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# HOMOGENEOUS CONTACT COMPACT MANIFOLDS AND HOMOGENEOUS SYMPLECTIC MANIFOLDS

BY

ANTONIO DIAZ MIRANDA and AGUSTI REVENTOS (*)

Univ. Autònoma de Barcelona

RÉSUMÉ. — On montre que la fibration de Boothby et Wang donne une correspondance bijective entre l'ensemble des variétés compactes homogènes de contact (à une équivalence près) et celui des variétés compactes, simplement connexes, homogènes symplectiques dont la forme symplectique est entière (à une équivalence près). Ceci complète certains résultats partiels de Boothby et Wang sur les variétés homogènes de contact. D'autres résultats sur les espaces homogènes symplectiques sont aussi prouvés.

ABSTRACT. — In this paper we prove that the Boothby and Wang fibration gives us a bijective correspondence between the set of homogeneous contact compact manifolds (up to equivalence) and the set of homogeneous symplectic, simply connected compact manifolds whose symplectic form is integral (up to equivalence). This completes some partial results of Boothby and Wang on homogeneous contact manifolds. Other results on homogeneous symplectic spaces are also proved.

## 1. Introduction

Let $B$ be a symplectic manifold such that the cohomology class of the symplectic form, $\Omega$, is integral. Then, from the results of KOBAYASHI [4], we know that there is a principal circle bundle, $M(B, S^1)$, over $B$, with projection $\pi$ and a connection form $\omega$, on $M$ such that $d\omega = \pi^* \Omega$ (the Lie algebra of $S^1$ is

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Antonio DIAZ MIRANDA, Sangenjo, 9, Madrid-34 (Espagne).

Agusti REVENTOS, Portolá 22, Barcelona 23 (Espagne).

canonically identified to $\mathbb{R}$). Moreover $\omega$ is a regular contact form on $M$ (*cf.* BOOTHBY and WANG [1]). In general this bundle is not unique.

On the other hand if $\omega$ is a *regular* contact form on a compact manifold $M$, then there is a principal circle bundle $M(B,S^1)$ as above and a constant $k\neq 0$ such that $d(k\omega)=\pi^*\Omega$ (*cf.* [1]).

In this paper we study the homogeneous contact compact manifolds. We prove that every homogeneous contact compact manifold $M$ is the total space of a principal circle bundle over a *simply connected* homogeneous Hodge manifold. A similar result, in case $M$ is simply connected, was obtained by BOOTHBY and WANG [1].

We also prove some results in the converse direction. Suppose that a Lie group acts transitively on a simply connected symplectic manifold of integral fundamental class by automorphisms of the symplectic structure. In this situation, we show the existence of a compact semisimple Lie group having a double transitive action. First, it acts by automorphisms of the contact structure on the total space of the bundle obtained by Kobayashi's method (the simply connectedness condition implies unicity, up to equivalence, of the bundle). Second, it acts by automorphisms of the symplectic structure on the base space. Moreover, these actions are related by the fact that the fibre bundle projection is equivariant. With natural equivalence relations for contact and symplectic manifolds the above construction induces a one to one map from the set of equivalence classes of homogeneous contact compact manifolds onto the set of equivalence classes of homogeneous symplectic simply connected compact manifolds with integral fundamental class.

The organization of the paper is as follows. In section 2 we collect some known definitions and results to be used in the following. In section 3 we prove some previous results, some of which are also of independent interest. There we show that the compact strongly symplectic homogeneous spaces are in fact the simply connected homogeneous Hodge manifolds, as well as the coadjoint orbits of compact semisimple Lie groups (up to equivalence).

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## 2. Preliminaries

All manifolds considered are assumed to be differentiable of class $C^\infty$, finite dimensional, Hausdorff, second countable and connected.

A *contact manifold* is a pair $(M, \omega)$ where $M$ is a differentiable manifold of dimension $2n+1$, and $\omega$ a 1-form on $M$ such that $\omega \wedge (d\omega)^n$ is a volume element on $M$. The unique vector field, $Z$, defined by $i_Z \omega = 1$ and $i_Z d\omega = 0$ is called associated to $\omega$.

A *homogeneous contact manifold* is a contact manifold $(M, \omega)$, such that there exists a Lie group which acts transitively on $M$ by $\omega$-preserving diffeomorphisms.

A *simplectic manifold* is a pair, $(B, \Omega)$, where $B$ is a manifold and $\Omega$ a non-degenerate closed 2-form on $B$, i. e., $d\Omega = 0$, $\dim B = 2n$ and $\Omega^n$ is a volume element on $B$.

If $(B, \Omega)$ is a symplectic manifold, then for any $f \in C^\infty(B)$ there exists a unique vector field on $B$, $X_f$, such that:

$$i_{X_f} \Omega = df.$$

Such a vector field is said to be a *globally Hamiltonian* one.

One defines the *Poisson bracket* of $f, g \in C^\infty(B)$ by:

$$\{f, g\} = X_f(g) = \Omega(X_g, X_f).$$

The space $C^\infty(B)$ is a Lie algebra under Poisson bracket.

We recall now some known facts about the coadjoint representation (*see* [5] for proofs).

Let $G$ be a Lie group, $\mathfrak{G}$ its Lie algebra and $\mathfrak{G}^*$ the dual of $\mathfrak{G}$. The *coadjoint representation* of $G$ is the homomorphism:

$$\mathrm{Ad}^* : G \to \mathrm{Aut}(\mathfrak{G}^*),$$

defined by:

$$\langle \mathrm{Ad}_g^* . \alpha, X \rangle = \langle \alpha, \mathrm{Ad}_{g^{-1}} . X \rangle, \qquad \forall X \in \mathfrak{G}, \quad \alpha \in \mathfrak{G}^*.$$

$\mathfrak{G}^*$, as a finite dimensional real vector space, has a canonical differentiable structure. If $X \in \mathfrak{G}$, then $X$ defines a differentiable function on $\mathfrak{G}^*$ by $X(\alpha) = \langle \alpha, X \rangle$.

Consider the (differentiable) action:

$$\psi : G \times \mathfrak{G}^* \to \mathfrak{G}^*,$$

given by $\psi(g, \alpha) = \mathrm{Ad}_g^* . \alpha$.

Let $\psi'_Y$ be the vector field on $\mathfrak{G}^*$ generated by $Y \in \mathfrak{G}$ (i. e., the flow of $\psi'_Y$ is $\{ \psi_{\mathrm{Exp}(-tY)} : t \in \mathbb{R} \}$).

On the differentiable functions cited above we have $\psi'_Y(X) = [Y, X]$, so $\psi'_Y$ will be denoted by $\mathrm{ad}_Y$.

If $\theta$ is the orbit of $\alpha \in \mathfrak{G}^*$ by this action, its tangent space at $\alpha$ can be canonically identified with the subspace of $T_\alpha \mathfrak{G}^*$ generated by the values at $\alpha$ of the vector fields $\{ \mathrm{ad}_X ; X \in \mathfrak{G} \}$.

The differential 2-form defined on $\theta$ by:

$$(\Omega_\theta)_\alpha((\mathrm{ad}_X)_\alpha, (\mathrm{ad}_Y)_\alpha) = \langle \alpha, [Y, X] \rangle,$$

is a symplectic form on $\theta$.

A symplectic manifold $(B, \Omega)$ is said to be *homogeneous symplectic* if there is a Lie group acting transitively on $B$ as a group of diffeomorphisms which leave $\Omega$ invariant. In case $\Omega$ corresponds to a Hodge metric, then $(B, \Omega)$ is said to be a *homogeneous Hodge* manifold.

A *strongly symplectic homogeneous space* is a homogeneous symplectic manifold such that the vector fields generated by the elements of $\mathfrak{G}$ are globally hamiltonian.

A *Hamiltonian* space is a strongly symplectic homogeneous space for which there exists a Lie algebra homomorphism, $\varphi$, from $\mathfrak{G}$ into the Lie algebra of differentiable functions on $B$ (under Poisson bracket), such that for every $X \in \mathfrak{G}$ one has:

$$i_{X_B}\Omega = d[\varphi(X)].$$

Here $X_B$ denotes the vector field generated on $B$ by $X$.

If $\theta$ is a coadjoint orbit, then $(\theta, \Omega_\theta)$ is a Hamiltonian space. The Lie algebra homomorphism $\varphi_\theta : \mathfrak{G} \to C^\infty(\theta)$, is given by $\varphi_\theta(X) = X|_\theta$.

Let $(B, \Omega)$ be a Hamiltonian space with corresponding homomorphism $\varphi$. Then the map:

$$\tau : B \to \mathfrak{G}^*$$

given by:

$$\langle \tau(m), X \rangle = (\varphi(X))(m),$$

is such that $\tau(B)$ is an orbit, $\theta$, of the coadjoint action. If $\tau$ is considered as a map onto $\theta$, then $\tau$ is a covering map and $\tau^* \Omega_\theta = \Omega$.

A homogeneous symplectic manifold of a semisimple Lie group is a Hamiltonian space.

Finally, if $r$ is the natural coordinate function on $\mathbb{R}$, the Lie algebra of the Lie group $\mathbb{R}$ is $\{s=s\,d/dr : s\in\mathbb{R}\}$ and can be identified with $\mathbb{R}$ by the isomorphism $\varphi(sd/dr)=s$. The map $E:\mathbb{R}\to S^1$ given by $E(t)=e^{2\pi it}$ is a Lie group homomorphism. If we put $A^s=dE(s)$, we have

$$\operatorname{Exp} t\,A^s=E\,(st)=e^{2\pi ist}.$$

In particular $dE$ is a isomorphism of Lie algebras. $\mathbb{R}$ is canonically identified with the Lie algebra of $S^1$ *via* $dE$.

If $\gamma:\mathbb{R}\to\mathbb{R}$ is a curve in the Lie group $\mathbb{R}$, then the value of $A^s$ at $E\circ\gamma(0)$ is the tangent vector to $E\circ\gamma$ at 0 if and only if $s=d\gamma/dr(0)$.

### 3. Simply connected homogeneous Hodge manifolds

THEOREM 1. — *Let $(B,\Omega)$ be a (connected) symplectic manifold. Then the following four conditions are equivalent:*

(1) $(B,\Omega)$ *is a compact strongly symplectic homogeneous space;*

(2) $(B,\Omega)$ *is a simply connected compact homogeneous symplectic manifold;*

(3) $(B,\Omega)$ *is isomorphic to a pair $(\theta,\Omega_\theta)$, where $\theta$ is a coadjoint orbit of a compact semisimple Lie group;*

(4) $(B,\Omega)$ *is a compact homogeneous symplectic manifold with finite fundamental group.*

If any of the above conditions holds, then all Betti numbers of odd order of $B$, $b_{2r+1}$, are zero and there exists on $B$ a Hodge metric whose corresponding symplectic form is invariant for the given action.

*Proof:*

*1 implies* 2. — Let $(B,\Omega)$ be a compact strongly symplectic homogeneous space of the Lie group $G$, whose Lie algebra will be denoted by $\mathfrak{G}$. We can suppose the group acts effectively (eventually one needs to pass to a quotient of $G$ by a normal subgroup).

If $G_0$ is the identity component of $G$ then $G_0$ also acts effectively and transitively on $B$.

Following LICHNEROWICZ [6] we can define on $\mathfrak{G}$ a adjoint-invariant scalar product, so that the riemannian metric obtained on $G_0$ by left traslation of this scalar product is also right invariant. Then $\mathfrak{G}$ is a reductive Lie algebra. LICHNEROWICZ also proves that (infinitesimal) transitivity implies the center of $\mathfrak{G}$ is reduced to zero, so that $\mathfrak{G}$ is semisimple.

The Ricci curvature of the riemannian metric on $G_0$ turns out to be positive (*cf.* MILNOR [7]) so that $G_0$ is compact.

Then $(B, \Omega)$ is a homogeneous symplectic space of a connected compact semisimple Lie group, $G_0$, acting effectively on $B$.

Theorem 1 in BOREL [2] then says that $B$ is simply connected so that 2 holds. On the other hand, the same theorem asserts that the isotropy subgroup of any point on $B$ is connected and the centralizer of a torus of $G_0$. Theorem 2 of BOREL then implies that $B$ is homogeneous Hodge.

2 *implies* 1. – A simply connected homogeneous symplectic space is a strongly symplectic one.

1 *implies* 3. – The proof above shows that $(B, \Omega)$ is a homogeneous symplectic space of a compact connected semisimple Lie group. Then $(B, \Omega)$ is a Hamiltonian space, so that $B$ covers a coadjoint orbit, $\theta$, of the group in such a way that the pull back of $\Omega_\theta$ by the covering map is $\Omega$. As $\theta$ is simply connected the covering map is a diffeomorphism.

3 *implies* 1. – Obvious.

2 *implies* 4. – Obvious.

4 *implies* 1. – Let $(B, \Omega)$ be a compact homogeneous symplectic space of the Lie group $G$, the fundamental group of $B$ being finite.

We denote by $K$ a maximal compact subgroup of $G_0$ and by $H$ the maximal semisimple subgroup of $K$.

Since the action of $G_0$ is transitive, so is that of $K$ (MONTGOMERY [8]) and that of $H$ (WANG [10]).

$(B, \Omega)$ being a homogeneous symplectic space of a semisimple Lie group, it is a strongly symplectic homogeneous space.

Our assertion on the Betti numbers can be proved as follows. We have seen that 1 implies that $B$ is diffeomorphic to a homogeneous space of the form $G_0/C$, where $G_0$ and $C$ are compact connected and $C$ is the centralizer of a torus $T$ in $G_0$. $T$ must be contained in his own centralizer, $C$; and if $T'$ is a maximal torus in $G_0$ which contains $T$, then $T'$ is also contained in $C$, so $G_0$ and $C$ have the same rank. It is a standard fact on the cohomology of homogeneous spaces that in these conditions all the Betti numbers of odd order are zero.

*Remark* 1. – There are compact homogeneous symplectic spaces with all Betti numbers of order lower than the dimension different from zero. The tori $T^{2n}$ are examples. No one of these can be strongly symplectic.

*Remark* 2. – If $(B, \Omega)$ is a compact (connected) homogeneous symplectic space such that $B$ admits a Kählerian structure and $b_{2r+1}(B)=0$ for some $2r+1<\dim(B)$, then any one of the conditions in Theorem 1 holds for $(B, \Omega)$. In fact we shall prove that $b_1(B)=0$ and so 1 holds. Let $\dim(B)=2n$ and $b_{2r+1}(B)=0$; then we know that there exists $2s+1\leqslant n$ such that $b_{2s+1}(B)=0$ (Poincaré duality). If $2s+1=1$, we have finished; if not, the existence of a Kählerian structure implies

$$0\leqslant b_{2s-1}(B)\leqslant b_{2s+1}(B)=0.$$

Hence, $b_{2s-1}(B)=0$ and we can continue this process to get $b_1(B)=0$.

## 4. Homogeneous compact contact manifolds

In this section $(B, \Omega)$ will be a compact simply connected homogeneous symplectic manifold with $[\Omega]\in H^2(B, \mathbb{R})$ integral, and $(M, \omega)$ will be a compact homogeneous contact manifold.

Definition. – We say that the pairs $(B, \Omega)$ and $(M, \omega)$ are related if there exist a positive constant, $T$, a map $\pi: M\to B$ and an action of $S^1$ on $M$ such that:

(*a*) $M(B, S^1)$ is a principal fibre bundle with projection $\pi$;

(*b*) $\omega/T$ is a connection form on $M$ such that $d(\omega/T)=\pi^*\Omega$.

(Here we identify $\mathbb{R}$ with the Lie algebra of $S^1$ by the canonical isomorphism). In particular, $\omega$ is then invariant by the action of $S^1$.

Definition. – We say that $(M, \omega)$ and $(M', \omega')$ are equivalent if there exists a diffeomorphism $f: M\to M'$ and a positive constant $k$ such that $f^*\omega'=k\,\omega$.

The set of equivalence classes will be denoted by $\mathscr{E}$.

Definition. – We say that $(B, \Omega)$ and $(B', \Omega')$ are equivalent if there exists a diffeomorphisms, $\varphi$, from $B$ onto $B'$ such that $\varphi^*\Omega'=\Omega$.

The set of equivalence classes will be denoted by $\mathscr{H}$.

Let us denote by $[M, \omega]$ the equivalence class of $(M, \omega)$ in $\mathscr{E}$ and by $[B, \Omega]$ the equivalence class of $(B, \Omega)$ in $\mathscr{H}$.

Now we define $\Phi$: $\mathscr{H}\to\mathscr{E}$ by $\Phi([B, \Omega])=[M, \omega]$ where $(M, \omega)$ is any homogeneous contact compact manifold related to $(B, \omega)$.

Theorem 2. – *$\Phi$ is a bijection from $\mathscr{H}$ to $\mathscr{E}$.*

To show that $\Phi$ is a well defined one-to-one and onto map, we give the following lemmas.

LEMMA 1. — *If* $(M, \omega)$ *is related to* $(B, \Omega)$, $(M', \omega')$ *is related to* $(B', \Omega')$ *and* $(M, \omega)$ *is equivalent to* $(M', \omega')$, *then* $(B, \Omega)$ *is equivalent to* $(B', \Omega')$.

*Proof.* — For each $s \in R$, $s \neq 0$, let $s^*$ (resp. $s^{**}$) be the vector field on $M$ (resp. $M'$) generated by $s$ as an element of the Lie algebra of $S^1$. Let $Z$ (resp. $Z'$) be the vector field associated to $\omega$ (resp. $\omega'$) and $T$ (resp. $T'$) a positive real number such that $\omega/T$ (resp. $\omega'/T'$) is a connection from in $M(B, S^1)$ (resp. $M'(B', S^1)$).

Since $i_{s^*}\omega = Ti_{s^*}(\omega/T) = T.s$ and $i_{s^*}d\omega = i_{s^*}(T\pi^*\Omega) = 0$, we have $s^* = (sT)Z$ and $s^{**} = (sT')Z'$, and so $Z = (1/T)^*$ and $Z' = (1/T')^*$.

Thus, if we denote by $\{Z_t\}$ the flow of $Z$, then $Z_t$ is the diffeomorphism associated to $e^{2\pi it/T}$ by the action of $S^1$. Analogously, if $\{Z'_t\}$ is the flow of $Z'$, $Z'_t$ is the diffeomorphism associated to $e^{2\pi it/T'}$ by the action of $S^1$. The fibres of $\pi$ (resp. $\pi'$) are images of integral curves of $Z$ (resp. $Z'$).

Let $F: M \to M'$ be a diffeomorphism such that $F^*\omega' = k\omega$.

The relations:

$$k = i_Z(k\omega) = i_Z F^*\omega' = F^* i_{F_*Z}\omega'$$

and:

$$0 = i_Z d(k\omega) = i_Z F^* d\omega' = F^* i_{F_*Z} d\omega',$$

imply $F_* Z = kZ'$.

Hence, $F$ is fibre preserving and it induces a unique bijection $f: B \to B'$ such that $f \circ \pi = \pi' \circ F$. $f$ is a diffeomorphism because $\pi$ and $\pi'$ admit local cross sections and $F$ is a diffeomorphism.

On the other hand, $F_* Z$ has a periodic flow with $T$ as periode and $kZ'$ has a periodic flow with $T'/k$ as periode. Then $T = T'/k$ and one has:

$$\pi^*(f^*\Omega') = F^*\pi'^*\Omega' = F^*((1/T')d\omega') = (1/T')d(k\omega) = (1/T)d\omega = \pi^*\Omega.$$

As $\pi^*$ is injective, $f^*\Omega' = \Omega$ and Lemma 1 is proved.

LEMMA 2. — *If* $(M, \omega)$ *is related to* $(B, \Omega)$, $(M', \omega')$ *is related to* $(B', \Omega')$ *and* $(B, \Omega)$ *is equivalent to* $(B', \Omega')$ *then* $(M, \omega)$ *is equivalent to* $(M', \omega')$.

*Proof.* — First we prove that if $(M, \omega)$ is related to $(B, \Omega)$, the first Betti number of $M$, $b_1(M)$, is zero.

The characteristic class of the bundle $M(B, S^1)$ is $[\Omega] \in H^2(B, \mathbb{Z})$ and $[\Omega] \neq 0$ because $B$ is compact and $\Omega$ symplectic. Hence the Gysin sequence becomes:

$$0 \to H^1(B, \mathbb{R}) \xrightarrow{\pi^\#} H^1(M, \mathbb{R}) \to \mathrm{H}^0(B, \mathbb{R}) \xrightarrow{L} H^2(B, \mathbb{R}) \ldots$$

where $L$ is left exterior multiplication by $\Omega$. Thus $L$ is an isomorphism of $H^0(B, R)$ into $H^2(B, R)$ and therefore $\pi^{\#}$ is an onto isomorphism.

As $\Pi_1(B)=0$, the first Betti number of $B$, $b_1(B)$, is zero. Thus $b_1(M)=b_1(B)=0$, as stated.

Now let $f$ be a diffeomorphism from $B$ onto $B'$ such that $f^*\Omega'=\Omega$.

Then, for the given action of $S^1$ on $M'$, $M'(B, S^1)$ is a principal circle bundle with bundle projection $\pi''=f^{-1}\circ\pi'$. The characteristic class of this bundle is $[\Omega]$ because $\omega'/T'$ is a connection form and

$$d(\omega'/T')=\pi'^*\Omega'=\pi'^*(f^{-1})^*\Omega=\pi''^*\Omega.$$

On the other hand, as $H_1(B, \mathbb{Z})=0$, the mapping which sends a equivalence class of principal circle bundles into the characteristic class of any one of its representatives is an isomorphism onto the image of the canonical map from $H^2(B, \mathbb{Z})$ to $H^2(B, \mathbb{R})$ (*cf.* Kobayashi [4]).

Since the bundles $M(B, S^1)$ and $M'(B, S^1)$ have the same image, $[\Omega]$, by this isomorphism they are equivalent as principal bundles.

Hence there exists a diffeomorphism $F: M\to M'$ equivariant for the actions of $S^1$ on $M$ and $M'$ and such that $\pi''\circ F=\pi$. In particular $F_* s^*=s^{**}$, $\forall s\in\mathbb{R}$, where $s^*$ (resp. $s^{**}$) is the vector field on $M$ (resp. $M'$) associated by the action to $s$ as $a$ element of the Lie algebra of $S^1$.

Now, $d((\omega/T)-F^*(\omega'/T'))=\pi^*\Omega-F^*\pi''^*\Omega=0$ and $b_1(M)=0$, and so there exists a differentiable function, $h$, on $M$ such that $(\omega/T)-F^*(\omega'/T')=dh$.

We claim that $h$ is constant on the fibres. In fact with the same notation as above we have

$s^*h=i_{s^*}dh=i_{s^*}((\omega/T)-F^*(\omega'/T'))=i_{s^*}(\omega/T)-F^*i_{s^{**}}(\omega'/T')=s-F^*s=0$ and $h$ is constant over the integral curves of $s$. But if $s\neq 0$, any fibre is the image of a integral curve of $s$ and so we are done.

Now consider the map $\eta: M\to M$ defined by $\eta(x)=x.e^{2\pi i h(x)}$ where the product means the action of $S^1$ on $M$. $\eta$ is a fibre-preserving diffeomorphism and $\eta^*dh=d(h\circ\eta)=dh$ because $h$ is constant over the fibres.

Now we prove that $\eta^*(\omega/T)=(\omega/T)+dh$.

Let $p\in M$, $v\in T_pM$ and $\sigma$ be a curve on $M$ such that $\sigma(0)=p$, $\dot\sigma_0=v$. Let us denote $a(t)=e^{2\pi i h(\sigma(t))}$.

Then:

$$(\eta^*(\omega/T))_p(v)=(\omega/T)_{\eta(p)}\cdot\dot{\overline{(\eta\circ\sigma)}}_0=(\omega/T)_{\eta(p)}[(R_{a(0)})_*\cdot v+(\varphi^{\sigma(0)})_*\dot a_0],$$

where $R_{a(0)}$, is the diffeomorphism associated by the action to $a(0)\in S^1$ and $\varphi^{\sigma(0)}: g\in S^1 \to \sigma(0).g\in M$.

But $\dot{a}_0=(A^s)_{a(0)}$ where $s=(d(h\circ\sigma)/dt)_{t=0}=(dh)_p.v$ (*cf.* section 2).

Then:

$$(\eta^*(\omega/T))_p(v)=(\omega/T)_p v+(\omega/T)_p[(\varphi^{\sigma(0)})_*(A^s)_{a(0)}]$$
$$=(\omega/T)_p v+(\omega/T)_{\eta(p)}(s^*_{\eta(p)})=(\omega/T)_p v+s=((\omega/T)+dh)_p v,$$

as desired.

Then we have:

$$(F\circ\eta)^*\omega'=\eta^* T'((\omega/T)-dh)=(T'/T)\omega$$

and Lemma 2 is proved.

LEMMA 3. — *For any pair $(B, \Omega)$ there exists a pair $(M, \omega)$ related to it.*

*Proof.* — The theorems of KOBAYASHI [4] give the existence of a circle bundle $M(B, S^1)$ with connection $\omega$, projection $\pi$ and structure equation $d\omega=\pi^*\Omega$.

Since $\omega(1^*)\neq 0$ and $(d\omega)^n$ cannot vanish on the element determined by the horizontal vectors, the contact condition $\omega\wedge(d\omega)^n=0$ is satisfied.

As $B$ and $S^1$ are compacts, $M$ is also compact.

It remains to show that there is a Lie group acting transitively on $M$ by $\omega$-preserving diffeomorphisms.

Let $G$ be a semisimple compact connected Lie group acting transitively on $B$ by $\Omega$-preserving diffeomorphisms (such a $G$ exists by the result in section 3). Let $\mathfrak{G}$ be the Lie algebra of $G$. Then $(B, \Omega)$ is a Hamiltonian space for this action, so that there exists a homomorphism of Lie algebras $\varphi: \mathfrak{G}\to C^\infty(B)$ such that $i_{X_B}\Omega=d(\varphi(X))$ for each $X\in\mathfrak{G}$.

Let $D(M)$ be the Lie algebra of vector fields on $M$ and let $\gamma: \mathfrak{G}\oplus\mathbb{R}\to D(M)$ be the map defined by:

$$\gamma(X+t)=\overline{X}_B-(\varphi(X)\circ\pi+t)Z,$$

where $\overline{X}_B$ is the horizontal lift of $X_B$, and $Z$ is the vector field associated to $\omega$. We claim that $\gamma$ is a homomorphism of Lie Algebras.

To prove this, note first the following relations:

$$Z(f\circ\pi)=0, \qquad \forall f\in C^\infty(B),$$
$$[\overline{X}_B, Z]=0, \qquad \forall X\in\mathfrak{G}.$$

The first one follows from the relation $Z=1^*$, which gives that $Z$ is vertical. The second one is direct consequence of the relations $i_{[\bar{X}_B, Z]}\omega=0$, $i_{[\bar{X}_B, Z]}d\omega=0$, and the fact proved above that $\omega$ is a contact form.

Now we show that $[\gamma(X+t), \gamma(Y+s)]=\gamma([X, Y]+0)$:

$$\begin{aligned}[\gamma(X+t), \gamma(Y+s)] &= [\bar{X}_B, \bar{Y}_B]-[\bar{X}_B, (\varphi(Y)\circ\pi+s)Z] \\ &\quad -[(\varphi(X)\circ\pi+t)Z, \bar{Y}_B]+[(\varphi(X)\circ\pi+t)Z, (\varphi(Y)\circ\pi+s)Z] \\ &= \overline{[X, Y]}_B+\omega([\bar{X}_B, \bar{Y}_B])Z-(\bar{X}_B(\varphi(Y)\circ\pi))Z+(\bar{Y}_B(\varphi(X)\circ\pi))Z \\ &= \overline{[X, Y]}_B-(\{\varphi(X), \varphi(Y)\}\circ\pi)Z=\gamma([X, Y]+0).\end{aligned}$$

Then, as $\gamma$ is linear, it is a homomorphism of Lie algebras and the claim is established.

Thus $\gamma(\mathfrak{G}\oplus\mathbb{R})$ is a finite dimensional Lie algebra of vector fields on the compact manifold $M$, which we shall denote by $\mathscr{L}$. By a Theorem of Palais (*cf.* [9], Thm. III, Chapt. IV), there is a connected Lie group $H$ and a smooth left action of $H$ on $M$ such that the Lie algebra of the generated vector fields coincides with $\mathscr{L}$. $H$ can be supposed to be a connected, simply connected Lie group with $\mathfrak{G}\oplus\mathbb{R}$ as Lie algebra.

We shall show that the action of $H$ on $M$ is transitive. For this it is enough to show the following "local transitivity" condition: for each $m\in M$ and $v\in T_m M$, there exists $A\in\mathscr{L}$ such that $A_m=v$. But taking horizontal and vertical parts one has $v=h(v)+\omega_m(v)Z_m$. Then if we take a $X\in\mathfrak{G}$ such that $h(v)=(\bar{X}_B)_m$ (obviously such $X$ exists), we have:

$$v=(\bar{X}_B)_m+\omega_m(v).Z_m=(\gamma(X-\varphi(X)\circ\pi(m)-\omega_m(v)))_m.$$

Finally, for each $X\in\mathfrak{G}$ and $t\in\mathbb{R}$ we have:

$$\begin{aligned}L_{\gamma(X+t)}\omega &= L_{\bar{X}_B}\omega-L_{(\varphi(X)\circ\pi)Z}\omega \\ &= i_{\bar{X}_B}d\omega-d(\varphi(X)\circ\pi)=\pi^*(i_{X_B}\Omega-d(\varphi(X)))=0.\end{aligned}$$

This shows that $\omega$ is invariant under the action of $H$ on $M$ and completes the proof of Lemma 3.

*Proof of the Theorem.* – The Lemmas above show that $\Phi$ is well defined and injective. It remains to show that $\Phi$ is an onto map.

Let $(M, \omega)$ be a homogeneous contact compact manifold for a Lie group $G$. Then $\omega$ is a regular contact form and there exist a positive real constant $k$, a symplectic manifold $(B, \Omega)$ and a map $\pi$ from $M$ onto $B$ such that $M(B, S^1)$ is a principal circle bundle with $\pi$ as bundle projection, $k\,\omega$ is a connection form on this bundle and $\Omega$ is the curvature of $k\,\omega$.

The cohomology class of $\Omega$ is the real characteristic class of the bundle (in particular it is integral) and the fibres of the bundle are the integral curves of the vector field $Z$ associated to $\omega$.

$Z$ is invariant by the transitive action of $G$ on $M$ because $\omega$ is. So, this action induces a transitive differentiable action of $G$ on $B$ which preserves $\Omega$ in such a way that $\pi$ is equivariant for these actions. Then $(B, \Omega)$ is a compact homogeneous symplectic manifold.

Now, let $\mathfrak{G}$ be the Lie algebra of $G$ and $X_M$ (resp. $X_B$) the vector field on $M$ (resp. $B$) generated by $X \in \mathfrak{G}$. The function $i_{X_M}\omega$ is constant over the fibres because the relation $L_{X_M}\omega=0$ implies $Z(i_{X_M}\omega)=i_Z\,di_{X_M}\omega=-i_Z\,i_{X_M}d\omega=0$.

Then, there is a differentiable function $f_X$ on $B$ such that $\pi^* f_X=i_{X_M}\omega$.

Moreover $\pi^* i_{X_B}\Omega=i_{X_M}d(k\,\omega)=-di_{X_M}(k\,\omega)=-\pi^* d(kf_X)$ because $X_B$ and $X_M$ are $\pi$-related and $L_{X_M}\omega=0$.

Since $\pi^*$ is injective, we get $i_{X_B}\Omega=d(-kf_X)$ and thus the compact homogeneous symplectic manifold $(B, \Omega)$ is a strongly symplectic homogeneous space. From section 3, $B$ is simply connected. This proves that $\Phi$ is onto.

COROLLARY 1. — *The fundamental group of a homogeneous contact compact manifold is abelian finite.*

*Proof.* — Let $(M, \omega)$ be a homogeneous contact compact manifold and let $(B, \Omega)$ be a simply connected homogeneous symplectic manifold related to $(M, \omega)$. The exact homotopy sequence of the principal bundle $M(B, S^1)$ implies the existence of a onto homomorphism, $f$ from $\mathbb{Z}=\Pi_1(S^1)$ onto $\Pi_1(M)$. Then $\Pi_1(M)$ is abelian so that, by Hurewicz isomorphism, $\Pi_1(M)\simeq H_1(M, Z)$. If the kernel of $f$ were trivial we would have $H_1(M, \mathbb{Z})\approx\mathbb{Z}$ in contradiction with $H^1(M, \mathbb{R})=0$ (*see* the proof of Lemma 2). Then the kernel of $f$ is not trivial, so that $\Pi_1(M)$ is finite.

The next result asserts in particular that any homogeneous contact compact manifold is a homogeneous space of a compact semisimple Lie group.

COROLLARY 2. — *Let $(M, \omega)$ be related to $(B, \Omega)$ with $\pi: M \to B$ as bundle projection. Then there exists a compact semisimple Lie group which acts transitivelly both on $M$ by $\omega$-preserving diffeomorphism and on $B$ by*

*$\Omega$-preserving diffeomorphisms, in such a way that $\pi$ is equivariant for these actions.*

*Proof.* — By Theorem 1 there exists a compact semisimple Lie group, $G$, which acts transitively on $B$ by $\Omega$-preserving diffeomorphisms. Let $\tilde{G}$ be the universal covering group of $G$ and $\mathfrak{G}$ its Lie algebra.

Then $\tilde{G}\times\mathbb{R}$ is the universal covering group of any connected Lie group whose Lie algebra is $\mathfrak{G}\oplus\mathbb{R}$.

On the other hand $\tilde{G}$, as any covering group of a compact semisimple Lie group, is compact semisimple.

MONTGOMERY [8], has proved that any maximal compact Lie subgroup of a Lie group which acts transitively on a connected compact manifold with finite fundamental group, is also transitive for the induced action.

In our case, the proof of Lemma 3 implies that $\tilde{G}\times\mathbb{R}$ acts transitively on $M$ by $\omega$-preserving diffeomorphisms in such a way that the vector field generated by $X+t\in\mathfrak{G}\oplus\mathbb{R}$ is $\gamma(X+t)$. $\tilde{G}$ is maximal compact in $\tilde{G}\times\mathbb{R}$, so that it acts transitively on $M$ in such a way that the vector field on $M$ generated by $X\in\mathfrak{G}$ is $\gamma(X)$.

If we consider the action of $\tilde{G}$ on $B$ canonically associated to that of $G$, we need only to prove that $\pi$ is equivariant for these actions of $\tilde{G}$.

But this is a consequence of the fact that $\pi$ maps the vector field on $M$ generated by $X\in\mathfrak{G}$, $\gamma(X)=\overline{X}_B-(\varphi(X)\circ\pi)Z$, into the vector field on $B$ generated by $X$, $X_B$.

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