)$.

A computation shows that the components of these vectors have irrational quotient, i.e. they induce dense linear flows in T^3.

Now we consider the compact manifold $T_A^4 = T^3 \times \mathbf{R}/\sim$, where $(x, t) \sim (A\cdot x, t+1)$. As the direction given by u_1 is invariant by A, it induces a global flow in T_A^4. This flow is transversely modeled over the Lie algebra of $\mathscr{G}_7$ with $k = \log\lambda_3/\log\lambda_2 < 0$. To verify this we observe that an invariant transverse parallelism in $T^3 \times \mathbf{R}$ is given by

$$e_1 = \xi^t u_2, \quad e_2 = \xi^{kt} u_3, \quad e_3 = -\frac{1}{\log\xi}\frac{\partial}{\partial t},$$

and it satisfies $[e_1, e_2] = 0$, $[e_1, e_3] = e_1$, $[e_2, e_3] = ke_2$.

Remark. We do not know any realization of $\mathscr{G}_8$ with $h \neq 0$ and $\operatorname{codim}\overline{\mathscr{F}} = 1$.

Case 4. $\operatorname{codim}\overline{\mathscr{F}} = 0$.

This is a trivial case because the transverse algebra coincides with the structural algebra and so it is abelian. Only $\mathscr{G}_1$ is realizable (a linear dense flow on T^4).

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