Some geometrical applications of Fourier series

A. Reventós

Homage to Xosé Masa. June 2018

The isoperimetric inequality

$$\Delta = L^2 - 4\pi F \geq 0$$

$F =$ area of a convex set K

$L =$ length of ∂K

Equality if and only if K is a circle

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Second part of the talk $\Delta \leq \bullet$ upper bounds

Bonnesen-style inequalities

Third part: the visual angle

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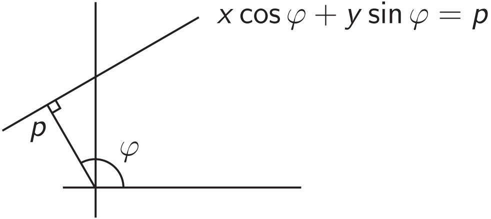
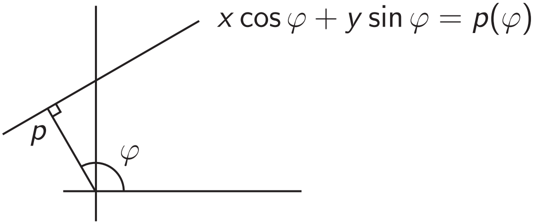
Preliminaries

The support function and the Steiner point

Support function

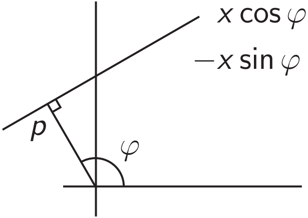


Support function



Uniparametric family of lines

Envelope



$$x\cos\varphi + y\sin\varphi = p(\varphi)$$
$$-x\sin\varphi + y\cos\varphi = p'(\varphi)$$

$$x = p\cos\varphi - p'\sin\varphi$$
$$y = p\sin\varphi + p'\cos\varphi$$

Arc length

$$dx = -(p + p'') \sin\varphi \, d\varphi$$

$$dy = (p + p'') \cos\varphi \, d\varphi$$

$$ds^2 = (p + p'')^2 \, d\varphi^2$$

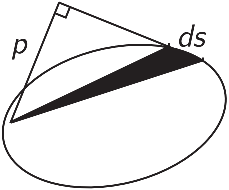
For convex sets $p + p'' > 0$, $ds = \rho \, d\varphi$

$$\rho = p + p'' = \text{curvature radius}$$

Length of the boundary of convex sets

$$L = \int ds = \int_0^{2\pi} (p + p'')\, d\varphi = \int_0^{2\pi} p\, d\varphi$$

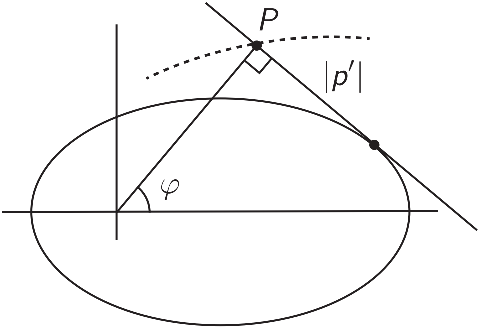
Area of convex sets



$$F = \frac{1}{2}\int_{\partial K} p\,ds = \frac{1}{2}\int_0^{2\pi} p(p + p'')\,d\varphi$$

$$= \frac{1}{2}\int_0^{2\pi} (p^2 - p'^2)\,d\varphi$$

Pedal curve



$$A = \frac{1}{2}\int_0^{2\pi} p^2 \, d\varphi$$

$$A \geq F$$

The pedal curve results from the orthogonal projection of a fixed point on the tangent lines of a given curve

Steiner point

The Steiner point is the centroid of K with respect to the mass distribution given by the curvature of ∂K.

$$x_M = \frac{\displaystyle\int_0^L x\,k\,ds}{\displaystyle\int_0^L k\,ds}, \qquad y_M = \frac{\displaystyle\int_0^L y\,k\,ds}{\displaystyle\int_0^L k\,ds}.$$

Steiner point

Substituting $x = p\cos\varphi - p'\sin\varphi$, and $ds = \rho\, d\varphi$,

$$x_M = \frac{\int (p\cos\varphi - p'\sin\varphi)k \cdot \rho \cdot d\varphi}{2\pi} = \frac{1}{\pi}\int_0^{2\pi} p \cdot \cos\varphi\, d\varphi$$

$$y_M = \frac{1}{\pi}\int_0^{2\pi} p \cdot \sin\varphi\, d\varphi$$

$$S(K) = (x_M, y_M), \quad S(K) \in K$$

Steiner point

The Steiner point is also characterized as being the point with pedal curve enclosing minimum area.

Fourier series

$$p(\varphi) = a_0 + \sum_{1}^{\infty} (a_n \cos n\varphi + b_n \sin n\varphi)$$

$$S(K) = (a_1, b_1)$$

Taking $S(K)$ as origin $\qquad p(\varphi) = a_0 + \sum_2^\infty (a_n \cos n\varphi + b_n \sin n\varphi)$

Fourier series

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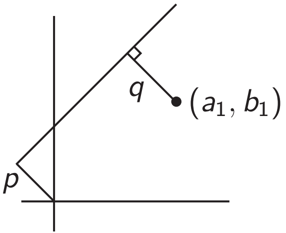

$$q = |a_1 \cos \varphi + b_1 \sin \varphi - p|$$

Taking $S(K)$ as origin $p(\varphi) = a_0 + \sum_2^{\infty} (a_n \cos n\varphi + b_n \sin n\varphi)$

Fourier series

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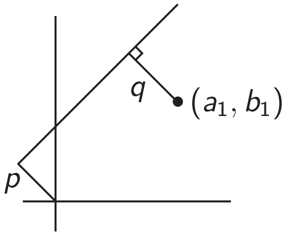

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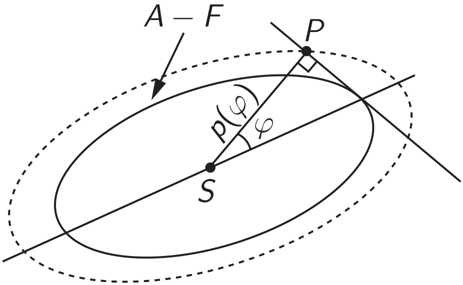
Taking $S(K)$ as origin

$$p(\varphi) = a_0 + \sum_{2}^{\infty} (a_n \cos n\varphi + b_n \sin n\varphi)$$

Theorem

$\Delta \geq 3\pi(A - F) \geq 0$

$$\Delta = L^2 - 4\pi F$$
$$A = \textit{area pedal curve w.r.t. Steiner point}$$



Theorem

$\Delta \geq 3\pi(A - F) \geq 0.$

Proof.

Parseval

$$\int_0^{2\pi} p^2\, d\varphi = 2\pi a_0^2 + \pi \sum_{n\geq 2} c_n^2; \qquad c_n^2 = a_n^2 + b_n^2,$$

$$\int_0^{2\pi} p'^2\, d\varphi = \pi \sum_{n\geq 2} n^2 c_n^2.$$

$$L = \int_0^{2\pi} p\, d\varphi, \qquad F = \frac{1}{2}\int_0^{2\pi} (p^2 - p'^2)\, d\varphi = A - \frac{1}{2}\int_0^{2\pi} p'^2\, d\varphi$$

$$L^2 - 4\pi F = \left(\int_0^{2\pi} p\,d\varphi\right)^2 - 2\pi\int_0^{2\pi}(p^2 - p'^2)\,d\varphi$$

$$= (2\pi a_0)^2 - 2\pi[2\pi a_0^2 + \pi\sum c_n^2] + 2\pi[\pi\sum_{n\geq 2} n^2 c_n^2]$$

$$= 2\pi^2\sum_{n\geq 2}(n^2 - 1)c_n^2$$

In particular $\Delta \geq 0$

$$L^2 - 4\pi F = \left(\int_0^{2\pi} p\, d\varphi\right)^2 - 2\pi \int_0^{2\pi} (p^2 - p'^2)\, d\varphi$$

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$$= 2\pi^2 \left(\sum (n^2 - 1) c_n^2\right)$$

$$\geq \frac{3\pi^2}{2} \sum_2 n^2 c_n^2 = \frac{3\pi}{2} \int_0^{2\pi} p'^2\, d\varphi = 3\pi(A - F)$$

□

Equality

$\Delta = 3\pi(A - F)$ if and only if ∂K is a circle or a curve parallel to an astroid (a 4-cusped hypocicloid) at distance $L/2\pi$.

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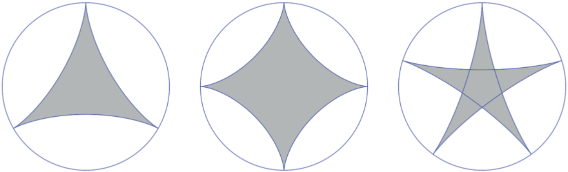
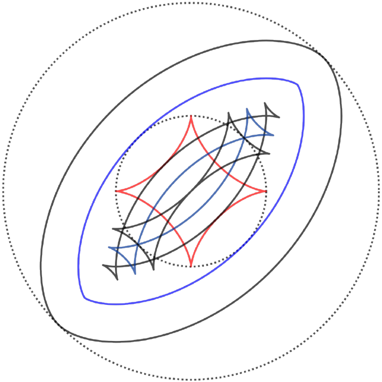


$k = 3$ $\quad$ $k = 4$ $\quad$ $k = \frac{5}{2}$

$R = kr$ The locus of a point on a circle of radius r when it rolls inside a circle of radius R.

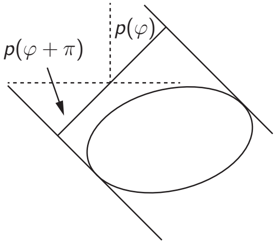
Parallel sets



red: astroid; blue: parallel to the astroid

Constant width

Width: $p(\phi) + p(\phi + \pi) = 2\sum_1^\infty \left(a_{2n}\cos(2n\phi) + b_{2n}\sin(2n\phi) \right)$



Constant width $\Rightarrow$ only odd terms

In this case we improve $\Delta \geq 3\pi(A - F)$ to

Theorem

$$\Delta \geq \frac{32}{9}\pi(A - F).$$

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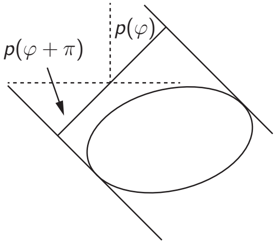

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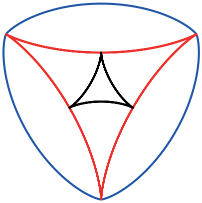
Theorem

$$\Delta \geq \frac{32}{9}\pi(A - F).$$

Equality holds if and only if C is a circle or a curve parallel to an hypocycloid of three cusps.

In this case the evolute of C and the interior parallel curve to C at distance $L/2\pi$, are similar with ratio 3.



Second part: upper bounds

$$L^2 = \left(\int_0^{2\pi} p \, d\varphi\right)^2 \le \int_0^{2\pi} p^2 \, d\varphi \cdot 2\pi = 4\pi A$$

$$\Rightarrow \Delta = L^2 - 4\pi F \le 4\pi(A - F)$$

$$\boxed{3\pi(A - F) \le \Delta \le 4\pi(A - F)}$$

Second part: upper bounds

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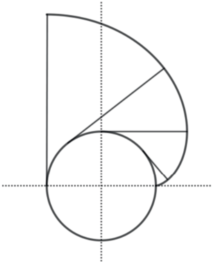
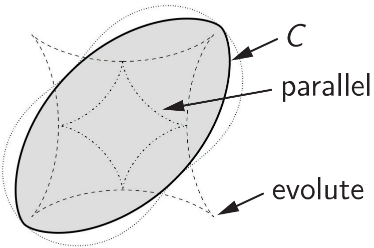
$$\boxed{3\pi(A - F) \leq \Delta \leq 4\pi(A - F)}$$

Hurwitz inequality

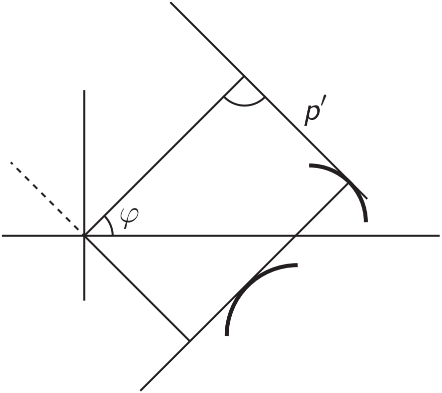
$$\Delta \leq \pi |F_e|$$

$$\begin{aligned} \text{Evolute} &= \text{envelope of normals} \\ &= \text{locus of centers of curvature} \end{aligned}$$





Generalized support function



$$p_e(\varphi) = -p'\left(\varphi + \frac{\pi}{2}\right)$$

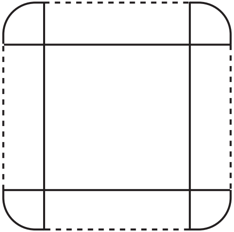
Algebraic area:

$$F_e = \frac{1}{2}\int_0^{2\pi} (p'^2 - p''^2)\, d\varphi$$

Hurwitz inequality is based on

1. Area of parallel sets
2. Wirtinger inequality

1. $F_r = F + L\,r + \pi r^2$



$$F_{-\frac{L}{2\pi}} = F + L\left(-\frac{L}{2\pi}\right) + \pi\frac{L^2}{4\pi^2} = F - \frac{L^2}{4\pi}$$

$$\Delta = 4\pi\left|F_{-\frac{L}{2\pi}}\right|$$

2. Wirtinger inequality

If $\int_0^{2\pi} q(\varphi)\, d\varphi = 0$, then $\int_0^{2\pi} q'^2\, d\varphi \geq \int_0^{2\pi} q^2\, d\varphi$.

In particular, $\int_0^{2\pi} q''^2\, d\varphi \geq \int_0^{2\pi} q'^2\, d\varphi$.

$$\Rightarrow F_e \leq 0$$

2. Wirtinger inequality

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In particular, $\int_0^{2\pi} q''^2\,d\varphi \geq \int_0^{2\pi} q'^2\,d\varphi$.

$$\Rightarrow F_e \leq 0$$

Proof of Hurwitz inequality

Given a convex set of length L and support function p we put
$q = p - \frac{L}{2\pi}$, and denote $W_q = \int_0^{2\pi} (q'^2 - q^2)\, d\varphi \geq 0$.

Thus, by the formula of algebraic area,

$F_{-\frac{L}{2\pi}} = -\frac{1}{2} W_q;$

$F_e = -\frac{1}{2} W_{q'}.$

Substituting

$$\pi|F_e| - \Delta = \frac{\pi}{2}(W_{q'} - 4W_q) \quad \text{[it can be proved that } \geq 0\text{]}.$$

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Lema. Let $q = q(\varphi)$ a 2π-periodic $\mathcal{C}^2$ function defined on $[0, 2\pi]$. Then

$$W_{q'} \geq 4W_q + \frac{2}{\pi}\left(\int_0^{2\pi} q \, d\varphi\right)^2.$$

Equality holds if and only if
$q(\varphi) = a_0 + a_1 \cos\varphi + b_1 \sin\varphi + a_2 \cos\varphi + b_2 \sin\varphi$

Corollary. $\pi|F_e| - \Delta \geq 0$ with equality if and only if ∂K is a circle or it is a curve parallel to an astroid at distance $L/2\pi$.

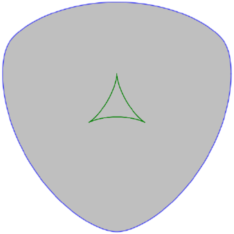
Hurwitz inequality [constant width]

For constant width we improve $\Delta \leq \pi |F_e|$ to

Theorem

$$\Delta \leq \frac{4}{9} \pi |F_e|$$

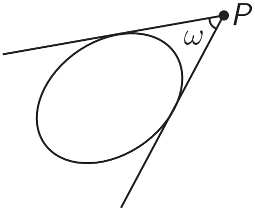
with equality when ∂K is parallel to a Steiner curve (hypocicloid with $k = 3$)



And introducing the visual angle

Theorem

$$\pi|F_e| - \Delta \geq \frac{5}{4}L^2 + 5\int_{P\notin K}\left(\omega - \sin\omega - \frac{2}{3}\sin\omega\right)\, dP \geq 0$$



Equality holds if and only if

1. K is a disc or it is bounded by a curve parallel to an astroid.
2. K is bounded by a curve parallel to a Steiner curve.
3. Minkowski sum of sets of the above type.

Third part: the visual angle

The visual angle was first considered by Crofton

On the theory of local probability 1868

$\omega - \sin\omega$ is the density of the intersection of lines that meet a given area

$$\int_{P \notin K} (\omega - \sin\omega)\, dP = \frac{L^2}{2} - \pi F$$

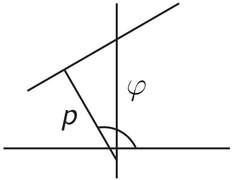
Integral Geometry

The set of lines of the plane is parametrized by p, φ.
The measure (invariant by Euclidean motions) is

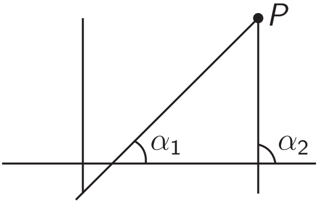
$$dG = dp \wedge d\varphi$$

For instance, for a convex set K with $L =$ length of ∂K we have

$$\int_{G \cap K \neq \emptyset} dG = L$$



Density for pairs of lines



A pair of intersecting lines is determined by the intersection point P and the angles α_1, α_2 with the x axis

$$dG_1 \wedge dG_2 = |\sin(\alpha_2 - \alpha_1)|\, d\alpha_1 \wedge d\alpha_2 \wedge dP$$

Integrating both sides

$$L^2 = 2\pi F + \int_{P \notin K} 2(\omega - \sin\omega)\, dP$$

Crofton formula

Masotti's integral via Integral Geometry

$$\int_{P \notin K} (\omega^2 - \sin^2 \omega)\, dP$$

$$\omega = \text{visual angle from } P \text{ to } K$$

Masotti, in 1954, gives without proof (she says that using Hurwitz approach) the value of this integral in terms of F, L, and the Fourier coefficients of the support function of ∂K.

We will calculate this integral in two different ways: using Integral Geometry and using a general integration formula that we will explain later.

Masotti's integral via Integral Geometry

Multiplying both sides of

$$dG_1 \wedge dG_2 = |\sin(\alpha_2 - \alpha_1)|\, d\alpha_1 \wedge d\alpha_2 \wedge dP$$

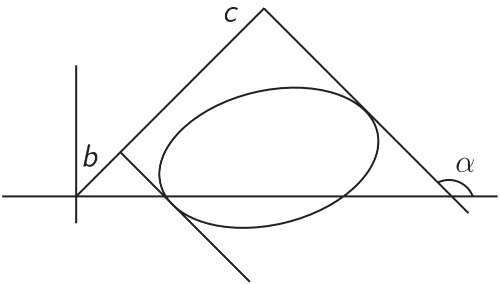
by $|\sin(\alpha_2 - \alpha_1)|$ we have

$$|\sin(\alpha_2 - \alpha_1)|\, dG_1 \wedge dG_2 = \sin^2(\alpha_2 - \alpha_1)\, d\alpha_1\, d\alpha_2\, dP$$

Integrating on the left

$$\int_{G_1 \cap K \neq \emptyset} |\sin(\alpha_2 - \alpha_1)| \, dG_1 = \int_0^{\pi} \int_{b(\alpha_1)}^{c(\alpha_1)} |\sin(\alpha_2 - \alpha_1)| \, d\alpha_1 \, dp_1$$

$$= \int_0^{\pi} a(\alpha_1) |\sin(\alpha_2 - \alpha_1)| \, d\alpha_1$$



$a(\alpha_1) = \text{width}$

Integrating on the left

Using the Fourier series of $a(\alpha)$ and integrating the product of Fourier series we get

$$\int_{G_1}\int_{G_2} |\sin(\alpha_2-\alpha_1)|\, dG_1\, dG_2 = \frac{2L^2}{\pi} - 4\pi \sum_{n\geq 1} \frac{1}{4n^2-1} c_{2n}^2$$

Integrating on the right The right hand side must be integrated over the points $P \in K$ and over the points $P \notin K$.

$$\int_{P\in K} \sin^2(\alpha_2 - \alpha_1)\, d\alpha_1\, d\alpha_2\, dP = \frac{\pi^2 F}{2}$$

$$\int_{P\notin K} \sin^2(\alpha_2 - \alpha_1)\, d\alpha_1\, d\alpha_2\, dP = \frac{1}{2}(\omega^2 - \sin^2\omega)$$

Masotti:

$$\int_{P\notin K} (\omega^2 - \sin^2\omega)\, dP = \frac{4L^2}{\pi} - \pi^2 F - 8\pi \sum_{n\geq 1} \frac{1}{4n^2 - 1} c_{2n}^2$$

Constant width

For constant width, $c_{2n} = 0$, $n \geq 1$,

$$\int_{P \notin K} (\omega^2 - \sin^2 \omega) \, dP \ = \frac{4L^2}{\pi} - \pi^2 F$$

Santaló in “Integral Geometry and Geometric Probability” gives lower and upper bounds for the Masotti integral. We improve the lower bound to

$$\left[\frac{4L^2}{\pi} - \pi^2 F\right] - \frac{4}{3}(A - F) \leq \int_{P \notin K} (\omega^2 - \sin^2\omega)\, dP \leq \left[\frac{4L^2}{\pi} - \pi^2 F\right]$$

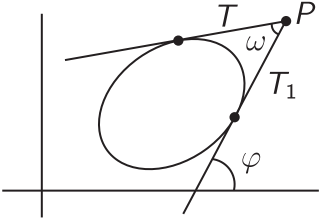
$A =$ area of the pedal curve

A formula for the integral of functions of the visual angle

$$\int_{P \notin K} f(\omega)\, dP$$

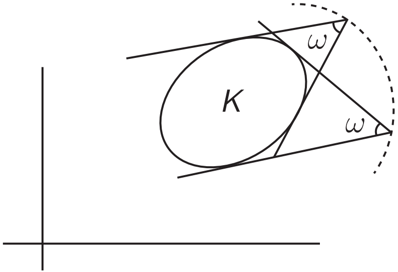
Since de domain of integration is not bounded, $f(\omega)$ must satisfy some conditions when $\omega \to 0$.

We can take (φ, ω) as coordinates outside K



$$dP = \frac{T \cdot T_1}{\sin \omega} \, d\varphi \wedge d\omega$$

We denote by $F(\omega)$ the area of the level set

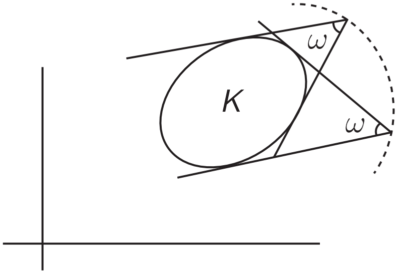


$$\sin^2\omega \cdot F(\omega) = \frac{L^2}{2\pi}(1+\cos\omega) + \pi \sum_{k\geq 2} g_k(\omega) c_k^2$$

$$c_k^2 = a_k^2 + b_k^2$$

$$g_k(\omega) = 1 + \frac{(-1)^k}{2}((k+1)\cos(k-1)\omega - (k-1)\cos(k+1)\omega)$$

We denote by $F(\omega)$ the area of the level set



$$\sin^2\omega \cdot F(\omega) = \frac{L^2}{2\pi}(1+\cos\omega) + \pi\sum_{k\geq 2} g_k(\omega)c_k^2$$

$$c_k^2 = a_k^2 + b_k^2$$

$$g_k(\omega) = 1 + \frac{(-1)^k}{2}((k+1)\cos(k-1)\omega - (k-1)\cos(k+1)\omega)$$

The integral formula

$$
\begin{aligned}
\int_{P \notin K} f(\omega)\, dP = &- [f(\omega)F(\omega)]_0^{\pi} + \frac{L^2}{2\pi} M(f) \\
&+ \pi \sum_{k \geq 2,\ \text{even}} \left(M(f) + 2 \sum_{j=1,\ \text{odd}}^{k-1} \int_0^{\pi} f'(\omega) j \cos(j\omega)\, d\omega \right) c_k^2 \\
&+ \pi \sum_{k \geq 3,\ \text{odd}} \left(-2 \sum_{j=2,\ \text{even}}^{k-1} \int_0^{\pi} f'(\omega) j \cos(j\omega)\, d\omega \right) c_k^2,
\end{aligned}
$$

with $M(f) = \int_0^{\pi} \frac{f'(\omega)}{1 - \cos\omega}\, d\omega$. [Does not depend on the convex set]

Examples

1. Crofton. $M(f) = \pi$. $[f(\omega)F(\omega)]_0^\pi = \pi F$.
2. Masotti. $M(f) = 8$. $[f(\omega)F(\omega)]_0^\pi = \pi^2 F$.

Examples

Let K be a compact convex set with boundary of class C^2 and length L. Write $c_k^2 = a_k^2 + b_k^2$ where a_k, b_k are the Fourier coefficients of the support function of ∂K. Then

$$\begin{aligned}\int_{P\notin K} \sin^m \omega \, dP &= M\frac{L^2}{2\pi} + \\ &+ \frac{m!\pi^2}{2^{m-1}(m-2)} \sum_{k\geq 2,\mathrm{even}} \frac{(-1)^{\frac{k}{2}+1}(k^2-1)}{\Gamma(\frac{m+1+k}{2})\Gamma(\frac{m+1-k}{2})} c_k^2,\end{aligned}$$

where $M = \int_0^\pi \frac{f'(\omega)}{1-\cos\omega} \, d\omega$. In particular for m odd the index k in the sum runs only from 2 to $m-1$.

Examples

$$\int_{P\notin K} \sin^3\omega \, dP = \frac{3L^2}{4} + \frac{9}{4}\pi^2 c_2^2 \qquad \text{Hurwitz}$$

$$\int_{P\notin K} \sin^5\omega \, dP = \frac{5L^2}{16} + \frac{5\pi^2}{4} c_2^2 - \frac{25\pi^2}{16} c_4^2$$

$$\int_{P\notin K} \sin^4\omega \, dP = \frac{4L^2}{3\pi} + \sum_{1}^{\infty} \frac{24\pi}{9-4p^2} c_{2p}^2$$

Examples

Constant width

$$\int_{P\notin K} \sin^3 \omega \, dP = \frac{3L^2}{4}$$

$$\int_{P\notin K} \sin^5 \omega \, dP = \frac{5L^2}{16}$$

$$\int_{P\notin K} \sin^4 \omega \, dP = \frac{4L^2}{3\pi}$$

$$\int_{P\notin K} \sin^m \omega \, dP = \frac{\pi\, m!}{2^{m-1}(m-2)\Gamma(\frac{m+1}{2})^2} \frac{L^2}{2\pi}.$$

Examples

Geometrical interpretation of c_k^2 [**Hurwitz functions**]

$$h_k(\omega) = \frac{(-1)^k}{4}[(k+1)^2\cos(k-2)\omega + (k-1)^2\cos(k+2)\omega - 2(k^2-3)\cos k\omega] + 2\cos\omega$$

$$\int_{P\notin K} h_k(\omega)\,dP = L^2 + (-1)^k\pi^2 c_k^2(k^2-1).$$

c_k^2 are invariants of the curve .

Examples

Geometrical interpretation of c_k^2 [**Hurwitz functions**]

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c_k^2 are invariants of the curve .

Theorem

The integral of the power of the sinus of the visual angle is a linear combination of the integrals of the Hurwitz's functions.

Proof.

$$\int_{P\notin K} \sin^m \omega \, dP = \frac{m!}{2^m(m-2)} \left(\frac{1}{\Gamma(\frac{m+1}{2})^2} - 2\sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{\Gamma(\frac{m+1}{2}+p)\Gamma(\frac{m+1}{2}-p)} \right) L^2$$

$$+ \frac{m!}{2^{m-1}(m-2)} \sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{\Gamma(\frac{m+1}{2}+p)\Gamma(\frac{m+1}{2}-p)} \cdot \int_{P\notin K} f_{2p}(\omega)\, dP.$$

Proof.

The sum in the coefficient of L^2 is (except for a constant factor) the hypergeometric function

$$F(\frac{3-m}{2},1;\frac{3+m}{2};1)$$

and

$$F(a,b;c;1)=\frac{\Gamma(-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)},\quad c-a-b>0.$$

Using this we see

$$\frac{1}{\Gamma(\frac{m+1}{2})^2}=2\sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{\Gamma(\frac{m+1}{2}+p)\Gamma(\frac{m+1}{2}-p)}$$

and the coefficient of L^2 vanishes. □


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