ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

M. LLABRÉS
A. REVENTÓS

Unimodular Lie foliations

Annales de la faculté des sciences de Toulouse 5e série, tome 9, nº 2 (1988), p. 243-255.

<http://www.numdam.org/item?id=AFST_1988_5_9_2_243_0>

© Université Paul Sabatier, 1988, tous droits réservés.

L'accès aux archives de la revue « Annales de la faculté des sciences de Toulouse » (http://picard.ups-tlse.fr/~annales/), implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/legal.php). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme Numérisation de documents anciens mathématiques
http://www.numdam.org/

Annales Faculté des Sciences de Toulouse Vol. IX, n°2, 1988

Unimodular Lie Foliations

M. LLABRÉS[(1)] and A. REVENTÓS[(1)]

RÉSUMÉ. — Soit $\mathcal{F}$ un $\mathcal{G}$-feuilletage de Lie de codimension n sur une variété compacte M.

Nous étudions la relation entre $H(\mathcal{G})$ et l'espace de cohomologie basique $H(M/\mathcal{F})$. Nous démontrons que en général $H^*(\mathcal{G}) \neq H^*(M/\mathcal{F})$; nous donnons des conditions suffisantes pour l'inclusion $H^*(\mathcal{G}) \subseteq H^*(M/\mathcal{F})$ et pour l'égalité $H^n(\mathcal{G}) = H^n(M/\mathcal{F})$.

Finalement, si $\mathcal{F}$ est un flot de Lie avec $H^n(M/\mathcal{F}) \neq 0$ nous caractérisons quand $\mathcal{F}$ est de type homogène en termes de sa classe d'Euler.

ABSTRACT. — Let $\mathcal{F}$ be a codimension n Lie $\mathcal{G}$-foliation on a compact manifold M.

We study the relation between $H(\mathcal{G})$ and the cohomology space $H(M/\mathcal{F})$. We show that in general $H^*(\mathcal{G}) \neq H^*(M/\mathcal{F})$ and we give sufficient conditions for the inclusion $H^*(\mathcal{G}) \subseteq H^*(M/\mathcal{F})$ and for the equality $H^n(\mathcal{G}) = H^n(M/\mathcal{F})$.

Finally, if $\mathcal{F}$ is a Lie flow with $H^n(M/\mathcal{F}) \neq 0$ we characterize when $\mathcal{F}$ is homogeneous in terms ot its Euler class.

1. Introduction

Let $\mathcal{F}$ be a foliation on a manifold M given by an integrable subbundle $L \subset TM$. The complex of basic forms is the subcomplex $\Omega^*(M/\mathcal{F}) \subset \Omega^*(M)$ of the De Rham complex given by the forms α satisfiying $L_X\alpha = 0$ and $i_X\alpha = 0$ for all $X \in \Gamma L$. The cohomology of this complex, $H^*(M/\mathcal{F})$, is called the basic cohomology of the foliated manifold (M/$\mathcal{F}$).

A. El KACIMI and G. HECTRO proved in [3], and independently V. SERGIESCU in [11], that for Riemannian foliations on compact manifolds

(1) Departament de Matemàtiques, Universitat Autònoma de Barcelona, Bellaterra, Barcelona - Spain.

the space of cohomology $H^*(M/\mathcal{F})$ satisfies Poincare duality if and only if $H^n(M/\mathcal{F}) \neq 0$ (where n = codim $\mathcal{F}$). In this case $\mathcal{F}$ will be called *unimodular.*

The Carriere's counterexample to Poincare duality (a linear flow on the hyperbolic torus T_A^3 (cf. [1])) is not only a Riemannian foliation but a Lie foliation. In fact, it is modeled on the affine Lie group which is not unimodular.

It seems therefore interesting to study the relation between the basic cohomology $H^*(M/\mathcal{F})$ of a Lie $\mathcal{G}$-foliation and the cohomology $H^*(\mathcal{G})$ of the Lie algebra $\mathcal{G}$.

For instance, if $\mathcal{F}$ is a dense Lie $\mathcal{G}$-foliation then $H^*(M/\mathcal{F}) = H^*(\mathcal{G})$ (cf. Remark 2.3). Thus, in this case, $\mathcal{F}$ is unimodular if and only if $\mathcal{G}$ is unimodular.

In this paper we obtain the following results:

THEOREM 3.1.— *Let $\mathcal{F}$ be an unimodular Lie $\mathcal{G}$-foliation on a compact manifold M. Then the Lie algebra $\mathcal{G}$ is unimodular.*

We don't know if the converse is true; but we have

THEOREM 3.2.— *Let $\mathcal{F}$ be a Lie $\mathcal{G}$-foliation on a compact manifold M with $\mathcal{G}$ unimodular. If the structural Lie algebra $\mathcal{H}$ of (M,$\mathcal{F}$) is an ideal of $\mathcal{G}$ then $\mathcal{F}$ is unimodular.*

The most important consequences of this are:

COROLLARY 3.3.— *Let $\mathcal{F}$ be a Lie $\mathcal{G}$-foliation with $\mathcal{G}$ a nilpotent Lie algebra. Then $\mathcal{F}$ is unimodular.*

COROLLARY 3.4.— *Let $\mathcal{F}$ be a Lie $\mathcal{G}$-foliation with codim $\overline{\mathcal{F}} = 1$. Then $\mathcal{F}$ is unimodular if and only if $\mathcal{G}$ is unimodular and the structural Lie algebra $\mathcal{H}$ is also unimodular.*

All this results are obtained by studying the cohomology spaces of maximum degree.

Paragraph 4 is dedicated to the study of the relation between the cohomology spaces of degree $r \leq n$.

Our first result is in the line of [5]:

PROPOSITION 4.1.— *Let $\mathcal{F}$ be an unimodular Lie $\mathcal{G}$-foliation. Then for* *all $r \leq n$ the map $i_r^* : H^r(\mathcal{G}) \longrightarrow H^r(M/\mathcal{F})$ induced by the canonical inclusion $i_r : \Omega_G^r(G) \longrightarrow \Omega_\Gamma^r(G)$ is injective.*

We give an exemple to show that i_r^* is not always exhaustive, i.e. $H^*(\mathcal{G}) \neq H^*(M/\mathcal{F})$, and we end this paragraph with

PROPOSITION 4.3.— *Let $\mathcal{F}$ be a $\mathcal{G}$-foliation with $\overline{\Gamma}$ a normal subgroup of G. Then the map i^* is an isomorphism for all $r \leq n$.*

Finally, in paragraph 5, we prove for one dimensional Lie foliations:

THEOREM 5.1.— *Let $\mathcal{F}$ be an unimodular Lie flow with Lie algebra $\mathcal{G}$. Then $\mathcal{F}$ is homogeneous if and only if the Euler class $e(\mathcal{F}) \in H^2(M/\mathcal{F})$ belongs to $H^2(\mathcal{G})$ ($H^2(\mathcal{G}) \subset H^2(M/\mathcal{F})$ by Proposition 4.1).*

We end this paragraph with an exemple of an unimodular Lie flow which is not homogeneous.

We are indebted with G. HECTOR, who suggested this problems and helped us in the proofs of theorems.

We also thank A. El KACIMI-ALAOUI for his help and suggestions.

2. Preliminaries

Let $\mathcal{F}$ be a smooth foliation of codimension n on a smooth manifold M given by an integrable subbundle $L \subset TM$. We denote by $\mathcal{L}(M,\mathcal{F})$ the Lie algebra of foliated vector fields, i.e. $X \in \mathcal{L}(M,\mathcal{F})$ if and only if $[X,Y] \in \Gamma L$ for all $Y \in \Gamma L$. ΓL is an ideal of $\mathcal{L}(\mathcal{M},\mathcal{F})$ and the elements of $\mathcal{X}(M,\mathcal{F}) = \mathcal{L}(M,\mathcal{F})/\Gamma L$ are called basic vector fields.

If there is a family $\{X_1,\dots,X_n\}$ of foliated vector fields on M such that the corresponding family $\{\overline{X}_1,\dots,\overline{X}_n\}$ of basic vector fields has rank n everywhere the foliation is called transversally parallelizable and $\{\overline{X}_1,\dots,\overline{X}_n\}$ a transvers parallelism. If the vector subspace $\mathcal{G}$ of $\mathcal{X}(M/\mathcal{F})$ generated by $\{\overline{X}_1,\dots,\overline{X}_n\}$ is a Lie subalgebra, the foliation is called a Lie foliation.

We shall use the following structure theorems (cf. [6] and [4]).

THEOREM 2.1.— *Let $\mathcal{F}$ be a transversally parallelizable foliation on a compact manifold M, of codimension n. Then*

a) There is a Lie algebra $\mathcal{H}$ of dimension $g \leq n$.

b) There is a locally trivial fibration $\pi : M \rightarrow W$ with compact fibre F and

$$dim\ W = n - g = m.$$

c) There is a dense Lie $\mathcal{H}$-foliation on F such that :

i) The fibres of π are the adherences of the leaves of $\mathcal{F}$.

ii) The foliation induced by $\mathcal{F}$ on a fibre F of π is isomorphic to the $\mathcal{H}$-foliation on F.

$\mathcal{H}$ is called the structural Lie algebra of $(M, \mathcal{F})$, π the basic fibration and W the basic manifold.

Let $T(\overline{\mathcal{F}})$ be the subbundle of $T(M)$ tangent to the fibres of π. A transvers parallelism on M determines a subbundle $N(\mathcal{F})$ of $T(M)$ satisfiying $T(M) = T(\mathcal{F}) \oplus N(\mathcal{F})$. A foliated vector field X is called pure horizontal (respectively, pure vertical) if $X \in \Gamma N(\overline{\mathcal{F}})$ (resp. $X \in \Gamma T(\overline{\mathcal{F}})$).

A basic form $\alpha \in \Omega^r(M/\mathcal{F})$ is called pure of (p,q)-type $(p+q=r)$ if $\alpha(Y_1, \ldots, Y_n) = 0$ for all family of r pure vector fields except that p of them are pure horizontal and q are pure vertical.

If we denote by $\Omega^{p,q}(M/\mathcal{F})$ the A(W)-module of pure forms of (p,q)-type we have the decomposition

$$\Omega^r(M/\mathcal{F}) = \bigoplus_{p+q=r} \Omega^{p,q}(M/\mathcal{F})$$

and the operator d of exterior derivative is decomposed as

$$d = d_{1,0} + d_{0,1} + d_{2,-1} \text{ (cf. [3])}$$

Let $\theta^1, \ldots, \theta^n$ denote the dual basis of a given transvers parallelism $X_1, \ldots, X_n$. That is $\theta^1, \ldots, \theta^n$ are basic 1-forms with $\theta^i(X_j) = \delta_{i,j}$. The generator $v = \theta^1 \wedge \ldots \wedge \theta^n \in \Omega^n(M/\mathcal{F})$ is called the basic volume form.

THEOREM 2.2.— *Let $\mathcal{F}$ be a Lie $\mathcal{G}$-foliation on a compact manifold M and let G be the connected simply connected Lie group with Lie algebra $\mathcal{G}$. Let $p : \widetilde{M} \longrightarrow M$ be the universal covering of M. Then there is a locally trivial fibration $D : \widetilde{M} \longrightarrow G$ equivariant by $Aut(p)$ (i.e. if $Dx = Dy$ then $Dgx = Dgy$ for all $x, y \in \widetilde{M}$ and $g \in Aut(p)$) such that the foliation $\widetilde{\mathcal{F}} = p^*\mathcal{F}$ is given by the fibres of D.*

The natural morphism $h : \pi_1 M \longrightarrow Diff(G)$ is such that $\Gamma = imh \subset G$, where the inclusion $G \subset Diff(G)$ is by right translations.

The space of differential forms on G, invariant under the right action of Γ is denoted by $\Omega^*_\Gamma(G)$ and the subspace of $\Omega^*(\widetilde{M}/\widetilde{\mathcal{F}})$ given by the forms invariant under the action of $\mathrm{Aut}(p)$ is denoted by $\Omega^*_I(\widetilde{M}/\widetilde{\mathcal{F}})$.

The map p^* gives an isomorphism between $\Omega^*(M/\mathcal{F})$ and $\Omega^*_I(\widetilde{M}/\widetilde{\mathcal{F}})$. Also D^* gives an isomorphism between $\Omega^*_\Gamma(G)$ and $\Omega^*_I(\widetilde{M}/\widetilde{\mathcal{F}})$.

So we have $H^*(M/\mathcal{F}) = H^*_\Gamma(G)$.

Remark 2.3.— In particular, since $\mathcal{F}$ is dense in M if and only if Γ is dense in G, the above equality shows that for dense Lie $\mathcal{G}$-foliations $H^*(\mathcal{G}) = H^*(M/\mathcal{F})$.

Finally, we say that a 1-dimensional Lie $\mathcal{G}$-foliation $\mathcal{F}$ (or a Lie flow) on a compact manifold M is homogeneous if and only if:

i) There is a Lie group H and a discrete Lie subgroup H_o of H such that $M = H/H_o$.

ii) There is a 1-dimensional subgroup K of H such that the leaves of $\mathcal{F}$ are the orbits of the left action of K on H.

Throughout this paper we also assume that K is normal in H.

3. The basic cohomology of maximum degree.

THEOREM 3.1.— *Let $\mathcal{F}$ be an unimodular Lie $\mathcal{G}$-foliation on a compact manifold M. Then the Lie algebra $\mathcal{G}$ is unimodular.*

Proof.— Let $\mathcal{F}$ be an unimodular Lie $\mathcal{G}$-foliation on a compact manifold M of codimension n. Let W be the basic manifold of $(M, \mathcal{F})$, of dimension m.

As $H^n(M/\mathcal{F}) \neq 0$, we have an isomorphism:

$$I : H^n(M/\mathcal{F}) \longrightarrow H^m(W)$$

given by $I([v]) = I([\omega])$, where ω is the volume element of W and v is the basic volume form.

Since $\bigwedge^n \mathcal{G}^* = \Omega^*_G G \subset \Omega^*_\Gamma G = \Omega^n(M/\mathcal{F})$, every $0 \neq \alpha \in \bigwedge^n \mathcal{G}^*$ can be considered as a nowhere zero basic n-form on M. Hence, $\alpha = fv$ for a nowhere zero basic function f.

Then $I([\alpha]) = I([fv]) = hI([v]) = h([\omega])$, where h is a function on W such that $f = h \circ \pi$; thus h is nowhere zero on W.

As W is compact, ω is the volume element of W and h is not zero everywhere, then $h\omega$ is not exact and $I[\alpha] \neq 0$. Therefore α is not the differential of a basic $(n-1)$-form on M. In particular $\alpha \neq d\beta$ for all $\beta \in \bigwedge^{n-1} \mathcal{G}^*$ and $\mathcal{G}$ is unimodular.

This proves Theorem 3.1.•

THEOREM 3.2.— *Let $\mathcal{F}$ be a Lie $\mathcal{G}$-foliation on a compact manifold M with $\mathcal{G}$ unimodular. If the structural Lie algebra $\mathcal{H}$ of $(M,\mathcal{F})$ is an ideal of $\mathcal{G}$ then $\mathcal{F}$ is unimodular.*

We begin with two lemmas:

LEMMA 3.2.1.— *In the hypothesis of Theorem 3.2, $\mathcal{G}/\mathcal{H}$ is an unimodular Lie algebra.*

Proof.— First, we verify that every Lie group G which admits a uniform discrete subgroup H is unimodular.

Let σ be a right-invariant n-form on G (n =dim G). σ is projectable, i.e. $\sigma = p^*(\widetilde{\sigma})$ where $p : G \longrightarrow G/H$.

Clearly, $L_g^*\sigma = f\sigma$ for a fixed $g \in G$; and $L_g^*\sigma$ is right-invariant.

Since $L_g^*\sigma = f\sigma$ and σ are right-invariant, then the function f is a constant k.

There is a natural left-action of G over the homogeneous space G/H. As G/H is compact, we can consider

$$\int_{G/H} \widetilde{\sigma} = \int_{G/H} \bar{L}_g^*\widetilde{\sigma} = \int_{G/H} k\widetilde{\sigma} = k \int_{G/H} \widetilde{\sigma}$$

then $k = 1$ and $\sigma = L_g^*\sigma$.

Thus σ is a bi-invariant n-form on G, and this is equivalent to that G is unimodular.

In the hypothesis of Theorem 3.2, the quotient $G/\overline{\Gamma}_e$ (where $\overline{\Gamma}_e$ is the connected component of $\overline{\Gamma}$ at the identity) is a Lie group and $\overline{\Gamma}/\overline{\Gamma}_e$ is a uniform discrete subgroup. Then $G/\overline{\Gamma}_e$ is unimodular and its associate Lie algebra, $\mathcal{G}/\mathcal{H}$, is also unimodular. •

LEMMA 3.2.2.— *In the hypothesis of Theorem 3.2, $d\beta = 0$ for all $(n-1)$-basic form β of $(m, g-1)$-type.*

Proof.— Since the estructural Lie algebra $\mathcal{H}$ is an ideal of $\mathcal{G}$, one can choose a transvers parallelism $\{\overline{Y}_1, \dots, \overline{Y}_n\}$ such that g of the foliated vector fields $\{Y_1, \dots, Y_n\}$ are tangent to $\overline{\mathcal{F}}$ (where $g = n-m = codim\mathcal{F} - codim\overline{\mathcal{F}}$).

Given this parallelism we assume that $Y_1, \dots, Y_g$ are tangent to $\overline{\mathcal{F}}$. Let v be the corresponding basic volume form.

Since $\mathcal{G}$ is unimodular

$$d\beta(Y_1, \dots, Y_n) = \sum_{i=1}^{n} (-1)^{i+1} Y_i \beta(Y_1, \dots, \widehat{Y_i}, \dots, Y_n)$$

but $\beta(Y_1, \dots, \widehat{Y_j}, \dots, Y_n) = 0 \ \forall j > g$, because β is of $(m, g-1)$-type. Thus

$$d\beta(Y_1, \dots, Y_n) = \sum_{i=1}^{g} (-1)^{i+1} Y_i \beta(Y_1, \dots, \widehat{Y_i}, \dots, Y_n) = 0$$

because f is a basic function and $Y_j(f) = 0$ for all $j \leq g$. •

Proof of Theorem 3.2 .— Let $\{Y_1, \dots, Y_n\}$ and v be as in the proof of Lemma 3.2.

Assume that $\mathcal{F}$ is not unimodular; this means that there exists a basic (n-1)-form α such that $v = d\alpha$.

Moreover we can consider that α is of $(m-1, g)$-type because we have the decomposition:

$$\Omega^{n-1}(M/\mathcal{F}) = \Omega^{m,g-1}(M/\mathcal{F}) \oplus \Omega^{m-1,g}(M/\mathcal{F}),$$

then $\alpha = \alpha^{(m,g-1)} + \alpha^{(m-1,g)}$ and, by Lemma 3.2.2, $d\alpha^{(m,g-1)} = 0$.

Let β be the contraction $i_{Y_1} \dots i_{Y_g} \alpha$, then β is a $(m-1)$ basic form of $(m-1, 0)$-type. Since $L_Y \beta = 0$ for all vector field Y tangent to $\overline{\mathcal{F}}$, β is projectable, i.e. there exists a $(m-1)$-form u on W such that $\beta = \pi^* u$.

Observe that:

$$du(X_1, \dots, X_m) = \sum_{i=1}^{m} (-1)^{i+1} X_i u(X_1, \dots, \widehat{X}_i, \dots, X_m)$$

for all $X_1, \dots, X_m$ corresponding to vectors of $\mathcal{G}/\mathcal{H}$, because $\mathcal{G}/\mathcal{H}$ is unimodular by Lemma 3.2.1.

Then we have:

$$1 = v(Y_1, \dots, Y_n) = d\alpha(Y_1, \dots, Y_n) = \sum_{i=g+1}^{n} (-1)^{i+1} Y_i \beta(Y_{g+1}, \dots, Y_n)$$

because for $i = 1, \ldots, g$ the Y_i are tangent to $\overline{\mathcal{F}}$.

Since $\beta(Y_{g+1}, \ldots, \widehat{Y}_i, \ldots, Y_n)$ are basic functions then

$$Y_i\beta(Y_{g+1}, \ldots, \widehat{Y}_i, \ldots, Y_n) = \pi_* Y_i \beta(Y_{g+1}, \ldots, \widehat{Y}_i, \ldots, Y_n).$$

So,

$$1 = \sum_{i=g+1}^{n} (-1)^{i+1} \pi_* Y_i \pi^* u(Y_{g+1}, \ldots, \widehat{Y}_i, \ldots, Y_n) \; ; \qquad \text{i.e.}$$
$$1 = du(\pi_* Y_{g+1}, \ldots, \pi_* Y_n).$$

Then we have a volume form on the compact manifold W which is exact. This is a contradiction and Theorem 3.2 is proved. •

COROLLARY 3.3.— *Let $\mathcal{F}$ be a Lie $\mathcal{G}$-foliation with $\mathcal{G}$ a nilpotent Lie algebra. Then $\mathcal{F}$ is unimodular.*

Proof.— Since $\overline{\Gamma}$ is a closed uniform subgroup of a nilpotent Lie group then, following [8], the connected component at the identity, $\overline{\Gamma}_e$, is a normal subgroup of G, this implies that the subalgebra $\mathcal{H}$ of $\mathcal{G}$ is an ideal. Then we are in the hypothesis of Theorem 3.2 and corollary follows.•

COROLLARY 3.4.— *Let $\mathcal{F}$ be a Lie $\mathcal{G}$-foliation with $codim\overline{\mathcal{F}} = 1$. Then $\mathcal{F}$ is unimodular if and only if $\mathcal{G}$ is unimodular and the structural Lie algebra $\mathcal{H}$ is also unimodular.*

Proof.— We have only to prove that $\mathcal{F}$ is unimodular if $\mathcal{G}$ is unimodular.

By Theorem 3.2, it suffices to prove that $\mathcal{H}$ is an ideal of $\mathcal{G}$.

Let $e_1, \ldots, e_n$ be a basis of $\mathcal{G}$ such that $e_1, \ldots, e_{n-1}$ is a basis of $\mathcal{H}$.

If we put $[e_i, e_j] = \sum_{k=1}^{n} c_{ij}^k e_k$ then $\mathcal{H}$ will be an ideal of $\mathcal{G}$ if and only if $c_{in}^n = 0$ for all $i < n$.

The assumption that $\mathcal{G}$ is unimodular implies that $\sum_{j=1}^{n} c_{ij}^j = 0$ for all i. But since $\mathcal{H}$ is unimodular , we have $\sum_{j=1}^{n-1} c_{ij}^j = 0$ for all $i < n$. This implies that $c_{in}^n = 0$ for all $i < n$, thus $\mathcal{H}$ is an ideal. •

COROLLARY 3.4.— *Let $\mathcal{F}$ be a $\mathcal{G}$-foliation with $\overline{\Gamma}$ a normal subgroup of G. Then $\mathcal{F}$ is unimodular if and only if $\mathcal{G}$ is unimodular.*

Proof.— We have only to prove that $\mathcal{F}$ is unimodular if $\mathcal{G}$ is unimodular.

Since $\overline{\Gamma}$ is a normal subgroup of G, the connected component at the identity, $\overline{\Gamma}_e$, is normal too. Then its associate Lie algebra is an ideal of $\mathcal{G}$. This proves that $\mathcal{H}$ is an ideal.

So we are in the hypothesis of Theorem 3.2 and Corollary 3.4 follows. •

4. The basic cohomology of arbitrary degree.

PROPOSITION 4.1.— *Let $\mathcal{F}$ be an unimodular Lie $\mathcal{G}$-foliation. Then, for all $r \leq n$, the map $i_r^* : H^r(\mathcal{G}) \longrightarrow H^r(M/\mathcal{F})$ induced by the canonical inclusion $i_r : \Omega_G^r(G) \longrightarrow \Omega_\Gamma^r(G)$ is injective.*

Proof.— Let $0 \neq [\alpha] \in H^r(\mathcal{G})$. Since $\mathcal{G}$ is unimodular there exists $0 \neq [\beta] \in H^{n-r}(\mathcal{G})$ such that $[\alpha \wedge \beta] \neq 0$. Suppose $i^*([\alpha]) = 0$. Then there is a $(r-1)$-basic form γ on M such that $\alpha = d\gamma$.

Since $d\beta = 0$, we have:

$$\alpha \wedge \beta = d\gamma \wedge \beta = d(\gamma \wedge \beta) + (-1)^r \gamma \wedge d\beta = d(\gamma \wedge \beta)$$

Then the n-form $\alpha \wedge \beta$ is exact as basic form, i.e. $[\alpha \wedge \beta] = 0$ in $H^n(M/\mathcal{F})$; but in the proof of Theorem 3.1 we have proved that if a n form is not exact in $\mathcal{G}$ then it is not exact as basic form. •

There exist unimodular Lie foliations for which i_r^* is not isomorphism:

Example 4.2 .— Let Σ be the double torus, and let $W = T_1\Sigma \rightarrow \Sigma$ be the unit tangent bundle over Σ. Let $\mathbf{H}$ denote the universal covering of Σ then $T_1\mathbf{H}$ is a covering of W and $T_1\mathbf{H} = PSL(2,\mathbf{R})$.

Let $\mathcal{F}$ be the foliation on W given by points, then $\mathcal{F}$ is a transvesally Lie foliation with Lie algebra $\mathcal{S}l(2,\mathbf{R})$.

In this case we have $H^r(W/\mathcal{F}) = H^r(W)$ and, in particular,

$$H^1(W/\mathcal{F}) = H^2(W/\mathcal{F}) = \mathbf{R}^4;$$

but $H^1(\mathcal{S}l(2,\mathbf{R})) = H^2(\mathcal{S}l(2,\mathbf{R})) = 0$. This means that in this case i_1^* and i_2^* are not exhaustives.

Now, we are interested to know when the map i_r^* is exhaustive for all $r \leq n$.

In that sense, we have only the following:

PROPOSITION 4.3.— *Let $\mathcal{F}$ be a $\mathcal{G}$-foliation with $\overline{\Gamma}$ a normal subgroup of G.*

Then the map i_r^* is an isomorphism for all $r \leq n$.

Proof.— Let Z_K denote the space of K invariant closed forms on G, where K is a subgroup of G.

As $Z_{\bar{\Gamma}}$ is a Frechet space, we can adapt the standart construction of the Haar measure on compact Lie groups (cf. for instance [9]) replacing the space $\mathcal{C}(W)$ of continuous functions on W by the space $\mathcal{C}(W, Z_{\bar{\Gamma}})$ of continuous functions on W in $Z_{\bar{\Gamma}}$ to obtain a $Z_{\bar{\Gamma}}$ valued Haar measure; that is, a G-invariant linear map:

$$\begin{aligned} \mathcal{C}(W, Z_{\bar{\Gamma}}) &\longrightarrow Z_{\bar{\Gamma}} \\ f &\longrightarrow \int_W f \end{aligned}$$

This measure induces a linear map:

$$\phi : Z_{\bar{\Gamma}} \longrightarrow Z_G$$

given by $\phi(\alpha) = \int_W \phi_\alpha$, where $\phi_\alpha : W \longrightarrow Z_{\bar{\Gamma}}$ denotes the map $\phi_\alpha(g) = l_g^* \alpha$.

As $\int_W \phi_\alpha$ belongs to the closure of the convex hull of the set of all left translates of ϕ_α (which belongs to $\mathcal{C}(W, Z_{\bar{\Gamma}})$ because $\bar{\Gamma}$ is normal) every traslate of ϕ_α is homotopic to α, we obtain for each $[\alpha] \in H_{\bar{\Gamma}}(G)$ an element $\int_W \phi_\alpha \in Z_G$ such that $[\alpha] = [\int_W \phi_\alpha] \in H_{\bar{\Gamma}}(G)$.

So the exact sequence:

$$0 \longrightarrow B_G \longrightarrow Z_G \longrightarrow H_{\bar{\Gamma}}(g)$$

admits a section and

$$H^*(M/\mathcal{F}) = H^*_{\bar{\Gamma}}(G) = H^*_G(G) = H^*(\mathcal{G}).\bullet$$

5. Unimodular Lie flows.

THEOREM 5.1.— *Let $\mathcal{F}$ be an unimodular Lie flow with Lie algebra $\mathcal{G}$. Then $\mathcal{F}$ is homogeneous if and only if the Euler class $e(\mathcal{F}) \in H^2(M/\mathcal{F})$ belongs to $H^2(\mathcal{G})$ ($H^2(\mathcal{G}) \subset H^2(M/\mathcal{F})$ by Proposition 4.1).*

Proof. — The assumption that $\mathcal{F}$ is unimodular is equivalent to that $\mathcal{F}$ is a flow of isometries (cf. [7]). This means that there exists a Riemannian metric g on M and a vector field Z tangent to $\mathcal{F}$ which generates a group of isometries (φ_t). We can assume that Z is a unit vector field.

In this situation, the characteristic 1-form of $\mathcal{F}$ with respect to (g, Z) is defined by:

$$\chi = i_Z g$$

and it satisfies the equations:

$$\chi(Z) = 1 \text{ and } i_Z d\chi = 0.$$

In particular, the 2-form $d\chi$ is basic for $\mathcal{F}$.

Following [10] one can define the Euler class of $\mathcal{F}$ with respectedto g by

$$e(\mathcal{F}) = [d\chi] \in H^2(M/\mathcal{F});$$

up to a non zero factor this class does not depend on the metric g.

First, assume that $e(\mathcal{F}) \in H^2(\mathcal{G}) \subset H^2(M/\mathcal{F})$ (see Proposition 4.1).

LEMMA 5.2. — *In this case, we can choose foliated vector fields $\widetilde{Y}_1, \ldots, \widetilde{Y}_n$, corresponding to a transvers parallelism, such that $Z, \widetilde{Y}_1, \ldots, \widetilde{Y}_n$ generates a Lie algebra $\mathcal{H}$.*

Proof. — Given a transvers parallelism $\overline{Y}_1, \ldots, \overline{Y}_n$ we can always consider that the foliated vector fields Y_i are such that $g(Z, Y_i) = 0$ because $Y_i = Y_i' - g(Y_i', Z)Z$ represents the same class in $\mathcal{X}(M/\mathcal{F})$.

Since Z generates a flow of isometries we have $[Z, Y_i] = 0$, because Y_i is foliated and $[Z, Y_i]$ must be orthogonal to Z.

The condition $e(\mathcal{F}) \in H^2(\mathcal{G})$, means that $d\chi = \alpha + d\beta$, where β is a basic 1-form and α is a basic 2-form which can be interpreted as a form on $\mathcal{G}$ by the inclusion

$$i : \Omega^*_G(G) \longrightarrow \Omega^*(M/\mathcal{F}).$$

We can modify the metric g to obtain another metric $\widetilde{g}$ such that Z is still Killing and unitary with respect to $\widetilde{g}$ and the corresponding characteristic 1-form $\chi_{\widetilde{g}}$ is such that $d\chi_{\widetilde{g}} = \alpha$.

Concretely the new metric is $\widetilde{g} = g - (\chi \otimes \beta + \beta \otimes \chi)$, and the foliated vector fields $\widetilde{Y}_i = Y_i + \beta(Y_i)Z$ are $\widetilde{g}$-orthogonal to Z. (Remark: using $Z, \widetilde{Y}_1, \ldots, \widetilde{Y}_n$ as a basis, one can easily verify that $\widetilde{g}$ is effectively a new metric.)

Then $[Z, \widetilde{Y}_i] = 0$ and

$$[\widetilde{Y}_i, \widetilde{Y}_j] = \sum_{k=1}^{n} c_{ij}^k \widetilde{Y}_k + b_{ij} Z,$$

thus we have only to prove that b_{ij} are constant.

But $\alpha(\widetilde{Y}_i, \widetilde{Y}_j) = constant$ and, on the other hand,

$$\alpha(\widetilde{Y}_i, \widetilde{Y}_j) = d\chi_{\widetilde{g}}(\widetilde{Y}_i, \widetilde{Y}_j) = -\chi_{\widetilde{g}}([\widetilde{Y}_i, \widetilde{Y}_j]) = -b_{i,j}\chi_{\widetilde{g}}(Z) = -b_{ij}.$$

This proves the lemma. •

So we have a Lie subalgebra $\mathcal{H} \subset \mathcal{X}(M)$ and therefore a Lie group H, with Lie algebra $\mathcal{H}$, acting on M in such a way that the Lie algebra of fundamental vector fields is $\mathcal{H}$.

Since $\mathcal{F}$ is transversally parallelizable and the Lie algebra of H is generated by a transvers parallelism of $\mathcal{F}$ and the vector field Z (tangent to $\mathcal{F}$), we can assume that the action of H on M is transitive.

In this case there is a diffeomorphism between M and H/H_o where H_o is the isotropy group of a point $m_o \in M$.

Then M is a homogeneous manifold and the leaves of $\mathcal{F}$ are the orbits of the action of the subgroup K of H on M, where K is the connected subgroup of H whose associated Lie algebra is the ideal generated by Z.

Thus $\mathcal{F}$ is a homogeneous flow.

Reciprocally:

Let $\mathcal{F}$ be an homogeneous flow on M (then $M = H/H_o$) and let $\mathcal{H}$ (resp. $\mathcal{G}$) be the Lie algebra of H (resp. of $G = H/H_o$); see §2 for notation.

The assumption that $\mathcal{F}$ is an unimodular Lie flow implies that $\mathcal{G}$ is an unimodular Lie algebra and $\mathcal{F}$ is an isometric flow.

We have to prove that $d\chi(Y_i, Y_j)$ is constant, but the Lie algebra $\mathcal{H}$ is generated by $Z, Y_1, \ldots, Y_n$. Thus $[Y_i, Y_j] = \sum_{k=1}^{n} c_{ij}^k Y_k + b_{ij} Z$, where c_{ij}, b_{ij} are constants.

Then $d\chi(Y_i, Y_j) = -b_{ij}$ is constant. Hence $d\chi$ can be considered as a 2-form on $\mathcal{G}$ and the Euler class of $\mathcal{F}$ is an element of $H^2(\mathcal{G})$.

Theorem 5.1 is proved. •

The following example proves that there exist unimodular Lie flows that are not homogeneous.

Example 5.3 **(G. Hector)** . — Let Σ, $\mathbf{H}$, W and $T_1\mathbf{H}$ be as in Example 4.2. Since $H^2(W, \mathbf{Z}) \neq 0$, there exists a non trivial fibre bundle over W with fibre $\mathbf{S}^1$:

$$\mathbf{S}^1 \longrightarrow M \longrightarrow W$$

Then we have a flow on M which is transversally a Lie flow with algebra $\mathcal{S}l(2,\ R)$.

By construction the Euler class is not zero:

$$0 \neq e(\mathcal{F}) \in H^2(M/\mathcal{F})$$

and since $H^2(\mathcal{S}l(2,\ R)) = 0$ this flow is not homogeneous (cf. Theorem 5.1).

References

[1] Carriere (Y.). — Flots riemanniens, *Journées S.M.F. sur les structures transverses, Astérisque*, t. **116**, 1984.

[2] El Kacimi -Alaoui (A.), Sergiescu (V.) and G.Hector (G.). — La cohomologie basique d'un feuilletage est de dimension finie, *Math. Z.*, t. **188**, 1985, p. 593-599.

[3] El Kacimi -Alaoui (A.) and G.Hector (G.). — Décomposition de Hodge basique pour un feuilletage riemannien, *Ann. Inst. Fourier*, t. **36-2**, 1986, p. 207-227.

[4] Fedida (E.). — Sur les feuilletages de Lie, *C.R. Acad. Sci. Paris*, t. **272**, 1971, p. 999-1002.

[5] Lehmann (D.). — Modèle minimal relatif des feuilletages, *In Lecture Notes in Mathematics*, t. **1183**, , p. 250-258.

[6] Molino (P.). — Géométrie globale des feuilletages riemanniens, *Proc. Kon. Nederl. Akad.*, t. **85 ser. A.1**, 1982, p. 45-76.

[7] Molino (P.) and Sergiescu (V.). — Deux remarques sur les flots riemanniens, *Manuscripta Math.*, t. **51 n°1-3**, 1985, p. 145-161.

[8] Raghunathan (M.S.). — Discrete subgroups of Lie Groups, *Springer-Verlag*, t. **68**, 1972.

[9] Rudin (W.). — *Functional Analysis.* — McGraw-Hill, 1973.

[10] Saralegui (M.). — *The Euler class for flows of isometries.* — In Research Notes in Mathematics, t. **131**.

[11] Sergiescu (V.). — Cohomologie basique et dualité des feuilletages riemanniens, *Ann. Inst. Fourier*, t. **35,3**, 1985, p. 137-158.

(Manuscrit reçu le 12 mai 1987)