NOTES

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An Interesting Property of the Evolute

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1. INTRODUCTION. The starting point of this note is the following inequality: if $C = \partial K$ is the boundary of a compact, convex set K of area F in $\mathbb{R}^2$, then

$$\int_C \frac{1}{k} ds \geqslant 2F, \tag{1}$$

where $k = k(s) > 0$ is the curvature function of C and ds signifies arclength measure on C. Equality holds if and only if C is a circle.

In [**1**], two proofs of this result are given: the first uses a polygonal approximation of the curve C; the second is based on ideas of Osserman [**4**]. In this note we give a very short new proof of (1), which has the advantage of providing a geometric interpretation of the difference $2F - \int_C k^{-1} ds$. To be precise, we prove that

$$\int_C \frac{1}{k} ds = 2(F - F_e),$$

where $F_e \leq 0$ is the (algebraic) area of the domain bounded by the evolute of C.

Inequality (1) is the two-dimensional analogue of Heintze and Karcher's inequality:

$$\int_S \frac{1}{H} dA \geqslant 3V,$$

where $H > 0$ is the mean curvature of a compact embedded surface S in $\mathbb{R}^3$ bounding a domain D of volume V. Equality holds if and only if S is a standard sphere [**5**]. This raises the obvious question: Is there a geometric interpretation of the difference $3V - \int_C H^{-1} dA$?

2. ENVELOPE OF A FAMILY OF LINES. An *evolute* is the locus of centers of curvature of a plane curve. The evolute E of a smooth curve C has singularities (called *cusps*; see Figure 1). The cusps of the evolute correspond to the points of C where the curvature takes extreme values. It can be seen that the evolute of C is the *envelope* of all the normals to this curve (i.e., the tangents to the evolute are the normals to C).

Recall, following [**6**], that a straight line G in the plane is determined by the angle ϕ that the direction perpendicular to G makes with the positive x-axis and the distance $p = p(\phi)$ of G from the origin. The equation of G then takes the form

$$x \cos\phi + y \sin\phi - p = 0. \tag{2}$$

Equation (2), when $p = p(\phi)$ varies with ϕ, is the equation of a family of lines. If we assume that the 2π-periodic function $p(\phi)$ is differentiable, the envelope of the family is obtained from (2) and the derivative of its left-hand side, as follows:

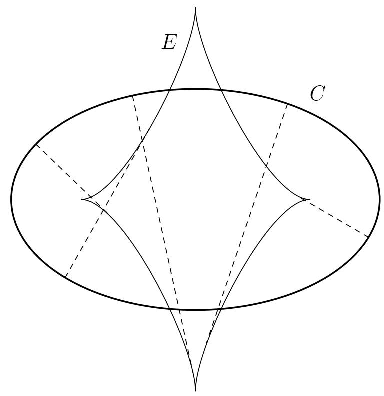

Figure 1.

$$-x \sin\phi + y\cos\phi - p' = 0, \quad p' = dp/d\phi. \tag{3}$$

From (2) and (3) we arrive at a parametric representation of the envelope of the lines (2):

$$x = p\cos\phi - p'\sin\phi, \qquad y = p\sin\phi + p'\cos\phi.$$

If the envelope is the boundary ∂K of a convex set K and the origin is an interior point of K, then $p(\phi)$ is called the *support function* of K (or the support function of the convex curve ∂K).

Since $dx = -(p + p'')\sin\phi\, d\phi$ and $dy = (p + p'')\cos\phi\, d\phi$ (we here assume that the function p is of class C^2), arclength measure on ∂K is given by

$$ds = \sqrt{dx^2 + dy^2} = |p + p''|\, d\phi \tag{4}$$

and the radius of curvature ρ by

$$\rho = \frac{ds}{d\phi} = |p + p''|. \tag{5}$$

It is well known that a necessary and sufficient condition for a periodic function p to be the support function of a convex set K is that $p + p'' > 0$. Finally, it follows from (4) that the length of a closed convex curve that has support function p of class C^2 is given by

$$L = \int_0^{2\pi} p\, d\phi.$$

The area of the convex set K is expressed in terms of the support function by

$$F = \frac{1}{2}\int_{\partial K} p ds = \frac{1}{2}\int_0^{2\pi} p(p + p'')\, d\phi \tag{6}$$

or, equivalently, by

$$dF = \frac{1}{2}\, p\, ds = \frac{1}{2}\, p\, \rho\, d\phi.$$

We also use the following well-known result (Wirtinger's inequality):

Lemma 1. *If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a C^2-function of period 2π, then*

$$\int_0^{2\pi} |f'|^2 d\phi \leq \int_0^{2\pi} |f''|^2 d\phi.$$

Equality holds if and only if $f(\phi) = a\cos\phi + b\sin\phi + c$ for constants a, b, and c.

See, for instance, [**7**, p. 81], or [**2**, p. 52], for an elementary geometric proof.

Finally we recall that if $h : \mathbb{R} \to \mathbb{R}$ is a C^1-function of period 2π, the *hedgehog* γ_h corresponding to h is parametrized by (see [**3**])

$$\gamma_h(\phi) = (h(\phi)\cos\phi - h'(\phi)\sin\phi, h(\phi)\sin\phi + h'(\phi)\cos\phi)\,.$$

Equivalently, the hedgehog is the envelope of the family of lines given by

$$x\cos\phi + y\sin\phi = h(\phi).$$

For instance, if $h(\phi) = \cos(25\phi)$, this envelope actually looks like a hedgehog (Figure 2).

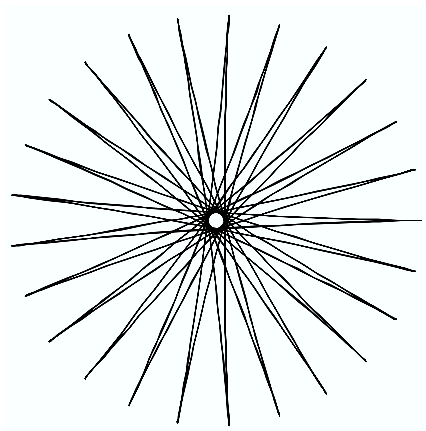

Figure 2.

The (algebraic) area F_h of the hedgehog corresponding to h is given by

$$F_h = \frac{1}{2}\int_0^{2\pi} h(h + h'')d\phi = \frac{1}{2}\int_0^{2\pi} (h^2 - h'^2)d\phi.$$

This quantity can be positive, negative, or zero. If h is the support function of a convex set, then F_h is the Euclidean area of this convex set (see (6)). In particular, this algebraic area is positive. We shall see that $F_h \leq 0$ whenever h is the support function of an evolute.

3. THE EVOLUTE OF A CONVEX CURVE. Let C be a closed convex curve in $\mathbb{R}^2$. We assume that C is the boundary of a convex set K and that the origin is in the interior of K. We assume further that the support function $p = p(\phi)$ of K is a C^3-function (of period 2π). A parametrization $\gamma_p(\phi) = (x(\phi), y(\phi))$ of C is then furnished by

$$x(\phi) = p(\phi)\cos\phi - p'(\phi)\sin\phi, \quad y(\phi) = p(\phi)\sin\phi + p'(\phi)\cos\phi.$$

Relative to this parametrization, the evolute $\tilde{\gamma}$ of γ_p is expressed as follows:

$$\tilde{\gamma}(\phi) = \gamma_p(\phi) + \rho(\phi)\frac{(-y'(\phi), x'(\phi))}{((x'(\phi))^2 + (y'(\phi))^2)^{1/2}},$$

where $\rho(\phi)$ is the radius of curvature of $\gamma_p(\phi)$. From (4) and (5) we have

$$\tilde{\gamma}(\phi) = \gamma_p(\phi) + (-y'(\phi), x'(\phi)).$$

Equivalently, $\tilde{\gamma}(\phi) = (\tilde{x}(\phi), \tilde{y}(\phi))$, with

$$\begin{aligned}\tilde{x}(\phi) &= x(\phi) - y'(\phi) = -p'(\phi)\sin\phi - p''(\phi)\cos\phi,\\ \tilde{y}(\phi) &= y(\phi) + x'(\phi) = p'(\phi)\cos\phi - p''(\phi)\sin\phi.\end{aligned}$$

We note the simplicity of this parametrization of the evolute.

Let $H(\phi) = -p'(\phi + \pi/2)$. Then H is a C^2-function of period 2π, and the hedgehog corresponding to it has the parametrization

$$\begin{aligned}\gamma_H(\phi) &= (H(\phi)\cos\phi - H'(\phi)\sin\phi,\, H(\phi)\sin\phi + H'(\phi)\cos\phi)\\ &= \left(-p'\left(\phi + \frac{\pi}{2}\right)\cos\phi + p''\left(\phi + \frac{\pi}{2}\right)\sin\phi,\right.\\ &\qquad \left.-p'\left(\phi + \frac{\pi}{2}\right)\sin\phi - p''\left(\phi + \frac{\pi}{2}\right)\cos\phi\right).\end{aligned}$$

In particular,

$$\begin{aligned}\gamma_H\left(\phi - \frac{\pi}{2}\right) &= (-p'(\phi)\sin\phi - p''(\phi)\cos\phi,\, p'(\phi)\cos\phi - p''(\phi)\sin\phi)\\ &= (\tilde{x}(\phi), \tilde{y}(\phi)) = \tilde{\gamma}(\phi).\end{aligned}$$

We thus see that the evolute $\tilde{\gamma}$ of γ_p is the hedgehog γ_H.

We conclude that the (algebraic) area F_e of the evolute of the convex curve supported by p is equal to the (algebraic) area F_H of the hedgehog corresponding to $H = H(\phi) = -p'(\phi + \pi/2)$. Stated differently,

$$F_e = \frac{1}{2}\int_0^{2\pi} H(H + H'')d\phi = \frac{1}{2}\int_0^{2\pi} p'(p' + p''')d\phi.$$

We now have the tools to establish the following result:

Theorem 1. *The integral with respect to arclength of the radius of curvature of a plane convex curve is twice the area of the domain it bounds minus the* (*algebraic*) *area of the domain bounded by its evolute:*

$$\int_C \rho\, ds = 2(F - F_e).$$

Proof. The area F_e enclosed by the evolute satisfies

$$F_e = \frac{1}{2}\int_0^{2\pi} p'(p' + p''')d\phi.$$

Integration by parts yields

$$\int_0^{2\pi} (p'' p)\, d\phi = -\int_0^{2\pi} (p')^2\, d\phi$$

and

$$\int_0^{2\pi} (p' p''')\, d\phi = -\int_0^{2\pi} (p'')^2\, d\phi.$$

Hence

$$\begin{aligned} F_e &= \frac{1}{2}\int_0^{2\pi} ((p')^2 - (p'')^2)d\phi \qquad (7) \\ &= -\frac{1}{2}\int_0^{2\pi} p''(p + p'')d\phi = -\frac{1}{2}\int_C p'' ds, \end{aligned}$$

since $ds = (p + p'')d\phi$. We conclude that

$$2F_e = -\int_C p'' ds = -\int_C (\rho - p)ds = 2F - \int_C \rho\, ds$$

or

$$\int_C \rho\, ds = 2(F - F_e).$$ ■

Corollary 1. *If the boundary $C = \partial K$ of a convex set K in the plane is a C^2-curve, then*

$$\int_C \rho\, ds \geq 2F,$$

where ds is arclength measure on C, $\rho = \rho(s)$ is the radius of curvature of C, and F is the area of K. Equality holds if and only if C is a circle.

Proof. The corollary is an immediate consequence of Theorem 1 and the fact that $F_e \leq 0$. That $F_e \leq 0$ follows from Lemma 1 and formula (7).

If equality holds, then $F_e = 0$ and we have equality in Wirtinger's formula. That is, $p(\phi) = a\cos\phi + b\sin\phi + c$. But this implies that C is a circle of center (a, b) and radius $|c|$. ■

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Rotation in a Normed Plane

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1. INTRODUCTION. What would it be like to live in a space with a non-Euclidean norm, where length depends on direction? Could you turn around? We'll show that in general the answer is no: you tend to get stuck. In a non-Euclidean normed plane, although a triangle can be fully rotated (all the way around), a typical rhombus with diagonal struts cannot. There are some exceptions.

Our main theorem (2) gives the rhombus results. Corollary 1 treats an isosceles triangle plus median, and Corollary 2 treats a right triangle plus median.

Our results strengthen a theorem of Day [**2**, Theorem 2.1, p. 321], which says that a norm is linearly equivalent to the Euclidean norm if *every* rhombus with diagonals can be fully rotated (although he did not use the language of rotation). Our results follow easily from results of Nordlander [**4**] and Alonso and Benítez [**1**].

2. DEFINITIONS AND PREVIOUS RESULTS. A *norm* $\|\cdot\|$ on $\mathbf{R}^2$ is a centrally symmetric positive function (except that $\|0\| = 0$) satisfying the *triangle inequality* ($\|a + b \leq \|a\| + \|b\|$) and *homogeneity* ($\|\Lambda a\| = |\lambda|\,\|a\|$). A *unit norm ball*, meaning a set of the type $\{x : \|x - p\| \leq 1\}$, is convex (by the triangle inequality) and centrally symmetric (by homogeneity). Examples of unit balls appear in Figure 1. The Euclidean norm on $\mathbf{R}^2$ is denoted by $\|\cdot\|_{\mathrm{E}}$. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are *linearly equivalent* if, for some linear map L, $\|x\|_2 = \|Lx\|_1$.

We consider (finite) connected graphs with positive prescribed edge-lengths satisfying a strict generalized triangle inequality: *for any cycle* (*closed path of distinct edges and vertices*), *the edge-lengths* $a_1, a_2, \ldots, a_n$ *satisfy*

$$a_n < a_1 + a_2 + \cdots + a_{n-1}.$$