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What did Gauss read in the *Appendix*?

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Abstract

In a clear analogy with spherical geometry, Lambert states that in an "imaginary sphere" the sum of the angles of a triangle would be less than π. In this paper we analyze the role played by this imaginary sphere in the development of non-Euclidean geometry, and how it served Gauss as a guide. More precisely, we analyze Gauss's reading of Bolyai's *Appendix* in 1832, five years after the publication of *Disquisitiones generales circa superficies curvas*, on the assumption that his investigations into the foundations of geometry were aimed at finding, among the surfaces in space, Lambert's hypothetical imaginary sphere. We also wish to show that the close relation between differential geometry and non-Euclidean geometry is already present in János Bolyai's *Appendix*, that is, well before its appearance in Beltrami's *Saggio*. From this point of view, one is able to answer certain natural questions about the history of non-Euclidean geometry; for instance, why Gauss decided not to write further on the subject after reading the *Appendix*.

Zusammenfassung

In einer deutlichen Analogie mit der Kugelgeometrie behauptet Lambert, dass in einer "imaginären Kugelfläche" die Summe der Winkel eines Dreiecks kleiner als π sein würde. In diesem Artikel analysieren wir die Rolle, die diese imaginäre Kugelfläche in der Entwicklung der nichteuklidischen Geometrie gespielt hat, und wie Gauss sie als Orientierungshilfe benutzte. Insbesondere analysieren wir die Lektüre, die Gauss 1832—fünf Jahren nach der Veröffentlichung der *Disquisitiones generales circa superficies curvas*—zu Bolyais *Appendix* gemacht hat, unter der Annahme, dass seine Forschung über die Grundlagen der Geometrie darauf gezielt hat, Lamberts hypothetische imaginäre Kugelfläche unter den Flächen im Raum zu finden. Wir möchten hiermit auch zeigen, dass der enge Zusammenhang zwischen der Differentialgeometrie und der nichteuklidischen Geometrie schon mit János Bolyais *Appendix* vorkommt, deutlich früher als Beltramis *Saggio*. In dieser Hinsicht können gewisse natürlich Fragen der Geschichte der nichteuklidischen Geometrie beantwortet werden; zum Beispiel: Warum entschloss sich Gauss nach der Lesung des *Appendix* dazu, nicht mehr über die nichteuklidische Geometrie zu schreiben?

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1. The classical problem

The Greek geometer Euclid began his *Elements*[1] with a list of 23 definitions, 5 logical rules, and 5 postulates. The fifth postulate refers to parallel lines, which are defined as those "straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction."[2] The fifth postulate states that: "If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles."

The 'problem of the fifth postulate' consists of demonstrating that this postulate is a consequence of the other four postulates of the *Elements*. Since this postulate is equivalent to the existence and uniqueness of a straight line parallel to a given straight line through a given point, research in this direction is called 'theory of parallels.'

Since Euclid, many mathematicians have tried to prove the fifth postulate. Posidonius attempted to solve the problem in the first century B.C., when he confused parallel straight lines with equidistant straight lines [Bonola, 1955, 2]. We can also mention Ptolemy (second century A.D.), Proclus (410–485), Nasîr-Eddîn (1201–1274), Giordano Vitali (1633–1711), John Wallis (1616–1703), and Adrien-Marie Legendre (1752–1833), among others.[3]

One of the most important works on the topic was Giovanni G. Saccheri's *Euclides ab omni naevo vindicatus*, published in 1733 [Saccheri, 1733]. He introduced what is nowadays called 'Saccheri's quadrilateral'—a quadrilateral in which two opposite sides are equal in length and perpendicular to the base. First he proved, using only the first four postulates (absolute geometry), that the summit angles of his quadrilateral are equal. Hence they can be right (Euclidean case), obtuse (Saccheri arrives at a contradiction), or acute. This third case is known as the 'hypothesis of the acute angle' or the 'third hypothesis.' We shall call 'new plane' a plane verifying the four first postulates of Euclid and, instead of the fifth, the hypothesis of the acute angle.

Saccheri's aim was to find a contradiction in this new plane and hence prove that the fifth postulate is, in fact, a theorem of Euclidean geometry. He proved, for example, the remarkable fact that under the hypothesis of the acute angle there are asymptotic straight lines (the word asymptotic is not used by Saccheri). At the end of [Saccheri, 1733], he claims that he had the contradiction he was looking for. Concretely, Proposition XXXIII says, "the hypothesis of the acute angle is absolutely false; because it is repugnant to the nature of the straight line."[4] The only error he committed was to consider points at infinity as ordinary points of the new plane, see [Dou, 1970, 391]. This was motivated by the fact that he disliked some of the results he had obtained because they went against his Euclidean intuition.

Johann H. Lambert in his *Theorie der Parallellinien* [1786] developed the theory of parallels in a way similar to that of Saccheri, but without claiming that he had arrived at a contradiction. Lambert used what is today called a 'Lambert quadrilateral'—a quadrilateral with three of its angles right angles. The fourth angle can be right (the Euclidean case), obtuse (Lambert arrives at a contradiction), or acute. He also obtained many interesting

[1] See, for instance, [Euclid, 1956, 154–155].
[2] Definition XXIII of Book I.
[3] For further details, see [Rosenfeld, 1988; Bonola, 1955].
[4] "Hypothesis anguli acuti est absolute falsa; quia repugnans naturae liniae rectae."

results, but his conclusion was, "I am tempted to conclude that the third hypothesis[5] holds for some imaginary sphere."[6]

The idea of considering a sphere of imaginary radius and its analogy with an ordinary sphere turned out to be the most important tool for the discovery of non-Euclidean geometry. We shall use the term 'Analogy' to denote the method by which, using formal substitution, R is replaced by the imaginary number Ri in all formulas that appear in the study of the geometry of a sphere of radius R, recalling that $\sin ix = i \sinh x$, and $\cos ix = \cosh x$.[7] The formulas thus obtained will be valid in the new plane.

However, the slow acceptance of complex numbers during the 18th and the early 19th century meant that the method of the Analogy was little discussed. Carl F. Gauss, however, was one of the exceptions. In [1831] he argued that complex numbers describe the plane (the basic example of a doubly extended quantity) in the same way that real numbers describe the line (the basic example of a simply extended quantity). In his famous letter on non-Euclidean geometry to Farkas Bolyai of 6 March 1832, Gauss suggested to Farkas that he should study complex numbers, thus relating non-Euclidean geometry and complex numbers (see Section 7, Letter 8).

The problem of the fifth postulate was resolved in the negative at the end of the 19th century. The definitive proof is attributed to Eugenio Beltrami (1835–1900) in his work *Saggio di interpretazione della geometria non-euclidea* [1868].[8] In this work he studies a "surface" given by the unit disc endowed with a length element, which he gives explicitly, with respect to which the curvature is constant and negative.[9] The *Saggio* can be read as the construction of a model of the new plane: its points are the points inside a circle in the Euclidean plane, the distances between these points are calculated from a length element defined by analogy with the length element of the real sphere, its straight lines are the chords, and parallel straight lines are chords meeting at a point on the circumference of the circle. In this way one obtains a geometry satisfying all of Euclid's postulates except the fifth. This geometry is the so-called non-Euclidean geometry.

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5 The third hypotheses of Lambert and Saccheri are, in fact, equivalent.

6 "Ich sollte daraus fast den Schlufs machen, die dritte Hypothese komme bey einer imaginären Kugelfläche vor" [Lambert, 1786, 354]. Although he does not say so explicitly, it is possible that this observation comes from the comparison of the formulas for the area of a triangle (spherical and non-Euclidean)

$$A = R^2(\alpha + \beta + \gamma - \pi),$$
$$A = R^2(\pi - \alpha - \beta - \gamma).$$

Lambert included a statement of the second formula in [1786, Section 82]. See Footnote 22. In 1980 Boris L. Laptev stated that Lambert also arrived at a contradiction; see [Rosenfeld, 1988, 101; Rodríguez, 2006].

7 We have developed the importance of the Analogy in the discovery of non-Euclidean geometry in [Reventós and Rodríguez, 2005; Rodríguez, 2006].

8 As Jeremy Gray remarked [1987b, 29], Beltrami was not aware of this, and it was Roberto Bonola who in fact noticed it when reading Beltrami. See [Bonola, 1955, 138, 177, 234]. The first to announce that Beltrami had shown the impossibility of proving the fifth postulate was Guillaume-Jules Hoüel (1823–1886) [1870], but his proof was incomplete, see [Voelke, 2005].

9 The observation that metric relations are independent of coordinates, important in Beltrami's work, appears clearly in Riemann's Habilitatsionsschrift [1867] and in Liouville's manuscripts of 1851; see [Lützen, 1990, 751]. But the *Saggio* was written before Beltrami read Riemann's work, posthumously published in 1868.

Many articles have been written about the history of non-Euclidean geometry. However, we believe that the close relation between classical and differential geometry and the key role played by the imaginary sphere in the discovery of non-Euclidean geometry have not been sufficiently emphasized.[10]

2. Lambert

Lambert, in section 11 of [1786], says,[11]

> The question is, can it [the fifth axiom] be correctly deduced from the Euclidean postulates together with the other axioms? Or, if these were not sufficient, can other postulates or axioms or both be given such that they have the same evidence as the Euclidean ones and from which the eleventh [fifth] axiom could be proved?
>
> For the first part of this question one can abstract from all that I have previously called *representation of the matter*. And since Euclid's postulates and remaining axioms are already expressed in words, it can and must be required that in the proof one never leans on the matter itself, but carries forward the proof in an absolutely symbolic way. In this respect Euclid's postulates are as so many algebraic equations, that one already has as previously given, and that must be solved for $x, y, z, \ldots$, without looking back to the matter itself.

We shall use the term 'Analytical Program' to refer to this idea by Lambert: the proof of the fifth postulate should not rely on any representation of the matter.

In our opinion, Gauss knew Lambert's work very well. One of the best experts on Lambert's work was Georg S. Klügel,[12] a close friend of Gauss's advisor Johann F. Pfaff (1765–1825). Some correspondence between Lambert and Klügel exists [Engels and Stäckel, 1895, 323]. Klügel was 11 years older than Lambert and his thesis, *Conatuum praecipuorum theoriam parallelarum demonstrandi recensio* (*Review of the main attempts to prove the theory of parallels*) [Klügel, 1763], which he defended on 20 August 1763, was supervised by Abraham G. Kaestner.[13] In his thesis, Klügel analyzes critically the

10 Gray, in [1979, 236], says, "The hyperbolic trigonometry of Lobachevskii and J. Bolyai was not generally taken as a conclusive demonstration of the existence of non-Euclidean geometry until it was given a foundation in the study of intrinsic Riemannian geometry."

11 A reprint of [Lambert, 1786] can be found in [Engels and Stäckel, 1895, 152–207]. The English translation, from which this extract is taken, is due to Albert Dou [1970, 401].

12 Georg S. Klügel (Hamburg 1739–Halle 1812). In 1760 he entered the University of Göttingen to study theology; but he soon came under the influence of Abraham G. Kaestner (see next footnote), who interested him in mathematics and encouraged him to write his thesis on the parallel postulate. Klügel was at the University of Göttingen until 1765 when he moved to Hannover, Helmstedt and finally to Halle. Therefore, at the time of his correspondence with Lambert he was still in Göttingen. See [Folta, 1973] and the Note of Klaus Thomas Volkert [2006] presenting the German translation of Klügel's thesis. The URL with the German translation is given in Footnote 15.

13 Abraham G. Kaestner (Leipzig 1719–Göttingen 1800). In 1756 he was appointed professor of mathematics and physics at the University of Göttingen, where he taught Gauss and Farkas Bolyai. Another of his students, Johann C. M. Bartels (1769–1836), taught Nikolai I. Lobachevsky (1792–1856). See [Goe, 1973].

experiments made so far to prove the parallel postulate. He believed in the independence of the fifth postulate and stated that Saccheri's results contradicted experience but not the axioms [Kline, 1972, 867–868].[14] For instance, when referring to equidistant straight lines, Klügel says, "It is quite clear that we use here that a line, which is at the same distance from a straight line, is itself a straight line. This can be concluded by experience and from the judgment of the eyes, not from the nature of the straight line" [Klügel, 1763, Section II].[15]

Pfaff's thesis was also supervised by Kaestner, and Gauss even stayed at Pfaff's house for several months [Dunnington, 2004, 415].

Thus, given these circumstances in Göttingen,[16] it seems unlikely that Gauss would have been unaware of Lambert's work[17] on the theory of parallels. The text was available to Gauss at the Göttingen University Library, although the records seem to show that Gauss did not withdraw it from the library.[18]

In his note of December 1818 on non-Euclidean geometry,[19] Ferdinand K. Schweikart (1780–1857) said, "That this sum [the sum of the three angles in a non-Euclidean triangle] becomes ever smaller, the more content the triangle encloses."[20] In his answer to Gerling (March 1819), Gauss said, "The defect of the angle sum in the plane triangle

[14] Perhaps Saccheri's work came into Gauss's hands via Klügel.

[15] "Satis apparet, sumi hic, lineam, quae a recta aequaliter semper distat, ipsam rectam esse, quod experientia et ex oculorum iudicio, non ex natura lineae rectae colligitur." The English translation is our own, made from the German translation due to Martin Hellmann accessible at http://www.uni-koeln.de/math-nat-fak/didaktiken/mathe/volkert/titel.htm.

[16] Dunnington [2004, 176] says, "When Gauss went to Göttingen, J. Wildt (1770–1844) gave a trial lecture on the theory of parallels (1795) ... and in 1801 Seyffer, the professor of astronomy, published two reviews of attempts to prove the parallel axiom. ... Gauss was very close to Seyffer, and their correspondence continued until the latter's death. Their conversations frequently touched on the theory of parallels."

[17] The famous entry [72] in his diary, "Plani possibilitatem demonstravi," is dated 28 July 1797, Göttingen.

[18] *Die Theorie der Parallellinie* [Lambert, 1786] was published posthumously as a paper in two parts in the first volume of the journal *Leipziger Magazin für die reine und angewandte Mathematik* in 1786. This journal appeared after the death of Christlieb B. Funck (1736–1786), who together with Nathanael G. Leske (1751–1786) and Carl F. Hindenburg (1741–1808), was editor for the journal *Leipziger Magazin für Naturkunde, Mathematik und Oekonomie*. Hindenburg, together with Johann Bernoulli (1667–1748), continued with the mathematical part of this journal (changing the second part of the title from *für Naturkunde, Oekonomie und Mathematik* to *für die reine und angewandte Mathematik*). The records show that Gauss withdrew three volumes of the journal *Leipziger Magazin für Naturkunde* in December 1795. Two months earlier, in October 1795 Gauss had also withdrawn Lambert's books *Beiträge zum Gebrauch der Mathematik (3 vols., Berlin, 1765–1772)* and in 1797 he withdrew the Lambert's book *Photometria* (see [Dunnington, 2004, 398–404] for the list of all books withdrawn by Gauss). Unfortunately the records for the beginning of 1796 are lost. We cannot know if Gauss withdrew exactly the volume of the journal in which we are interested, but we know that he was aware of the existence of the journal and many other works by Lambert.

[19] This note was sent to Gauss by Christian L. Gerling (1788–1864).

[20] "dass die Summen immer kleiner werden, je mehr Inhalt das Dreieck umfasst." [Halsted, 1900].

from 180° becomes, for example, not just greater as the area becomes greater, but it is exactly proportional to it."[21] It is possible that Gauss learned of this result through Lambert.[22]

3. The *Disquisitiones*

The relationship between the *Disquisitiones generales circa superficies curvas*[23] and non-Euclidean geometry can be analyzed according to the two following hypotheses, which enable certain natural questions to be answered.

1. Gauss was aware that definitive solution to the problem of the independence of the hypothesis of the acute angle was not possible with the material representation of points, lines, and planes given by diagrams. For this reason Gauss adopted Lambert's Analytical Program as the correct method for solving this problem definitively.
2. Gauss was determined to find a surface that could play the role of the imaginary sphere introduced by Lambert.[24]

Before the methods and results obtained from Lambert's Analytical Program can be generalized and applied to the study of any curved surface, an analytic treatment of spherical

[21] "Der Defect der Winkelsumme im ebenen Dreieck gegen 180° ist z.B. nicht bloss desto grösser, je grösser der Flächeninhalt ist, sondern ihm genau proportional." [Gauss, 1870–1927, vol. VIII, 182].

[22] Lambert states, "If the third hypothesis is true, ... then for each triangle the excess of 180° over the sum of its three angles is proportional to the area." ("Wenn es bey der dritten Hypothese möglich wäre, ... dass bey jedem Triangel der Ueberschuss von 180 Gr. über die Summe seiner drey Winkel dem Flächenraume des Triangles proportional wäre." [Lambert, 1786, Section 82].) Nevertheless Lambert's proof is far from rigorous. In fact, Gauss gave a synthetic proof of this in 1832 (Section 7, letter 8) assuming that the area of an ideal triangle is finite. A proof using differential methods had already been given in the *Disquisitiones*; see Footnote 23.

[23] Henceforth referred to simply as the *Disquisitiones*. It can be found in [Gauss, 1828; Dombrowski, 1979]. In this work, the Egregium theorem (cf. Footnote 27), the defect theorem (which relates the total curvature and the angles of a geodesic triangle), and the intrinsic theory of surfaces, which was the germ of Riemannian geometry, all appear for the first time.

[24] Among Gauss's manuscripts written between 1823 and 1827 there is the explicit formula for the pseudosphere,

$$y = R\sin\varphi$$
$$x = R\cos\varphi + \log\tan\frac{1}{2}\varphi$$
$$s = R\log\frac{1}{\sin\varphi},$$

preceded by the words, "For the curves whose revolution generates the opposite of the sphere, it is satisfied" ("Für die Curve, durch deren Revolution das Gegenstück der Kugel entsteht, ist:") [Gauss, 1870–1927, vol. VIII, 265]. From this expression it is clear that the curvature is $-\frac{1}{R^2}$, since the curvature of the surface of revolution obtained by rotating the curve $(x(s), y(s))$ about the x-axis, where s is the arc length, is given by

$$K = -\frac{1}{y}\frac{d^2y}{ds^2}.$$

Observe that this note on the "opposite of the sphere" was written while the *Disquisitiones* were being prepared.

geometry is required. The *Disquisitiones* constitute an attempt by Gauss in this direction, and for this reason Sections 1 and 2 are devoted to the sphere. The main result is Theorem VI of article 2 in the *Disquisitiones*.[25] This theorem, which seems unimportant in the 1827 version, plays an important role in the unpublished version of 1825.[26] In fact, the whole of spherical trigonometry can be analytically deduced from it.

The possibility of obtaining the results of spherical geometry without the use of diagrams strengthens even further the idea of looking for a surface analogous to the sphere but on which the hypothesis of the acute angle holds: the Lambert imaginary sphere. However, which surface does this imaginary sphere represent? How are triangles represented on it and what is the sum of their angles?

Perhaps the hope of finding such a surface was one of the reasons that led Gauss to write the *Disquisitiones*. Moreover, at the same time, his discoveries on the theory of surfaces, especially those relating to the possibility of developing one surface onto another, could be applied to geodesy, so that the *Disquisitiones* can also be considered as a first chapter on "advanced geodesy," as Gauss stated in his letter to Heinrich C. Schumacher (1780–1850), dated 21 November 1825.

If Gauss had understood, as Riemann did, that the plane can be 'curved' on itself, without being embedded in ordinary space, he could have developed the geometry corresponding to the hyperbolic length element. By applying the Analogy, we obtain this length element directly from the length element of the sphere.

Franz Taurinus (1794–1874) had made much progress in this direction, as can be seen from his writings of 1825 and 1826 on *logarithmic-spherical geometry* [Engels and Stäckel, 1895, 255–286; Rodríguez, 2006]. But Taurinus did not realize that the triangles he was considering were the geodesic triangles of the geometry of the hyperbolic length element.

Why did Gauss not take this step either? We believe that the most natural explanation is that Gauss was looking for this surface, the Lambert imaginary sphere, within ordinary space. We shall develop this idea in Section 8.

During his attempt to find the imaginary sphere, Gauss found the intrinsic geometry of surfaces. This brilliant discovery was included in the *Disquisitiones*, mainly in Section 12 in the Egregium theorem: "If a curved surface is developed upon any other surface whatever, the measure of curvature in each point remains unchanged."[27]

[25] The theorem is stated as follows: If L, L', L'', L''' denote four points on the sphere, and A the angle which the arcs LL', $L''L'''$ make at their point of intersection, then

$$\cos LL'' \cdot \cos L'L''' - \cos LL''' \cdot \cos L'L'' = \sin LL' \cdot \sin L''L''' \cdot \cos A.$$

[26] Gauss wrote in the 1825 version, "We shall add here another theorem, which has appeared nowhere else, as far we know, and which can often be used with advantage." ("Wir fügen noch ein anderes Theorem bei, welches unseres Wissens sonst nirgends vorkommt und öfters mit Nutzen gebraucht werden kann.") See [Gauss, 1870–1927, vol. VIII, 416] or [Gauss, 1902, 88] for the English version.

[27] "Si superficies curva in quamcunque aliam superficiem explicatur, mensura curvaturae in singulis punctis invariata manet." English translation from [Dombrowski, 1979, 38].

4. Gauss's isolation

In 1794, Legendre published his *Eléments de géomètrie*. In this work, and in later editions, he gave several proofs of the fifth postulate [Legendre, 1794, 1833; Bonola, 1955, 55–60]. Irrespective of whether these proofs were correct or not, it is clear that Legendre was convinced not only of the certainty of this postulate, but also that "he had finally removed the serious difficulties surrounding the foundations of geometry" [Bonola, 1955, 60]. Due to Legendre's great influence, mainly on French mathematicians, we believe that the problem of the theory of parallels was not sufficiently considered.[28] Gauss did really believe in the importance of this problem, as evidenced by his correspondence with Friedrich Bessel (1784–1846), Friedrich L. Wachter[29] (1792–1817), Farkas Bolyai, Gerling, Heinrich W. Olbers (1758–1840), and Schumacher. It was thanks to them that he received news of the important works by Schweikart, Taurinus, and János Bolyai, all of whom were either outsiders or amateur mathematicians. Lobachevsky, the other important person in this story,[30] was a professor of mathematics at the peripheral University of Kazan. Although in 1829 he had already published a text about the theory of parallels in Russian, it was not until 1840 (perhaps because his ideas on the theory of parallels were ridiculed by his Russian colleagues) that his book *Geometrischen Untersuchungen zur Theorie der Parallellinien* [Lobachevsky, 1955] appeared, which was read and immediately appreciated by Gauss.

5. The three ds^2 of Bolyai's *Appendix*

In this section we analyze the reading Gauss may have made of Bolyai's *Appendix* [2002] in 1832, taking into account that this was done 5 years after the publication of *Disquisitiones* and 10 years after Gauss wrote the formula for the curvature of a surface with respect to some conformally Euclidean chart. This formula appears in his personal notes with the

[28] Lützen in [1990, 739], mentioning Karin Reich, says that it was principally due to Joseph Liouville (1809–1882) that Gauss's ideas on differential geometry became known in France: "To be sure, Sophie Germain had read Gauss's *Disquisitiones generales circa superficies curvas* [1828], but during the following 15 years Lamé's theories of systems of orthogonal surfaces dominated the French scene, and Gauss's work was forgotten. In 1843, in a paper in Liouville's Journal on this subject, Bertrand admitted that 'After having written this memoir, I have learned about a memoir by Mr. Gauss entitled *Disquisitiones generales...*' [Bertrand, 1843]. The following year, Bonnet also referred to Gauss. It is not impossible that Liouville himself had called the attention of these two young talents to the *Disquisitiones*, and it is certain that when the interest in Gauss's ideas spread in France after 1847 it was due to Liouville." This happened 20 years after the publication of *Disquisitiones*.

[29] Wachter was a student of Gauss. In 1816 Wachter suggested to Gauss that a sphere of infinite radius in non-Euclidean space has Euclidean geometry; see [Gauss, 1870–1927, vol. VIII, 175–176]. Nevertheless, Wachter's explanations were quite obscure—for instance, when saying, "Then came the discomfort that the parts of this surface [sphere of infinite radius] are merely symmetric but not, as with the plane, congruent; which means the radius towards one side is infinite but towards the other imaginary." ("Es entsteht zwar die eine Unbequemlichkeit daraus, dass die Theile dieser Fläche bloss symmetrisch, nicht, wie bei der Ebene, congruent sind; oder dass der Radius nach der einen Seite hin unendlich, nach der andern imaginär ist.")

However, as Gray says, in [Dunnington, 2004, 463], "Gauss did not claim to possess knowledge of a new geometry, which surely means that even the ideas he was discussing with Wachter he considered to be hypothetical, and capable of turning out to be false."

[30] The question of priority has been widely studied; see for instance [Gray, 1989, 111].

title “The state of my investigations on the transformation of surfaces”;[31] that is, after he had acquired a deep knowledge of the role played by the line element *ds* in geometry.

As is well known, when Gauss wrote to Schumacher on 17 May 1831 about the theory of parallels, and more particularly about one equivalent formulation of the fifth postulate, he said,[32] “In the last few weeks I have begun to put down a few of my own meditations, which are already to some extent nearly 40 years old. These I have never put in writing, so that I have been compelled 3 or 4 times to go over the whole matter afresh in my head. I did not wish it to perish with me.”[33]

Nevertheless, some months later, in February 1832, Gauss read Bolyai’s *Appendix* and decided to write nothing further on the subject. In a letter to Gerling (14 February 1832), he said, “In addition, I note that in recent days I have received a short work from Hungary on non-Euclidean geometry in which I find all of my ideas and results developed with great elegance, although in a concentrated form that is difficult for one to follow who is not familiar with the subject. The author is a very young Austrian officer, the son of a friend of my youth with whom I had often discussed the subject in 1798, although my ideas at that time were much less developed and mature than those obtained by this young man through his own reflections. I consider this young geometer, v. Bolyai, to be a genius of the first class.”[34] Would Gauss have said this if he had thought that the work of Bolyai was a mere formal manipulation of concepts, along the lines of Taurinus,[35] without consistency?

Would Gauss have stopped writing his notes if he had not considered the problem completely solved?

Moreover, in the above letter to Farkas Bolyai (6 March 1832), he said,

> Now something about the work of your son. If I commenced by saying that I am unable to praise this work, you would certainly be surprised for a moment. But I cannot say otherwise. To praise it, would be to praise myself. Indeed the whole contents of the work, the path taken by your son, the results to which he is led, coincide almost entirely with my meditations, which have occupied my mind partly for the last thirty or thirty-five years. So I remained quite stupefied. So far as my own work is concerned, of which up till now I have put little on paper, my intention was not to let it be published during my lifetime. Indeed the majority of people have not clear ideas upon the question of

[31] “Stand meiner Untersuchung über der Flächen.” [Gauss, 1870–1927, vol. VIII, 374–384].

[32] Gauss’s letters on non-Euclidean geometry are discussed, using the Analogy, in [Reventós and Rodríguez, 2005]. See also [Reventós, 2004].

[33] “Von meinen eigenen Meditationen, die zum Theil schon gegen 40 Jahr alt sind, wovon ich aber nie etwas aufgeschrieben habe, und daher manches 3 oder 4 mal von neuem auszusinnen genöthigt gewesen bin, habe ich vor einigen Wochen doch einiges aufzuschreiben angefangen. Ich wünschte doch, dass es nicht mit mir unterginge.” [Gauss, 1870–1927, vol. VIII, 220].

[34] “Noch bemerke ich, dass ich dieser Tage eine Schrift aus Ungarn über die Nicht-Euklidische Geometrie erhalten habe, worin ich alle meine eigenen Ideen und Resultate wiederfinde, mit grosser Eleganz entwicklet, obwohl in einer für jemand, dem die Sache fremd ist, wegen der Concentrirung etwas schwer zu folgenden Form. Der Verfasser ist ein sehr junger österreichischer Officier, Sohn eines Jugendfreundes von mir, mit dem ich 1798 mich oft über die Sache unterhalten hatte, wiewohl damals meine Ideen noch viel weiter von der Ausbildung und Reife entfernt waren, die sie durch das eigene Nachdenken dieses jungen Mannes erhalten haben. Ich halte diesen jungen Geometer v. Bolyai für ein Genie erster Grösse.” [Gauss, 1870–1927, vol. VIII, 221].

[35] Taurinus developed non-Euclidean geometry formally using the imaginary sphere. The results were correct, but it was first necessary to prove that the imaginary sphere really existed. This work was commented on extensively by Gauss in his letter to Taurinus (see Section 7, letter 4).

> which we are speaking, and I have found very few people who could regard with any special interest what I communicated to them on this subject. To be able to take such an interest it is first of all necessary to have devoted careful thought to the real nature of what is wanted and upon this matter almost all are most uncertain. On the other hand it was my idea to write down all this later so that at least it should not perish with me. It is therefore a pleasant surprise for me that I am spared this trouble, and I am very glad that it is just the son of my old friend, who takes the precedence of me in such a remarkable manner.[36]

It is in this letter that Gauss suggests the name "parasphere" for the surface called only F by János Bolyai and "horosphere" by Lobachevsky. He says, "For instance, the surface and the line your son calls F and L might be named parasphere and paracycle, respectively: they are, in essence, the sphere and circle of infinite radii. One might call hypercycle the collection of all points at equal distance from a straight line with which they lie in the same plane; similarly for hypersphere."[37] Bolyai introduces the surface F, cited in Gauss's letter above, in Section 11 of the *Appendix*.

5.1. The first ds^2

In later sections of the *Appendix*, specifically in Section 24, Bolyai proves that the relation between the length z of the paracycle (horocycle) cd, the length y of the paracycle ab, and the length x of the straight line ac (see Figure 1) is given by

$$z = ye^{-x/R},$$

where R is the constant denoted i by Bolyai (the radius of the imaginary sphere for us).

From this it is easy to see that[38]

[36] "Jetzt einiges über die Arbeit Deines Sohnes. Wenn ich damit anfange, "dass ich solche nicht loben darf": so wirst Du wohl einen Augenblick stutzen: aber ich kann nicht anders; sie loben hiesse mich selbst loben: denn der ganze Inhalt der Schrift, der Weg, den Dein Sohn eingeschlagen hat, und die Resultate, zu denen geführt ist, kommen fast durchgehends mit meinen eigenen, zum Theile schon seit 30–35 Jahren angestellten Meditationen überein. In der That bin ich dadurch auf das Äusserste überrascht. Mein Vorsatz war, von meiner eigenen Arbeit, von der übrigens bis jetzt wenig zu Papier gebracht war, bei meinen Lebzeiten gar nichts bekannt werden zu lassen. Die meisten Menschen haben gar nicht den rechten Sinn für das, worauf es dabei ankommt, und ich habe nur wenige Menschen gefunden, die das, was ich ihnen mittheilte, mit besonderm Interesse aufnahmen. Um das zu können, muss man erst recht lebendig gefühlt haben, was eigentlich fehlt, und darüber sind die meisten Menschen ganz unklar. Dagegen war meine Absicht, mit der Zeit alles so zu Papier zu bringen, dass es wenigstens mit mir dereinst nicht unterginge.

Sehr bin ich also überrascht, dass diese Bemühung mir nun erspart werden kann und höchst erfreulich ist es mir, dass gerade der Sohn meines alten Freundes es ist, der mir auf eine so merkwürdige Art zuvorgekommen ist." [Gauss, 1870–1927, vol. VIII, 220–221]. English translation from [Gray, 2004, 53–54].

[37] "So könnte z. B. die Fläche, die Dein Sohn F nennt, eine Parasphäre, die Linie L ein Paracykel genannt werden: es ist im Grunde Kugelfläche, oder Kreislinie von unendlichem Radius. Hypercykel könnte der Complexus aller Punkte heissen, die von einer Geraden, mit der sie in Einer Ebene liegen, gleiche Distanz haben; eben so Hypersphäre." [Gauss, 1870–1927, vol. VIII, 221]. English translation from [Kárteszi, 1987, 35].

[38] Observe that this metric, with the change $u = e^{x/R}, v = y/R$, is the metric of the Poincaré half-plane. This could be the "extra work" mentioned by Gray in his comments to the *Appendix*, referred to earlier [Gray, 2004, 123]. Could Gauss have done this "extra work"?

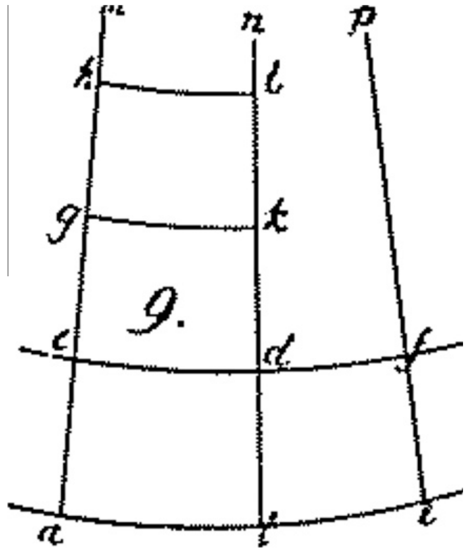

Figure 1. Figure 9 in Bolyai's *Appendix*.

$$ds^2 = dx^2 + e^{-2x/R} dy^2. \tag{1}$$

This computation, given later, could have been performed by a person with Gauss's knowledge. It is also important to point out that this expression is obtained without trigonometry and without resorting to three dimensions.

Moreover, it hardly seems possible to look at Bolyai's Figure 9 without seeing a system of local coordinates (see Figure 1).[39]

In fact, it is clear that the length element, in the sense used by Gauss, can be written in x, y coordinates as

$$ds^2 = dx^2 + f^2(x)dy^2$$

for a certain function $f(x)$, since

- this coordinate system is orthogonal[40] (so the term $dxdy$ does not appear),
- the lines $y =$ constant are geodesics parametrized by the arc length (so the coefficient of dx is 1), and
- it is invariant under translation in the y direction (so $f(x, y) = f(x)$).

To find $f(x)$, one takes the curve $\gamma(t) = (x, t)$, for a constant value of x, with $0 \leqslant t \leqslant y$ (a portion of a horocycle). The length L of γ is given by

$$L = \int_0^y |\gamma'(t)|dt = \int_0^y f(x)dt = yf(x).$$

However, since $L = ye^{-x/R}$, we have $f(x) = e^{-x/R}$.

[39] See Figure 4 for all 23 figures in the *Appendix*.

[40] The paracycles are orthogonal to the family of parallel straight lines.

5.2. The second ds^2

In Section 30, Bolyai gives the length of a circle in terms of its radius r. This relation is[41]

$$L(r) = 2\pi R \sinh \frac{r}{R}.$$

Nevertheless, calculations similar to those done to obtain (1) imply[42] that the metric in cyclic coordinates (r, θ) is given by

$$ds^2 = dr^2 + R^2 \sinh^2 \frac{r}{R} d\theta^2. \tag{2}$$

Indeed, it is clear that

$$ds^2 = dr^2 + f^2(r) d\theta^2$$

for a certain function $f(r)$, since this coordinate system is orthogonal[43] (so the term $drd\theta$ does not appear), $\theta =$ constant are geodesics parametrized by the arc length (so the coefficient of dr is 1), and it is invariant under rotation (so $f(r, \theta) = f(r)$).

To find $f(r)$, one takes the curve $\gamma(t) = (r, t)$, for a constant value of r, with $a \leqslant t \leqslant b$ (a portion of the circle). The length L of $\gamma(t)$ is given by

$$L = \int_a^b |\gamma'(t)| dt = \int_a^b f(r) dt = (b - a) f(r).$$

However, since $L(r) = 2\pi R \sinh \frac{r}{R}$, the length of γ is

$$L = (b - a) R \sinh \frac{r}{R}.$$

Hence, $f(r) = R \sinh \frac{r}{R}$, and the metric of the Bolyai plane in cyclic coordinates is the metric of the imaginary sphere.

Note that the metric of the sphere in cyclic coordinates is given by

$$ds^2 = dr^2 + R^2 \sin^2 \frac{r}{R} d\theta^2.$$

Applying here the Analogy, we obtain expression (2).

Did Gauss see this in Section 30 of the *Appendix*? Although we are unable to prove it, we are convinced that the answer to this question is affirmative, since Gauss had all the

[41] This formula is given without proof by Gauss in his letter to Schumacher in 1831; see [Gauss, 1870–1927, vol. VIII, 218]. We suggest that the approach adopted by Gauss to prove the formula was the inverse of that taken by Bolyai: Gauss obtained the length of the circumference of the circle from the line element of the imaginary sphere. How Gauss arrived at this formula, so easy to explain according to our hypothesis, has not been sufficiently explained in the literature. Less influence between differential geometry and the discovery of non-Euclidean geometry than we suppose is admitted in much of the literature. For example, Gray in Gray [2006, 63] says, "there is no evidence that Gauss derived the relevant trigonometric formulas from the profound study of differential geometry that occupied him in the 1820s." See also the section *Differential geometric foundations of non-Euclidean Geometry* in [Gray, 1987a, 2007, Chapter 20]. For a slightly different point of view on this topic see [Scholz, 2004] in German or its Spanish translation by José Ferreirós in [Scholz, 2005]. The relation between non-Euclidean and differential geometry is usually believed to have first appeared in Beltrami's work [1868].

[42] This computation does not appear in the *Appendix*; but Gauss would have found it easy to do.

[43] Gauss's lemma, proved in [Gauss, 1828].

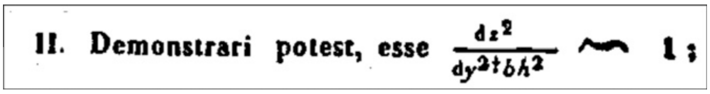
II. Demonstrari potest, esse $\frac{dz^2}{dy^2+bh^2} \sim 1$;

Figure 2. The metric of the *Appendix*.

necessary knowledge of differential geometry to perform these computations, and also because he stopped writing his notes on non-Euclidean geometry after reading Bolyai's work.

We remark that expressions (1) and (2) do not appear explicitly in Bolyai's work.

5.3. *The third ds^2*

In Section 32 of the *Appendix* a metric appears explicitly; see Figure 2.

Bolyai says,

$$\text{It can be proved that} \quad \frac{dz^2}{dy^2 + bh^2} \sim 1;$$

which, using the computation of bh given in Section 27 of the *Appendix*, is equivalent to

$$\frac{ds^2}{dy^2 + \cosh^2 \frac{y}{R} dx^2} = 1;$$

that is

$$ds^2 = dy^2 + \cosh^2 \frac{y}{R} dx^2, \tag{3}$$

which is the expression of the metric in hypercyclic coordinates.[44]

In fact, expression (3) would be apparent to anyone (Gauss, for instance) who knew the local theory of surfaces well.[45]

Specifically, it is clear that

$$ds^2 = dy^2 + f^2(y)dx^2$$

for a certain function $f(y)$, independent of x, since, by Gauss's lemma, this coordinate system is orthogonal (so the term $dxdy$ does not appear), the lines $x =$ constant are geodesics (so the coefficient of dy is 1), and it is invariant under translation in the x direction (so $f(x, y) = f(y)$).

To find $f(y)$, one takes the curve $\gamma(t) = (t, y)$ for a constant value of y, with $a \leqslant t \leqslant b$ (a portion of an equidistant line). The length of $\gamma(t)$ is

$$L = \int_a^b |\gamma'(t)| dt = \int_a^b f(y) dt = f(y)(b - a).$$

---

[44] See Appendix A for details of hypercyclic coordinates.

[45] Unfortunately János Bolyai never knew Gauss's work on the theory of surfaces: Kárteszi, in [1987, 32], says, "Even of Gauss' results only a small proportion was known to him; for example, he has not heard of the investigations of Gauss in surface theory contained in the work *Disquisitiones generales circa superficies curvas* throughout his life." This fact may explain how Gauss might have recognized that Bolyai had solved the problem of the theory of parallels, while Bolyai himself did not, and so Gauss gave up writing his notes on the subject. One can only assume that it would have been clear to Gauss that expression (3) is a length element corresponding to a surface of constant negative curvature.

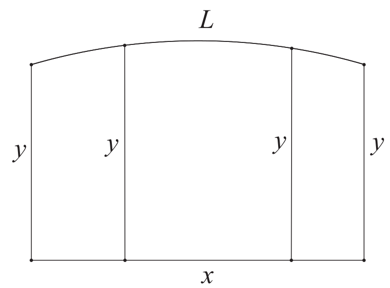

Figure 3. Mixed quadrilateral of base x and height y.

However, in Section 27 of the *Appendix*, Bolyai gives the formula for the length L of the equidistant in terms of the length x of the base and the length y of the height of the mixed quadrilateral (Figure 3). This relation is

$$L = x \cosh \frac{y}{R}.$$

Hence, $f(y) = \cosh \frac{y}{R}$, as we wished to demonstrate.

Note that the metric of the sphere in hypercyclic coordinates is given by $ds^2 = dy^2 + \cos^2 \frac{y}{R} dx^2$. Applying here the Analogy, we obtain expression (3).

Bolyai realizes the importance of the "third ds^2" and finishes Section 32, III, with these words: "The surfaces of bodies may also be determined in S,[46] as well as the *curvatures, the involutes, and evolutes* of any lines, etc."[47]

5.4. Curvature

The curvature formula

$$k = -\frac{1}{\sqrt{G}} \frac{\partial^2 \sqrt{G}}{\partial^2 r},$$

known by Gauss since his first version of the *Disquisitiones* in 1825, can be applied to the expressions (1), (2), and (3), with $G = e^{-2x/R}$, $G = R^2 \sinh^2 \frac{y}{R}$, and $G = \cosh^2 \frac{y}{R}$, respectively, to prove that Bolyai's plane is represented by a surface of constant negative curvature $-1/R^2$.

Gauss may have seen that Bolyai's expressions for the metric, equations (2) and (3), could be directly obtained by Analogy from the metric on the sphere written with regard to the cyclic and hypercyclic coordinates, respectively; especially (2), which gives the length of the circle directly, a formula well known to Gauss (letter to Schumacher, 17 May 1831). However, the expression for the metric in paracyclic coordinates, Eq. (1), cannot be obtained by Analogy, since the concept of paracycle is characteristic of hyperbolic geometry. However, Gauss's manuscripts on the theory of parallels of 1831 may be the beginning of a synthetic approach to finding this paracyclic metric; see [Gauss, 1870–1927, vol. VIII, 202–209].

In Appendix A, we make the change between cyclic, paracyclic, and hypercyclic coordinates explicit.

[46] S is the notation used by Bolyai to refer to the new plane.

[47] "Superficies quoque corporum in S determinari possunt, nec non *curvaturae, evolutae, evolventesque* linearum qualiumvis etc."

5.5. Relation with consistency

Did Gauss see that the hypercyclic coordinates on the new plane were global, unlike on the sphere, where they are not? In particular, did he see that the imaginary sphere can be covered with only one chart? Did Gauss see a proof of consistency in the *Appendix*? The letters to Gerling and Farkas Bolyai referred to above lead us to conjecture that he did. But did he have a clear concept of the problem of consistency?

Gauss, who could have done the computations that we perform in this article, did not realize that the problem of consistency had been solved, perhaps because he was trying to answer the question, 'Which surface in ordinary space has one of these metrics?' However, there does not exist such a surface in ordinary space. Although Gauss was the founder of the intrinsic geometry of surfaces, all the length elements (metrics) he used came from the Euclidean metric of ordinary space.

This epistemological mistake is very understandable: he was discovering a new world, and like many other pioneers he happened to overlook something very important. For instance, Beltrami also made the same mistake (see Footnote 8).

If one assumes that Gauss used the Analogy to find the ds^2 of the imaginary sphere, it is easy to explain all the results of the new geometry that Gauss in his letters showed that he knew. It also explains why he did not include proofs: the use of imaginary numbers was not sufficiently accepted. If Gauss had had any doubt about the consistency problem, given that he had used the Analogy and, therefore, complex numbers, it would have disappeared after he read the *Appendix*, because he found there all the necessary results deduced axiomatically and without any reference to imaginary numbers.

6. The diagrams in the *Appendix*: revision of some of Gray's comments

While we agree with Gray's comments on Bolyai's "Section 32" [Gray, 2004, 123–127],[48] we would nevertheless like to make some further remarks, which we trust will contribute to extending the recognition of Bolyai's work, which has already been acknowledged by Gray.

First of all, the coordinates used by Bolyai are the hypercyclic coordinates (the lines $x =$ constant are straight lines, while the lines $y =$ constant are equidistant lines). Thus, the expression "usual system of Cartesian (x, y) coordinates," used by [Gray [2004, 123] to denote the system of coordinates used by Bolyai, should not be misinterpreted by the reader.

Some of Gray's remarks can be considered as a moderate criticism of Bolyai's work; for instance:

— "Without as much as a hint in the direction just outlined, Bolyai supposed that his readers would recognise these arguments" [Gray, 2004, 124].
— "but it requires an interpretation that Bolyai was unwilling to provide" [Gray, 2004, 124].
— "Bolyai escaped the pedagogic problem, not for the first or only time in the *Appendix* by saying: "It can be demonstrated"" [Gray, 2004, 126].

[48] In this chapter, Gray comments on Section 32 of Bolyai's *Appendix*, stating that "János Bolyai made a series of observations whose significance seems to have escaped him and most of his commentators." [Gray, 2004, 123]. These observations are related to the method of resolving problems in the new geometry.

These comments are perfectly understandable if we accept the hypothesis that the *Appendix* was written with Gauss himself in mind. In fact, the *Appendix* was sent to Gauss in 1831[49] and the *Tentamen* was published in 1832. János Bolyai sent a first version of his work to his former professor Herr Johann Walter von Eckwehr in 1825,[50] and "on the prompting of his father" he translated it from German into Latin for publication in *Tentamen*, which was issued in Latin.[51] Given the friendship between Gauss and Farkas, it is logical to assume that Farkas had already[52] decided to send a copy to Gauss.

The writing in the *Appendix* is very concise. We do not know whether this was a result of financial difficulties[53] or whether it was for mathematical reasons. In his letter to Gerling (see above), Gauss says, "[the results of the *Appendix* are developed] in a concentrated form that is difficult for one to follow who is not familiar with the subject."

The 23 diagrams in the *Appendix*, with the caption "Tabula Appendicis" at the top on the right, reproduced here from [Bolyai, 2002, 29] in Figure 4, should not be interpreted as diagrams in the Euclidean plane, as might be erroneously inferred from Gray's remark in [2004, 124]: "He drew a picture of a curve *ABC* in the familiar Cartesian plane with x- and y-axes and outlined an interpretation of it as a picture of non-Euclidean geometry drawn in a Euclidean plane."

These figures play the same role as the figures that appear in the majority of versions of Euclid's *Elements*: they are guides for the proofs. In fact, Bolyai does not use the Euclidean plane at all. Note that in his few notes on the subject Gauss uses similar diagrams.

Nevertheless, a valid objection to Bolyai's diagrams is that he represents non-Euclidean segments in the same way that we usually represent Euclidean segments. This problem was skillfully solved by Battaglini,[54] and was the basis for the proof of consistency given by Beltrami, using a model where non-Euclidean segments are represented by Euclidean ones.

As Gray says [2004, 123], it is a pity that Bolyai did not find the hyperbolic half-plane model: "With a bit of extra work, he could have shown that the entire picture of non-Euclidean two-dimensional geometry could appear in the right half-plane (the region defined by $x > 0$), and that in his new space straight lines were curves of a certain appearance."

However, to prove consistency, it is not necessary to have this specific model of hyperbolic geometry. It suffices to have a 'plane' with an appropriate metric, as Bolyai had. But this presupposes the idea of an abstract Riemannian manifold, which was Riemann's great contribution many years later.

[49] It seems that this copy never arrived; see for instance, [Bonola, 1955, 100; Gray, 1989, 97; 1987b, 18].

[50] This manuscript has not been found. It seems that it was not returned to János. Perhaps for this reason it was not sent to Gauss until 1831.

[51] See [Bonola, 1955, XXVIII of Halsted's introduction].

[52] Perhaps as early as 3 November 1823, when he received a letter from his son in which János said, "I am determined to publish a work on parallels as soon as I can put it in order. ... All I can say now is that I have created a new and different world out of nothing." English translation from [Gray, 2004, 52].

[53] Halsted in [Bonola, 1955, XXVIII of Halsted's introduction] remarks that János contributed 104 florins and 50 kreuzers to the printing of the *Appendix*. (The yearly salary of a university professor was about 1300 florins; 60 krazers were equivalent to 1 florin.) In the opinion of Barna Szénássy, although the two Bolyais had financial difficulties throughout their lives, economy was not the main reason for the conciseness of the *Appendix*, see [Kárteszi, 1987, 224].

[54] See [Battaglini, 1867; Montesinos, 1994].

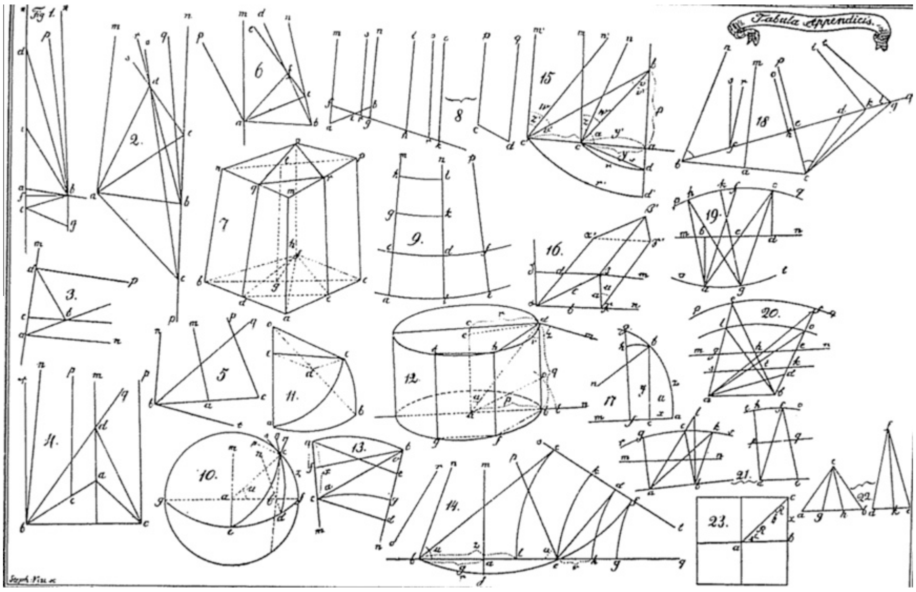

Figure 4. The diagrams in the *Appendix*.

Finally, we completely agree with Gray [2004, 126] when he says, "But the fact that Bolyai got as close as he did to formulating the elements of his new geometry in terms of the calculus is striking testimony to his insight, and seems not to have been appreciated sufficiently in his day or since."

7. The problem of the independence of the fifth postulate

In Gauss's time the consistency of Euclidean geometry was accepted without discussion. But this was not the case with the geometry arising from the negation of the fifth postulate. Perhaps because of the surprising results, a proof of the consistency of this new geometry was needed.

There are some letters written or received by Gauss in which "astral" geometry or "non-Euclidean" geometry is discussed,[55] and from which we can deduce that Gauss was convinced of the consistency of this new geometry. We mention the following:

1. Gauss to Olbers. Göttingen, 28 April 1817. "Wachter has written a short note on the foundations of the geometry. … I am becoming more and more convinced that the necessity of our geometry cannot be proved, at least not by human reason nor for human reason. Perhaps in another life we will be able to obtain insight into the nature of space, which is now unattainable. Until then we must place geometry not in the same class with arithmetic, which is purely a priori, but with mechanics." [56]

[55] See [Gauss, 1870–1927, vol. VIII, 159–225], Grundlagen der Geometrie, Nachträge zu Band IV, for Gauss's complete correspondence on the subject.

[56] "Wachter hat eine kleine Piece drucken lassen über die ersten Gründe der Geometrie. … Ich komme immer mehr zu der Überzeugung, dass die Nothwendigkeit unserer Geometrie nicht bewiesen werden kann, wenigstens nicht vom menschlichen Verstande noch für den menschlichen Verstand. Vielleicht kommen wir in einem andern Leben zu andern Einsichten in das Wesen das Raums, die uns jetzt unerreichbar sind. Bis dahin müsste man die Geometrie nicht mit der Arithmetik, die rein a priori steht, sondern etwa mit der Mechanik in gleichen Rang setzen." [Gauss, 1870–1927, vol. VIII, 177]. English translation from [Gray, 2007, 91].

2. Schweikart's Note to Gauss. Marburg, December 1818. "There are two kinds of geometry: a geometry in the strict sense—the Euclidean; and an astral geometry." [57]
3. Gauss to Gerling. Marburg, 16 March 1819. "The letter of Herr Professor Schweikart has given me extraordinary pleasure, ... because although I can really imagine that the Euclidean geometry is not correct, ..."[58]
4. Gauss to Taurinus. Göttingen, 8 November 1824. "The assumption that the sum of the three angles (of a triangle) is less than 180° leads to a special geometry, quite different from ours (Euclidean), which is absolutely consistent and which I have developed quite satisfactorily for myself, so that I can solve every problem in it, with the exception of the determination of a constant which cannot be found out a priori. ... All of my efforts to find a contradiction, an inconsistency in this non-Euclidean geometry have been fruitless."[59]
5. Gauss to Bessel. Göttingen, 27 January 1829. "My conviction that we cannot base geometry completely a priori has, if anything, become even stronger."[60]
6. Bessel to Gauss. Königsberg, 10 February 1829. "Through that which Lambert said, and what Schweikart disclosed orally, it has become clear to me, that our geometry is incomplete, and should receive a correction, which is hypothetical and, if the sum of the angles of the plane triangle is equal to a hundred and eighty degrees, vanishes."[61]
7. Gauss to Bessel. Göttingen, 9 April 1830. "According to my most intimate conviction, the theory of space has a completely different position with regards to our knowledge a priori, than the pure theory of magnitudes. Our knowledge of the former lacks completely that absolute conviction of its necessity (and therefore of its absolute truth) which is characteristic of the latter."[62]

[57] "Es gibt eine zweifache Geometrie,—eine Geometrie im engern Sinn—die Euklidische; und eine astralische Grössenlehre." [Gauss, 1870–1927, vol. VIII, 180]. English translation from [Gray, 2007, 92].

[58] "Die Notiz von Hrn. Prof. Schweikart hat mir ungemein viel Vergnügen gemacht, ... denn obgleich ich mir recht gut die Unrichtigkeit der Euklidischen Geometrie denken kann, ..." [Gauss, 1870–1927, vol. VIII, 181]. English translation partly from [Gray, 1989].

[59] "Die Annahme, dass die Summe der 3 Winkel kleiner sei als 180°, führt auf eine eigene, von der unsrigen (Euklidischen) ganz verschiedene Geometrie, die in sich selbst durchaus consequent ist, und die ich für mich selbst ganz befriedigend ausgebildet habe, so dass ich jede Aufgabe in derselben auflösen kann mit Ausnahme der Bestimmung einer Constante, die sich a priori nicht ausmitteln lässt. ... Alle meine Bemühungen, einen Widerspruch, eine Inconsequenz in dieser Nicht-Euklidischen Geometrie zu finden, sind fruchtlos gewesen." [Gauss, 1870–1927, vol. VIII, 187]. English translation partly from [Dunnington, 2004, 181].

[60] "Meine Überzeugung, dass wir die Geometrie nicht vollständig a priori begründen können, ist, wo möglich, noch fester geworden." [Gauss, 1870–1927, vol. VIII, 200]. English translation from [Gray, 2007, 95].

[61] "Durch das, was Lambert gesagt hat, und was Schweikart mündlich äusserte, ist mir klar geworden, dass unsere Geometrie unvollständig ist, und eine Correction erhalten sollte, welche hypothetisch ist und, wenn die Summe der Winkel des ebenen Dreiecks = 180° ist, verschwindet." [Gauss, 1870–1927, vol. VIII, 201]. English translation from [Halsted, 1900, 252].

[62] "Nach meiner innigsten Überzeugung hat die Raumlehre in unserm Wissen a priori eine ganz andere Stellung, wie die reine Grössenlehre; es geht unserer Kenntniss von jener durchaus diejenige vollständige Überzeugung von ihrer Nothwendigkeit (also auch von ihrer absoluten Wahrheit) ab, die der letztern eigen ist." [Gauss, 1870–1927, vol. VIII, 201]. English translation from [Ferreirós, 2007b, 209–210].

8. Gauss to Farkas Bolyai.[63] Göttingen, 6 March 1832. "That it is impossible to decide a priori between Σ and *S* is the clearest evidence of the mistake Kant had made when stating that space was merely the form of our looking at things. I pointed out another, equally strong, reason in a short paper to be found in the year 1831 volume of the Göttingischen Gelehrten Anzeigen as item 64 on p. 625.[64] Perhaps it will not be a disappointment if you try to procure that volume of the G.G.A. (which may be accomplished through any bookseller in Vienna or Buda[65]), as you also find there, developed in a few pages, the essence of my views concerning imaginary quantities."[66]

The arguments put forward by Gauss in these letters for the belief in the consistency of non-Euclidean geometry were of an inductive and physical type. Inductive: no matter how much he had searched for an inconsistency with the hypothesis of the acute angle, he had been unable to find it. Physical: although Euclidean geometry was a very good candidate for the geometry of physical space, a "non-Euclidean" geometry with a small negative curvature could also provide the answer.

Did Gauss have the concept of a mathematical model? Probably not, but it is not unrealistic to think that he may have entertained the idea that a surface in the space of three dimensions, with constant negative curvature and without singularities (the "opposite of the sphere"[67] mentioned in Footnote 24 had singularities), could be a proof of the possibility of a new plane. We completely agree with [Burago et al., 2001, 158] on this point.[68]

In the above-mentioned letter of 1832 to Farkas Bolyai, Gauss says that he had obtained the same results as Farkas's son and in similar ways. Nevertheless, in his letter to

[63] This letter is commented on in Sections 1 and 5 of this paper. See also Footnote 73.

[64] See [Gauss, 1831]. In this paper Gauss gives the geometrical interpretation of complex numbers.

[65] The German name for Buda is Ofen. Budapest became a single city with the unification in 1873 of Buda and Óbuda (Old Buda) together with Pest.

[66] "Gerade in der Unmöglichkeit, zwischen Σ und *S* a priori zu entscheiden, liegt der klarste Beweis, dass Kant Unrecht hatte zu behaupten, der Raum sei nur Form unserer Anschauung. Einen andern ebenso starken Grund habe ich in einem kleinen Aufsatze angedeutet, der in den Göttingischen Gelehrten Anzeigen 1831 steht Stück 64, pag. 625. Vielleicht wird es Dich nicht gereuen, wenn Du Dich bemühest Dir diesen Band der G.G.A. zu verschaffen (was jeder Buchhändler in Wien oder Ofen leicht bewirken kann), da darin unter andern auch die Quintessenz meiner Ansicht von den imaginären Grössen auf ein Paar Seiten dargelegt ist." [Gauss, 1870–1927, vol. VIII, 224]. English translation from [Kárteszi, 1987, 35].

[67] "Gegenstück der Kugel."

[68] In his letters, Gauss did express his personal belief that there is no contradiction in the axioms of non-Euclidean geometry. He had an ill-fated, though extremely wise idea of how to construct a model: he wanted to realize non-Euclidean geometry as the intrinsic geometry of some surface in ordinary space—the same way the geometry of the 'obtuse angle' is realized by Euclidean spheres. Gauss even found small embedded regions with the desired properties (so-called pseudo-spheres), but he was unable to realize the whole plane. This led him to suspect that a contradiction was still hidden somewhere.

Schumacher, dated 1846, he says that Lobachevsky had obtained the same results but in a different way:[69] "in the work of Lobachevsky I did not find new results, but the development follows a different approach to the one I took, and indeed Lobachevsky carried out the task in a masterly fashion and in a truly geometric spirit"[70] [Reventós and Rodríguez, 2005, 106]. Perhaps this "different approach" refers to the use of the length element of the imaginary sphere, which he obtained by Analogy (whereas János Bolyai had deduced one of these length elements explicitly, and the other two implicitly). However, as Gauss was unable to show a complete surface[71] in a space of three dimensions with this length element, he did not publish anything. The synthetic rewriting of the theory of parallels, which Gauss began in 1831, was far surpassed by the *Appendix*, a complete and masterly synthetic deduction of hyperbolic arc length.

Perhaps Gauss thought that non-Euclidean geometry could emerge by using the geometrical interpretation of complex numbers;[72] that would explain the suggestion made to Farkas Bolyai at the end of the letter. János indeed read Gauss's paper,[73] and developed independently a conception of complex numbers that applied to number theory. As far as we know, János Bolyai did not relate the new geometrical conception of complex numbers to the problem of consistency of the new geometry.

It is also possible that Gauss made the same suggestion to Riemann; but Riemann was by this time occupied with other mathematical and physical problems that would lead him to the discovery of Riemann surfaces (the first example of what today is called an abstract manifold: a topological manifold which is not a submanifold of an Euclidean space!) as well as to a conception of physical space as a perfectly elastic and massless medium formed by an elastic fluid, affected by the energy–momentum of the physical fields within it. Klein compared Riemann with Faraday, who had described the electromagnetic field with the idea of "lines of force." With Riemann, geometry became a physical geometry.

8. Looking for an imaginary sphere in ordinary space

Perhaps the most crucial mistake committed by Gauss in this matter was to look for an imaginary sphere in ordinary space. In fact, there exists no imaginary sphere, in the sense of

[69] Gauss refers to the German version [Lobachevsky, 1955], which does not use differential calculus.

[70] "Materiell für mich Neues habe ich also im LOBATSCHEWSKYschen Werke nicht gefunden, aber die Entwickelung ist auf anderm Wege gemacht, als ich selbst eingeschlagen habe, und zwar von LOBATSCHEWSKY auf eine meisterhafte Art in ächt geometrischem Geiste." [Gauss, 1870–1927, vol. VIII, 239].

[71] A surface without singularities, where the straight lines can be extended indefinitely.

[72] In Appendix B we sketch an elementary proof of consistency using complex numbers.

[73] In his letter of 6 March 1832, Gauss recommended Farkas to read his note [Gauss, 1831] on imaginary quantities, but he does not mention [Gauss, 1832], "despite the fact that Farkas had asked him many times for his detailed work on imaginary quantities." [Kiss, 1999, 73]. János read [Gauss, 1831] and quoted it in many places, but he was unaware of Gauss's more important work [Gauss, 1832].

a surface of constant negative curvature, in ordinary space.[74] Therefore, the search for an imaginary sphere proved to be a futile struggle.[75] It is possible that in 1831 Gauss was aware that he would come to a dead end, and decided to take the deductive point of view. But it was too late: János Bolyai had already followed this path in the *Appendix*. The impossibility of finding a complete surface of constant negative curvature in ordinary space could have caused Gauss to doubt his belief in the consistency of non-Euclidean Geometry, and may be the main reason that he made no effort to publicize the *Appendix*.

The *Appendix* proves that the problem of consistency is almost the same in both geometries: the parasphere (a surface of non-Euclidean space) has Euclidean geometry (see Footnote 29). Hence, for symmetry, it seems reasonable to look for an imaginary sphere in ordinary space.

It would be interesting to answer the following question: Why did Gauss only look at surfaces in three-dimensional space?

A possible answer is that numbers and geometry were understood on different levels, in the sense that the identification of the real numbers $\mathbb{R}$, and the sets $\mathbb{R}^2$ and $\mathbb{R}^3$ as geometrical objects had yet to be clearly made. Gauss and other contemporary mathematicians did not identify the set of pairs of real numbers with the points of the plane, as is done today. It was necessary to wait for Dedekind for the foundation of real numbers; he probably learned from Riemann the importance of thinking about mathematics conceptually, in order to take the definitive step towards the geometrization of $\mathbb{R}^n$.

As Ferreirós has observed, it is precisely with Riemann that the idea of a conceptual mathematics arises: a mathematics that studies manifolds and their mappings [Ferreirós, 2000, 93–95; 2007a]. Riemann took this giant step because he needed to extend geometric intuition to areas of mathematics different from geometry. However, at the same time, he also found the study of multiply extended quantities useful for thinking about geometry without any spatial intuition [Ferreirós, 2000, 94]; Riemann coincides on this point with Lambert and his Analytical Program, which was introduced with the hope of solving the classical problem of the Euclidean theory of parallels [Reventós and Rodríguez, 2005, 16]. This program was completed by Hilbert in his fundamental work on the foundations of geometry, using set theory introduced by Cantor [Hilbert, 1899]. As Hilbert said: “No one shall expel us from the paradise that Cantor has created for us.”[76]

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[74] The interpretation of Gaussian curvature as the product of principal curvatures, and hence equal to $1/R^2$ for a sphere of radius R, appears in the first version of *Disquisitiones* in 1825. In fact, Olinde Rodrigues essentially proved it in 1815 when he proved what today we know as the Gauss–Bonnet theorem, see [Rodrigues, 1815]. Indeed, Rodrigues in his study of the integral of the product of the principal curvatures says, “Let us imagine a sphere with radius equal to the unit; and let us move the radius of this sphere in a way such that it will be parallel to all the normals of the piece of the surface which we want to integrate. The area described by the endpoint of this radius will coincide with the value of the desired integral.” (“Concevons une sphère d’un rayon égal à l’unité; puis faisons mouvoir le rayon de cette sphère, de manière qu’il soit successivement parallèle à toutes les normales de la portion de surface sur laquelle on veut prendre l’intégrale, l’aire sphérique décrite par l’extrémité de ce rayon, sera la valeur de l’intégrale cherchée.”)

[75] Hilbert [1901] proved that there exists no complete regular surface of constant negative curvature immersed in $\mathbb{R}^3$. Fifty-four years later, Kuiper [1955] proved that such a surface does exist if we change regular to $\mathcal{C}^1$.

[76] “Aus dem Paradies, das Cantor uns geschaffen, soll uns niemand vertreiben können.” [Hilbert, 1926, 170].

9. Non-Euclidean geometry as absolute Euclidean geometry on a reduced scale

A note by Gauss dated about 1840–1846 [Gauss, 1870–1927, vol. VIII, 255–257] was found in his copy of Lobachevsky's work *Geometrischen Untersuchungen zur Theorie der Parallellinien* [1955]. This note is quite short and Gauss did not give it a title.[77] However, it is referred to as "The spherical and the non-Euclidean geometry"[78] by Stäckel, who carefully commented on it. It is also commented upon in [Reichardt, 1985, Section 2.3].

Although this note was written many years after the Gauss–Bolyai relation discussed in the previous sections, we draw attention to it because it gives a clue as to how Gauss used differential geometry in order to consider problems of non-Euclidean geometry. Perhaps Gauss was trying to arrive at the same conclusions as Lobaschevsky, but by his own method.

Before continuing, we would like to point out that the formulas that head this note (p. 255) relating the angles and the sides of a triangle with two unknown functions f, g, and from which f and g are computed,[79] hold for absolute geometry. Indeed they can be deduced solely from absolute geometry—where the side–angle–side criterion holds—, under the hypothesis that in this absolute geometry trigonometric formulas exist that relate the sides and the angles of a triangle, and assuming that this absolute geometry is Euclidean on a reduced scale.

These two hypotheses, together with the relations between the sides of a Saccheri quadrilateral, imply the rectificability of equidistants and the rectificability of a circle (which are true in absolute geometry and were well known by Gauss at that time).[80] Using these hypotheses and relations, Gauss's note can be understood without any great difficulty. Although Stäckel's explanations in [Gauss, 1870–1927, vol. VIII, 257–264] are totally clear and can be followed easily, we give here, for the benefit of the reader, the deduction of the first formula, adapted to our approach.

Let $\triangle ABC$ be a right-angle triangle with sides a, b, c, and assume that it is changing with time in such a way that the right angle C remains constant. Because of the SAS Theorem (side–angle–side), every trigonometric relation between A, B, C, a, b, c can be reduced to a relation between b, c, A. Thus we can assume that the relation $F(b(t), c(t), A(t)) = 0$ holds for each t. Hence, differentiating, we obtain

$$F_b \frac{db}{dt} + F_c \frac{dc}{dt} + F_A \frac{dA}{dt} = 0.$$

From the first diagram (Figure 5) we deduce

[77] In this note, not written for publication, Gauss reveals a part of his method for doing differential geometry: he applies classical geometry to a small variation of a diagram.

[78] "Die Sphärische und die Nicht-Euklidische Geometrie."

[79] The formulas are

$$\begin{aligned} ga.\partial b - \sin B.\partial c + fc.\cos B.\partial A &= 0 \\ gb.\partial a - \sin A.\partial c + fc.\cos A.\partial B &= 0 \\ \sin B.\partial a - ga.\cos B.\partial b - fc.\partial A &= 0 \\ gb.\cos A.\partial a - \sin A.\partial b + fc.\partial B &= 0. \end{aligned}$$

[80] The upper side of a Saccheri quadrilateral of equal sides a and base b is equal to $g(a)b$ for some function g, and the length of a circular sector of α radians and radius r is $f(r)\alpha$ for some function f.

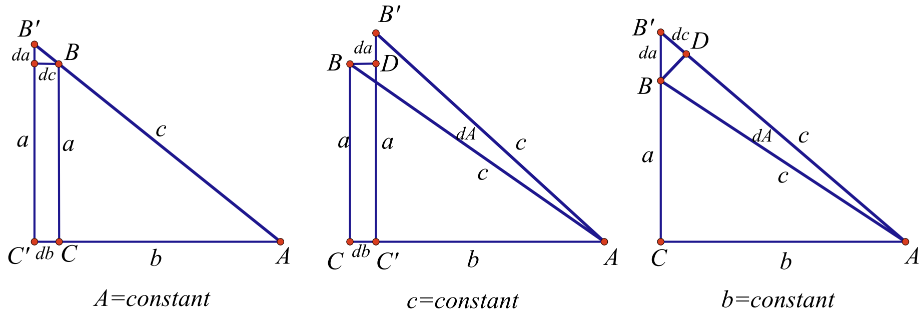

Figure 5. Stäckel's diagram redrawn, [Gauss, 1870–1927, vol. VIII, 259].

$$\sin B = \frac{g(a)db}{dc},$$

since the small triangle with hypotenuse dc can be considered as Euclidean, and one of the catheti is the top side of a Saccheri quadrilateral with base db and height a.

From the second diagram we deduce

$$\cos B = \frac{BD}{BB'} = \frac{-g(a)db}{f(c)dA},$$

since the small triangle $\triangle BB'D$ can be considered as Euclidean and BB' as the arc length of a circular sector of radius c and angle dA. The minus sign comes from the relative position between C and C'.

From the third diagram we deduce

$$\tan B = \frac{BD}{dc} = \frac{f(c)dA}{dc},$$

since the small triangle $\triangle BB'D$ can be considered as Euclidean, and BD as the arc length of a circular sector of radius c and angle dA.

From these three relations we easily compute the partial derivatives F_a, F_b, F_c (up to a constant) and obtain the first Gauss formula in [Gauss, 1870–1927, vol. VIII, 255]:

$$g(a)db - \sin B dc + f(c) \cos B dA = 0. \tag{4}$$

In fact, these three steps, followed by Stäckel, can be viewed together in the diagram in Figure 6, since formula (4) says only that

$$FD = FE + ED.$$

From (4) and similar expressions, Gauss with his characteristic genius arrives at a second-order differential equation, which allows the functions $f(c)$ and $g(a)$ to be computed easily. Gauss assumes that the integration constant is negative ($-kk$ in Gauss's notation), thereby obtaining

$$f(x) = \frac{\alpha}{k} \sin kx,$$
$$g(x) = \cos kx,$$

and, in particular, the spherical trigonometric formulas for a sphere of radius $1/k$. Nevertheless, if we assume that the integration constant is positive, we obtain, without the introduction of imaginary numbers, the non-Euclidean trigonometric formulas (those corresponding to a sphere of radius i/k). We also remark that by arriving at these formulas Gauss obtains two of the ds^2 in the *Appendix* : expressions (2) and (3).

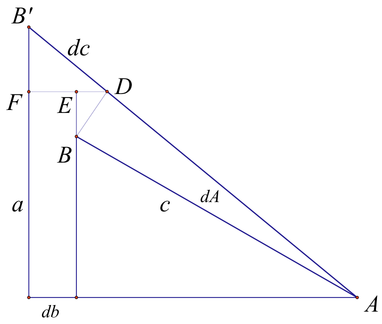

Figure 6. Representation of the three differentials.

In Section 19 of the *Disquisitiones*, Gauss computes $f'(0)$, obtaining $f'(0) = 1$; i.e., the constant α introduced by Gauss in the preceding computations is 1 if there is a tangent plane in A. In fact $\alpha \neq 1$ only if A is a singular point, such as the vertex of a cone.

In fact Gauss says [1828, Section 19]: "Generally speaking, $m[m = \sqrt{G}]$[81] will be a function of p, q and mdq the expression for the element of any line whatever of the second system. But in the particular case where all the lines p go out from the same point ... for an infinitely small value of p the element of a line of the second system (which can be regarded as a circle described with radius p) is equal to pdq, we shall have for an infinitely small value of p, $m = p$, and consequently, for $p = 0$, $m = 0$ at the same time, and $\frac{\partial m}{\partial p} = 1$."[82,83]

Why was it clear to Gauss in 1827 that the "element of any line" of the second system is pdq, while in 1840 this element is αpdq? The reason could be that in the *Disquisitiones* the argument used is that the metric is defined in the singular point $p = 0$, which is guaranteed because the metrics on the surfaces considered in the *Disquisitiones* come from the ambient metric of ordinary space.[84]

10. Conclusion

In this paper we analyze a crucial moment in the history of the discovery of non-Euclidean geometry: Gauss's reading of Bolyai's *Appendix* in 1832. We assume what we believe to be the plausible hypothesis that Gauss was following Lambert's Analytical Program and that he was looking, among surfaces in ordinary space, for Lambert's hypothetical imaginary sphere.

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[81] Note that, under the hypothesis of radial symmetry, we are assuming $m = f$.

[82] Spivak, in his comments on the *Disquisitiones*, gives a more complete proof of this fact; see [Spivak, 1979, vol. 2, 84 and 120].

[83] "Generaliter loquendo m erit functio ipsarum p, q atque mdq expressio elementi cuiusvis lineae systematis. In casu speciali autem, ubi omnes lineae p ab eodem puncto proficiscuntur, manifesto pro $p = 0$ esse debet $m = 0$; porro si in hoc casu pro q adoptamus angulum ipsum, quem elementum primum cuiusvis lineae primi systematis facit cum elemento alicuius ex ipsis ad arbitrium electae, quum pro valore infinite parvo ipsius p, elementum lineae secundi systematis (quae considerari potest tamquam circulus radio p descriptus), sit $= pdq$, erit pro valore infinite parvo ipsius p, $m = p$, adeoque, pro $p = 0$ simul $m = 0$ et $\frac{dm}{dp} = 1$."

[84] Many years later, the introduction of conical singularities into the study of hyperbolic manifolds led to the introduction of metrics of the type $ds^2 = dp^2 + \alpha^2 R^2 \sin^2 \frac{p}{R} dq^2$, which verify $f'(0) = \alpha$. All these metrics, for different values of α, have the same curvature, but the length of the element of a line $p =$ constant is αpdq.

Gauss placed this reading on record in two letters; one to Gerling in February 1832 and another longer letter to Bolyai's father in March 1832.

In the letter to Farkas, Gauss says,

1. "The way followed by your son, and the results he obtained agree almost from beginning to end with the meditations I had been engaged in partly for 30–35 years already."
2. "I had planned to write down everything in the course of time . . . now I can save myself this trouble."
3. "Perhaps it will not be a disappointment if you try to procure that volume . . . as you also find there, developed in a few pages, the essence of my views concerning imaginary quantities."

We answer some natural questions arising from these statements by Gauss:

1. What approach was adopted by Gauss in his meditations? Was it the same as that adopted by Bolyai?
2. Why did Gauss feel that there was no longer any need to write anything more about it?
3. What is the relation between imaginary quantities and the problem of the theory of parallels?

In Section 5 we saw how Bolyai axiomatically deduces a formula for ds^2 in the hypercyclic coordinate system, which is the system most similar to that of rectangular coordinates in Euclidean geometry. This shows that he wanted to follow the method of the differential geometry of his time: he was looking for an arc length element in the new plane. Gauss says in his letter that the approach adopted by Bolyai is the same as his. However, Bolyai was not familiar with the *Disquisitiones* and did not recognize the two first ds^2. Nevertheless, it seems clear that they were indeed recognized by Gauss.

In 1831, Gauss gave up searching for Lambert's imaginary sphere in ordinary space and opted for the deductive method: after studying the transitivity of parallelism, he described synthetically the paracycle [Gauss, 1870–1927, vol. VIII, 202–209; Bonola, 1955, 67–74]. However, in January 1832, after reading the *Appendix*, he gave up writing about such a difficult subject: the son of his old friend Farkas had "wonderfully outmatched him."

In the *Appendix*, Bolyai gives the rectification of the paracycle. This should have allowed him to deduce the first ds^2, but he failed to notice it. The new geometry can be deduced from this arc length element with the methods of the *Disquisitiones*; Bolyai had to deduce the third ds^2 to arrive at this conclusion.

The problem of consistency still remained to be solved. This would have been possible had Lambert's imaginary sphere been found. However, this depended on the consistency of imaginary quantities, a question resolved by Gauss in his note of 1831 (that he recommended for reading to F. Bolyai). Gauss was right: in Appendix B we see how the Analogy and the complex plane as conceived by Gauss lead naturally to the Poincaré disc model of the hyperbolic plane. This shows that non-Euclidean geometry is as consistent as Euclidean geometry.

Why did Gauss not publish a review of the *Appendix*? This would have attracted the attention of the mathematical community to this important work. His fear of the Kantians, who were led by his colleague Lotze, is not a convincing reason. We provide another possible reason: Gauss hoped to find the imaginary sphere. He knew the pseudosphere, but this surface was not geodesically complete.[85] If the singularities were in some sense inevitable, Bolyai's plane would also be inconsistent. Furthermore, the *Appendix* fails to satisfy Lambert's Analytical Program: it contains 23 diagrams (see Section 6), which is far from the idea that in the proof "we should never rely on any representation of the matter."

Gauss made the mistake of expecting the imaginary sphere to be immersed in ordinary space. In 1901, Hilbert proved that there exists no complete regular surface of constant negative curvature immersed in ordinary space. For Gauss, the length element ds is always the length element of a surface in the space. The notion of an abstract surface had yet to appear.

Finally, for completeness, we showed how Gauss uses differential geometry in considering problems of non-Euclidean geometry. We also showed how Gauss again addresses singularities and eventually finds the last two ds^2 of the *Appendix*.

Acknowledgments

We are grateful to Professor Jeremy Gray for his valuable comments after a reading of a first draft of this paper and for drawing our attention to the references [Reichardt, 1985] and [Gauss, 1870–1927, vol. VIII, 175–178 and 250–265]. We are indebted to the anonymous reviewers and to the editor, June Barrow-Green, for all their interesting and enriching comments and suggestions. We thank also Doris Potosí for helpful comments and suggestions. This research was partially supported by FEDER/Micinn through Grant MTM2009-0759.

Appendix A. Coordinate systems

In the hyperbolic plane, apart from the polar or "cyclic" coordinates and the cartesian or "hypercyclic" coordinates, there are also the "paracyclic" or "horocyclic" coordinates in which one of the distances is measured on paracycles. See Figure 7.

Cyclic (r, α). Here, r is the distance between the point P and the origin O; and α is the angle between the straight line PO and a given straight line through O. Observe that $r =$ constant is a hyperbolic circle.

Hypercyclic $(\bar{x}, \bar{y})$. Here, $\bar{x}$ is the distance between the origin O and the point Q, the intersection of the straight line through P orthogonal to a given straight line through O; and $\bar{y}$ is the distance between the point P and Q. Observe that $\bar{y} =$ constant is a hypercycle (equidistant).

Both cyclic and hypercyclic coordinates were introduced and widely used by Gauss in the *Disquisitiones*.

Paracyclic (x, y). Here, x is the distance between the origin O and the point Q, given as the intersection of a given straight line through O and the horocycle through P with axis this line; and y is the length of the horocycle OR, where R is the intersection of the axis through P with the horocycle of this family through O. Observe that $x =$ constant is a paracycle (horocycle).

Recall that three points of the hyperbolic plane determine a straight line, a circle, a hypercycle or a paracycle. The assumption that three points not on a line determine a circle is equivalent to the fifth postulate. In fact, this was the mistake made by Farkas Bolyai in his proof of this postulate.

[85] A surface is geodesically complete when the geodesics are defined for all values of the arc-length parameter.

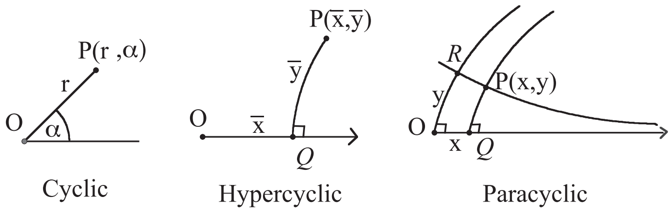

Figure 7. Three coordinate systems.

A.1. Hypercyclic–cyclic

The change of coordinates cyclic–hypercyclic is immediate by applying trigonometry to a right triangle of sides $\bar{x}, \bar{y}$ and hypotenuse r (see [Reventós and Rodríguez, 2005, 120]):

$$\cosh\frac{r}{R} = \cosh\frac{\bar{x}}{R}\cosh\frac{\bar{y}}{R},$$
$$\sinh\frac{\bar{y}}{R} = \sinh\frac{r}{R}\sin\theta.$$

From this system we can write $x = x(r,\theta)$, $y = y(r,\theta)$. In particular,

$$d\bar{y}^2 + \cosh^2\frac{\bar{y}}{R}d\bar{x}^2 = dr^2 + R^2\sinh^2\frac{r}{R}d\theta^2.$$

A.2. Hypercyclic–paracyclic

Let us assume that the point P has hypercyclic coordinates $(\bar{x}, \bar{y})$, and paracyclic coordinates (x, y).

In Figure 8, CO and PA are arcs of horocycles orthogonal to the parallel straight lines CP, OA. The hypercyclic coordinates are given by $\bar{x} = OB$, $\bar{y} = PB$; the paracyclic coordinates are given by $x = OA$, $y = CO$.

The relation between the length z of the horocycle PA and the length $\bar{y}$ of the straight line PB is

$$z = ye^{-x}. \tag{A.1}$$

Also

$$z = \sinh\bar{y} \tag{A.2}$$

and

$$e^a = \cosh\bar{y}, \tag{A.3}$$

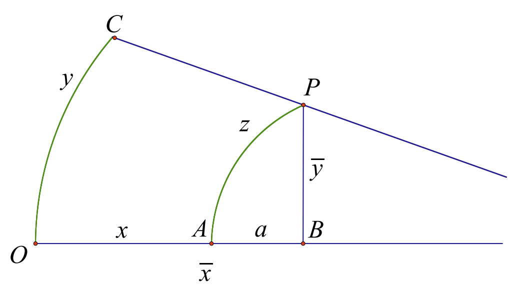

Figure 8. Relation between hypercyclic and paracyclic coordinates.

where $a = AB$. We remark that Eqs. (A.1), (A.2), and (A.3) are given directly in the *Appendix*: Eq. (A.1) in Section 24 and Eqs. (A.2) and (A.3) in Section 32. Bolyai writes $z = i \cot CBN$, which in our notation is $z = \cot \Pi(\bar{y})$, (we are assuming curvature $= -1$, i.e., $i = 1$), but it is easy to see that $\cot \Pi(\bar{y}) = \sinh \bar{y}$, and thus we have Eq. (A.2).

From these equations we can make the change of coordinates explicit:

$$\bar{x} = x + \frac{1}{2} \ln(1 + y^2 e^{-2x}),$$
$$\bar{y} = \ln(y e^{-x} + \sqrt{y^2 e^{-2x} + 1}).$$

In particular,

$$d\bar{y}^2 + \cosh^2 \bar{y} d\bar{x}^2 = dx^2 + e^{-2x} dy^2.$$

Appendix B. A wasted opportunity

As we commented in Section 2, as a result of Legendre's influence, the French mathematicians were not interested in the classical problem of the theory of parallels. Moreover, Lagrange's analytical point of view spread rapidly throughout the European mathematical community and the synthetical approach remained buried until Poincaré unearthed it again.

The wasted opportunity is revealed in the following argument which provides a construction of the Poincaré disc model of non-Euclidean geometry. The construction uses only the stereographic projection and the Analogy. So it could easily have been performed by Monge or his school in the École Polytechnique, 30 years before the *Appendix*, but they did not do it. This school had as its leitmotif the translation of geometric properties using geometric transformations; in particular, stereographic projections of quadrics over the plane. For instance, Michel F. Chasles (1793–1880) in [1937, 191] says, "From then on, Monge's students successfully cultivated Geometry of a really new kind, which has often been rightly referred to as the 'Monge School', which as we have said consists in introducing into plane Geometry considerations of three dimensional Geometry."[86]

The stereographic projection between the sphere S_R of radius R and the plane that contains the equator is given by

$$p = \frac{Rx}{R - z},$$
$$q = \frac{Ry}{R - z},$$

with $x^2 + y^2 + z^2 = R^2$.

Equivalently, the image of the point $(x, y, z) \in S_R$ is the complex number $w = p + iq$.

Let us translate the geometry of S_R to the extended complex plane $\mathbb{C}$ via this stereographic projection. First we note that the equator is given by

$$w\bar{w} = R^2.$$

Moreover, if w, w' are the images under the stereographic projection of antipodal points, then

[86] "Depuis, les élèves de Monge cultivèrent avec succès cette Géométrie, d'un genre vraiment nouveau, et à laquelle on a souvent donné, avec raison, le nom d'école de Monge, et qui consiste, comme nous venons de dire, à introduire dans la Géométrie plane des considerations de Géométrie à trois dimensions."

$$w' = -\frac{R^2}{\bar{w}}. \tag{B.1}$$

Since stereographic projection takes circles to circles, the image of a meridian is a circle in the complex plane. Hence, if $P, Q \in \mathbb{C}$, the 'straight line' PQ is the circle determined by the three points $P, Q, -P^*$, where P^* is the inverse point of P with respect to the circle $w\bar{w} = R^2$.

The 'angles' of this geometry on $\mathbb{C}$ are the angles in S_R. 'Congruent' relations can also be derived in this way. It is the geometry of the sphere considered as $\mathbb{C} \cup \{\infty\}$.

If we now apply the Analogy by formally replacing R by Ri in (B.1), we obtain

$$w' = \frac{R^2}{\bar{w}}.$$

What are the straight lines of this new geometry? If $P, Q \in \mathbb{C}$, the new straight line PQ is the circle determined by the three points P, Q, P^*. Since this circle is orthogonal to the circle $w\bar{w} = R^2$, the new straight lines are circles orthogonal to the boundary of the disc of radius R.

Note that we are obliged to exclude the case $P = P^*$ because the three points must be different. However, the set of points P with $P = P^*$ is the boundary of the disc. Therefore, this boundary does not belong to the new geometry.

Thus we have the open disc and its complement, which are 'equal' through inversion. If we consider the open disc with the straight lines defined above, and further assume that 'movements' are generated by inversions, we have the classical Poincaré disc. In other words, we have a model of non-Euclidean geometry, and the problem of consistency is solved. In fact, an inconsistency in non-Euclidean geometry would be translated into an inconsistency in inversion geometry, and hence into an inconsistency in Euclidean geometry. Non-Euclidean geometry is thus as consistent as Euclidean geometry.

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