ON THE NUMBER OF FIXED POINTS FOR A CONTINUOUS MAP OF A FINITE CONNECTED GRAPH

by

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ABSTRACT.

Let F_n be a bouquet of n circles. For an arbitrary continuous map f: $F_n \to F_n$ we shall define a non-negative integer m (f), easily computable in terms of the induced homomorphism $f_*: \pi_1 (F_n) \to \pi_1 (F_n)$. This integer is the best lower bound of the number of fixed points for the homotopy class of f. This result generalizes the well known fact that a continuous map f of the circle into itself has at least m (f) = |1 − degree (f)| fixed points.

§ 1. INTRODUCTION.

Let F_n be a bouquet of n circles, that is, the quotient space of [0,n] obtained by identifying all points of integer coordinates to a single point p.

This paper is related with the following question. If f: $F_n \to F_n$ is a continuous map, what can be said about the number of fixed points of f? Theorem A gives a complete answer for all maps homotopic to f rather than just for the map f itself, as is usual in fixed point theory (see [1]). In fact, we generalize to a bouquet of circles the well known result ([1,p 107]) that a continuous map f of the circle into intself has at least |1 − degree (f)| fixed points.

For an arbitrary continuous map f: $F_n \to F_n$ we shall define a non-negative integer m (f), easily computable in terms of the induced homomorphism

$$f_*: \pi_1 (F_n) \to \pi_1 (F_n) \text{ (see §2).}$$

Let N(f) be the Nielsen number of the map f (see [1] or §4). Our main results are the following.

Theorem A. Let f: $F_n \to F_n$ be a continuous map. Then the following hold.

(i) The map f has at least m (f) fixed points.
(ii) Let g: $F_n \to F_n$ be a continuous map homotopic to f, then m(g) = m(f).
(iii) There exists a continuous map g: $F_n \to F_n$ homotopic to f such that g has exactly m (f) fixed points.
(iv) We have N (f) $\leqslant$ m (f).

Theorem B. Let f: $F_1 \to F_1$ be a continuous map. Then N (f) = m (f).

Example C. There exists a continuous map f: $F_2 \to F_2$ such that N (f) < m (f).

From (iv) of Theorem A and Example C it follows that the number m (f) gives, for a map f: $F_n \to F_n$, more information that the number N (f).

Let X_j denote the quotient space of the interval [j − 1, j] identifying the points j − 1 and j to the point p, and let τ_j: [j − 1, j] $\to X_j$ be the natural map defined by this identification. Then, for any integer n > 1 we have that F_n is homeomorphic to the union of n circles $X_1, X_2, \ldots, X_n$ that intersect at a point p and only at this point.

The fundamental group of F_n based at p, Π (F_n, p), is isomorphic to the free group on n generators. We denote by $x_1, x_2, \ldots, x_n$ the n generators of Π (F_n, p). We can assume that x_j is represented by the loop

$$\tau_j: [j - 1, j] \to X_j, \text{ i.e. } x_j = \{\tau_j\}.$$

Let a ϵ X_j and b ϵ X_j with a $\neq$ b. We write [a, b] to denote the closed arc from a counterclockwise to b. Suppose that f (p) $\neq$ p and f (p) ϵ X_j. Then, let γ: [0, 1] $\to X_j$ be a path such that

$$\gamma(0) = p, \gamma(1) = f(p) \text{ and } \gamma([0, 1]) = [p, f(p)].$$

If f (p) = p, then we define the path γ: [0, 1] $\to F_n$ by γ ([0, 1]) = {p}.

Since F_n is arcwise connected, the fundamental froup Π (F_n, q) is isomorphic to Π (F_n, p), for all q ϵ F_n. We take as generators of Π (F_n, f (p)) the classes $\{\gamma^{-1} \tau_j \gamma\}$ where $1 \leqslant j \leqslant n$.

Let f: $F_n \to F_n$ be a continuous map. Then, we denote by

$$f_*: \Pi(F_n, p) \to \Pi(F_n, f(p))$$

the usual homomorphism induced by f from Π (F_n, p) to Π (F_n, f (p)). This homomorphism is known if we have, for all $1 \leq j \leq n$, the unique expression of f_* (x_j) (whenever f_* (x_j) is not the unit element $e = \{ \gamma^{-1} \gamma \}$) in the form:

$$f_* \{\tau_j\} = \{\gamma^{-1} \tau_{j,1}^{a(j,1)} \ldots \tau_{j,m(j)}^{a(j,m(j))} \gamma\},$$

where $\tau_{j,k} \in \{\tau_1, \tau_2, \ldots, \tau_n\}$ and two consecutive loops $\tau_{j,k}$ are always differents.

Theorem D. Let $f: F_n \to F_n$ be a continuous map. Suppose

$$f_* \{\tau_j\} = \{\gamma^{-1} \tau_j^{a(j)} \gamma\}$$

for all $j = 1, \ldots, n$, where $a(j)$ is an integer. Then $N(f) = m(f)$.

A one-dimensional simplicial complex is called a graph. If the simplicial complex is finite, then the graph is called finite. For a continuous map f: K → K, where K is a finite connected graph, we shall define a non-negative integer M' (f) (see § 6).

Theorem E. Let K be a finite connected graph and let f: K → K be a continuous map. Then the following hold.

(i) The map f has at least M' (f) fixed points.

(ii) Let g: K → K be a continuous map homotopic to f, then M' (g) = M' (f).

Theorem A follows from Theorem 1 and Lemma 2. Theorem 1 is announced in section 2 and proved in section 3. Theorem B, Example C, Theorem D and E are proved in sections 3, 4, 5 and 6, respectively.

§ 2. PRELIMINARY DEFINITIONS AND RESULTS.

We define a retraction r_j of F_n to X_j by sending all of $F_n - X_j$ to p.

We associate to each continuous map f: $F_n \to F_n$ a non-negative integer M (f), defined by

$$M(f) = \sum_{j=1}^{n}{}' M_j(f) ,$$

where

$$M_j(f) = \begin{cases} \sum_{k=1}^{m(j)} M_j^k(f) & \text{if } f_*\{\tau_j\} \neq e, \\ 1 & \text{if } f_*\{\tau_j\} = e \text{ and } r_j f(p) \neq p, \\ 0 & \text{if } f_*\{\tau_j\} = e \text{ and } r_j f(p) = p. \end{cases}$$

The integers M_j^k (f) are defined in Table I. Note that, essentially, M_j (f) is the number of fixed points of f on X_j. Furthermore, we have introduced the convention

$$\sum_{j=1}^{n}{}' M_j(f) = \begin{cases} 1 + \sum_{j=1}^{n} M_j(f) & \text{if } f(p) = p, \\ \sum_{j=1}^{n} M_j(f) & \text{if } f(p) \neq p. \end{cases}$$

The following theorem will be proved in the next section.

Theorem 1. Let $f: F_n \to F_n$ be a continuous map. Then the following hold.

(i) The map f has at least M (f) fixed points.

(ii) Let $g: F_n \to F_n$ be a continuous map homotopic to f, then

$$M(g) \cdot M(f) \in \{-2, -1, 0, 1, 2\}.$$

(iii) There exists a continuous map $g: F_n \to F_n$ homotopic to f such that g has exactly M (f) fixed points.

We define m (f) as the infimum of the numbers M (g), where $g: F_n \to F_n$ is a continuous map homotopic to f. Then, from Theorem 1 statements (i), (ii) and (iii) of Theorem A follow immediately.

From the following lemma it follows statement (iv) of Theorem A.

Lemma 2. Let $f: F_n \to F_n$ be a continuous map. Then the following hold.

(i) The map f has at least N (f) fixed points.

(ii) Let $g: F_n \to F_n$ be a continuous map homotopic to f, then N (g) = N (f).

For a proof see [1] p. 87 and 95.

Table I

				$a(j, k)$	$M_j^k(f)$
$m = k = 1$		$\tau_{j,k} = \tau_j$	$f(p) = p$	> 2	$\lvert 2 - a(j, k)\rvert$
				$0, 1, 2$	0
				< 0	$\lvert a(j, k)\rvert$
			$r_j f(p) \neq p$		$\lvert 1 - a(j, k)\rvert$
			$f(p) \notin X_j$		$\lvert a(j, k)\rvert$
		$\tau_{j,k} \neq \tau_j$	$r_j f(p) \neq p$		1
			$r_j f(p) = p$		0
$m > 1$	$k = 1$	$\tau_{j,k} = \tau_j$	$f(p) = p$	> 0	$\lvert 1 - a(j, k)\rvert$
				< 0	$\lvert a(j, k)\rvert$
			$r_j f(p) \neq p$		$\lvert 1 - a(j, k)\rvert$
			$f(p) \notin X_j$		$\lvert a(j, k)\rvert$
		$\tau_{j,k} \neq \tau_j$	$r_j f(p) \neq p$		1
			$r_j f(p) = p$		0
	$k = m$	$\tau_{j,k} = \tau_j$	$f(p) = p$	> 0	$\lvert 1 - a(j, k)\rvert$
			otherwise		$\lvert a(j, k)\rvert$
		$\tau_{j,k} \neq \tau_j$			0
	$k \notin \{1, m\}$	$\tau_{j,k} = \tau_j$			$\lvert a(j, k)\rvert$
		$\tau_{j,k} \neq \tau_j$			0

Let deg f be the degree of the continuous map $f: F_1 \to F_1$. The degree of f is an integer (for a definition, see [3] p. 196). We shall use the following lemma, which is proved in [1] p. 107.

Lemma 3. Let $f: F_1 \to F_1$ be a continuous map. Then $N(f) = |1 - \deg f|$.

From the definition of m (f) and Lemma 3 it follows Theorem B.

§ 3. PROOF OF THEOREM 1.

We shall use the following lemma.

Lemma 4. Let $f, g: F_n \to F_n$ be two continuous maps. Then f and g are homotopic if and only if f_* is isomorphic to g_*.

The proof of this lemma follows easily from Theorem 8 of [3] p. 141.

Let $f: F_n \to F_n$ a continuous map. We shall construct a continuous map $g: F_n \to F_n$ such that:

(a) $g(p) = f(p)$,

(b) g_* is equal to f_*,

(c) g has exactly M (f) fixed points.

From Lemma 4 and (b), we have that f and g are homotopic. Hence, statement (iii) of Theorem 1 follows.

Let $\tau: [0, n] \to F_n$ be the continuous map defined by $\tau(t) = \tau_j(t)$ if $t \in [j-1, j]$. We shall construct, for each $1 \leqslant j \leqslant n$, a continuous map $g_j: X_j \to F_n$ such that:

(1) $g_j(\tau(j-1)) = g_j(\tau(j)) = f(p)$,

(2) Let M_j be the number of fixed points of g_j without counting the point p, if p is a fixed point (note that we can define a fixed point of g_j because X_j is contained in F_n), then $M_j = M_j(f)$,

(3) $g_{j*}\{\tau_j\} = f_*\{\tau_j\}$.

We define the map $g: F_n \to F_n$ by $g(\tau(t)) = g_j(\tau(t))$ if $t \in [j-1, j]$. From (1), (2) and (3), it follows immediately that g satisfies (a), (b) and (c).

Now, we separate, the construction of the map g_j, into five cases.

Case 1. $f(p) = p$.

Case 2. $f(p) \neq p$, $f(p) \in X_j$, $\tau_{j,1} = \tau_j$ and $a(j, 1) > 0$.

Case 3. $f(p) \neq p$, $f(p) \in X_j$, $\tau_{j,1} = \tau_j$ and $a(j, 1) < 0$.

Case 4. $f(p) \neq p$, $f(p) \in X_j$, $\tau_{j,1} \neq \tau_j$.

Case 5. $f(p) \neq p$, $f(p) \notin X_j$.

Suppose $f(p) = p$. If $f_*\{\tau_j\} = e$, then we define g_j by $g_j(x) = p$ for all $x \in X_j$. Now, we can assume that

$$f_*\{\tau_j\} = \{\tau_{j,1}^{a(j,1)} \dots \tau_{j,m(j)}^{a(j,m(j))}\}$$

where $a(j, k) \neq 0$ for all $1 \leqslant k \leqslant m(j)$.

Let $P = \{t_0, t_1, \dots, t_q\}$ a partition of the interval $[j-1, j]$ such that

$$j-1 = t_0 < t_1 < \dots < t_q = j \text{ and } q = \sum_{k=1}^{m(j)} |a(j,k)|.$$

For a given integer $0 \leqslant i \leqslant q$, there exists an unique integer $1 \leqslant s(i) \leqslant m(j)$ such that

$$i \leqslant \sum_{k=1}^{s(i)} |a(j,k)| \text{ and if } s(i) > 1, \text{ then } i > \sum_{k=1}^{s(i)-1} |a(j,k)|.$$

Let r_{ik}^+ be the segment which joins the point $(t_i, k-1)$ with the point (t_{i+1}, k) on the square $Q = [0, n] \times [0, n]$. Similarly, the segment r_{ik}^- joins the points (t_i, k) and $(t_{i+1}, k-1)$. Then we define a map $g_j^1: [j-1, j] - P \to [0, n]$ in the following way: $g_j^1(t)$ is such that $(t, g_j^1(t)) \in r_{ik}^+$ or $(t, g_j^1(t)) \in r_{ik}^-$ if $t \in (t_i, t_{i+1})$, $\tau_{j,s(i)} = \tau_k$ and $a(j, s(i)) > 0$ or $a(j, s(i)) < 0$, respectively.

We define $g_j: \tau([j-1, j]) \to F_n$ by

$$g_j(\tau(t)) = \begin{cases} \tau(g_j^1(t)) & \text{if } t \notin P, \\ f(p) & \text{if } t \in P. \end{cases}$$

It is clear that g_j is a continuous map such that $g_{j*}\{\tau_j\} = f_*\{\tau_j\}$. Moreover, M_j equals the crossings of g_j^1 with the diagonal of the squre Q. Then, from Table 1 we have $M_j = M(f)$. This completes the proof of case 1.

Now, suppose that we are in case 2. Then

$$f_*\{\tau_j\} = \{\gamma^{-1} \tau_{j,1}^{a(j,1)} \dots \tau_{j,m(j)}^{a(j,m(j))} \gamma\}$$

where $a(i, k) \neq 0$ for all $1 \leqslant k \leqslant m(j)$. It is clear that there exists an unique $t_p \in (j-1, j)$ such that $\tau(t_p) = f(p)$.

Let $P = \{t_0, t_1, \dots, t_{q+1}\}$ a partition of the interval $[j-1, j]$ such that

$$j - 1 = t_0 < t_1 < \ldots < t_{q+1} = j \text{ and } q = \sum_{k=1}^{m(j)} |a(j, k)|.$$

We denote by r the segment which joins the point (t_0, t_p) with the point (t_1, j) on the square Q. Similarly, the segment s joins the points $(t_q, j - 1)$ and (t_{q+1}, t_p). We define a map g_j^1: $[j - 1, j] \to P \to [0, n]$ in the following way: $g_j^1(t)$ is such that $(t, g_j^1(t))$ belongs to r or s if $t \in (t_i, t_{i+1})$ and $i = 0$ or $i = 1$, respectively; and $g_j^1(t)$ is such that $(t, g_j^1(t))$ belongs to r_{ik}^+ or r_{ik}^- if $t \in (t_i, t_{i+1})$, $\tau_{j, s(i)} = \tau_k$ and a (j, s (i)) > 0 or a (j, s (i)) < 0, respectively. Here, r_{ik}^+, r_{ik}^- and s (i) are the sames of the above case.

We define g_j as in the preceding case. Then case 2 follows. The proofs of the remaining three cases are similar. In short, we have proved (iii) of Theorem 1.

From the proof of (iii) of Theorem 1, we have (roughly speaking) that the number of the fixed points of f is equal to the crossings of the graph of f with the diagonal of the square Q. Then, from the geometric interpretation of f_*, (i) of Theorem 1 follows.

Finally, from Lemma 4 and the definition of M (f) it is long but straightforward to obtain (ii) of Theorem 1.

§ 4. Example C.

We recall here the definition of the Nielsen number (for more details see [1] p. 87). The Nielsen number is usually defined for a continuous map on a compact ANR (see [1] p. 37). Since a polyhedron is a compact ANR (see [1] p. 39), we have that F_n is a compact ANR.

Let f: $F_n \to F_n$ be a continuous map, we say that fixed points x and y of f are f-equivalent if there is a path C: $[0, 1] \to F_n$ such that C (0) = x, C (1) = y, and for the path fC: $[0, 1] \to F_n$ we have $\{fC\} = \{C\}$, i.e. the paths fC and C are homotopic. Let Fix(f) denote the set of all fixed points of f. The equivalence classes are called fixed points classes of f. It is known that f has a finite number of fixed point classes. We denote by $F_1, \ldots, F_n$ the fixed point classes of f, then for each j = 1, . . . , n there is an open set U_j in F_n such that $F_j \subset U_j$ and cl $(U_j) \cap$ Fix(f) = F_j, where "cl" denotes closure. Let *i* be the index on the collection C_A of connected compact ANRs. Then we can consider the index of the triple (X, f, U_j), i.e. *i* (X, f, U_j). We define the index *i* (F_j) of the fixed points class F_j by *i* (F_j) = *i* (X, f, U_j). The definition of *i* (F_j) is independent of the choice of the open $U_j \subset X$ such that $F_j \subset U_j$ and cl $(U_j) \cap$ Fix (f) = F_j.

For a continuous map f: $F_n \to F_n$, a fixed point class F of f is said to be essential if *i* (F) $\neq 0$ and inessential if *i* (F) = 0. The Nielsen number N (f) of the map f is defined to be the number of fixed point classes of f that are essential.

We know that $N(f) \leqslant m(f)$. Now, we give an example with $N(f) < m(f)$. Let $f: F_2 \to F_2$ be a continuous map with

$$f(p) = p \text{ and } f_* \{\tau_1\} = \{\tau_2 \tau_1^2 \tau_2 \tau_1^{-1}\}\ ,\ f_* \{\tau_2\} = \{\tau_1\}$$

Then we have $M(f) = 4$. To compute $m(f)$, let $h: F_2 \to F_2$ be a continuous map homotopic to f. If $h(p) = p$, then we have $M(h) = 4$. If $h(p) \neq p$ and $h(p) \in X_1$, then $h_* \{\tau_1\} = \{\gamma^{-1} \tau_2 \tau_1^2 \tau_2 \tau_1^{-1} \gamma\}$ and $h_* \{\tau_2\} = \{\gamma^{-1} \tau_1 \gamma\}$ where γ is defined as above. We obtain $M(h) = 4$. Finally, if $h(p) \neq p$ and $h(p) \in X_2$, then we also obtain $M(h) = 4$. So, we have $m(f) = 4$.

To compute $N(f)$ we consider the following map g homotopic to f. Let $P = \{t_1, \ldots, t_5\}$ a partition of $[0, 2]$ such that

$$0 < t_1 < t_2 < t_3 < t_4 < 1 < t_5 < 2.$$

Take $g': [0, 2] - P \to [0, 2]$ defined by $g'(t)$ equals

$$\begin{array}{ll}
\dfrac{t}{t_1} + 1 & \text{if } 0 < t < t_1, \\[2ex]
\dfrac{t_1 - t}{t_1 - t_2} & \text{if } t_1 < t < t_2, \\[2ex]
\dfrac{t_2 - t}{t_2 - t_3} & \text{if } t_2 < t < t_3, \\[2ex]
\dfrac{t_4 - t}{t_3 - t_4} + 2 & \text{if } t_3 < t < t_4, \\[2ex]
\dfrac{t - 1}{t_4 - 1} & \text{if } t_4 < t < 1, \\[2ex]
\dfrac{1 - t}{1 - t_5} & \text{if } 1 < t < t_5, \\[2ex]
1 & \text{if } t_5 < t < 2;
\end{array}$$

and consider $g: F_2 \to F_2$ obtained from g' as in the proof of Theorem A (see fig. 1). Clearly g is homotopic to f.

Let

$$a = \frac{t_1}{t_1 - t_2 + 1} \quad \text{and} \quad b = \frac{1}{2 - t_4}.$$

We will now show that the fixed points $\tau(a)$ and $\tau(b)$ of g, are g-equivalent. Let us first define

$$\gamma'_1 : [0, a] \to [0, 2],\ \gamma'_2 : [a, b] \to [0, 2] \quad \text{and} \quad \gamma'_3 : [b, 2] \to [0, 2]$$

by

$$\gamma'_1\,(t) = a - t,\ \gamma'_2\,(t) = 0 \quad \text{and} \quad \gamma'_3\,(t) = 2 + b - t.$$

Then, $\gamma_1 = \tau \cdot \gamma'_1$, $\gamma_2 = \tau \cdot \gamma'_2$ and $\gamma_3 = \tau \cdot \gamma'_3$ are paths on F_2. Let

$$C = \gamma_1 \cdot \gamma_2 \cdot \gamma_3 .$$

Then we have that C is a path on F_2 such that C (0) = a and C (1) = b (see fig. 2). Product of paths is defined as usually (see [2] p. 57). Hence, it is easy to see that $g\,C = \sigma_1 \cdot \sigma_2 \cdot \sigma_3 \cdot \sigma_4 \cdot \sigma_5 \cdot \sigma_6$ where $\sigma_i = \tau \cdot \sigma'_i$ and $\sigma'_1, \ldots, \sigma'_6$ are paths defined by:

$$\sigma'_1 : [0, a - t_1] \to [0, 2] \qquad \text{and} \quad \sigma'_1\,(t) = \frac{t + t_1 - a}{t_1 - t_2},$$

$$\sigma'_2 : [a - t_1, a] \to [0, 2] \qquad \text{and} \quad \sigma'_2\,(t) = \frac{a - t}{t_1} + 1,$$

$$\sigma'_3 : [a, b] \to [0, 2] \qquad \text{and} \quad \sigma'_3\,(t) = 1,$$

$$\sigma'_4 : [b, 2 + b - t_5] \to [0, 2] \qquad \text{and} \quad \sigma'_4\,(t) = 1,$$

$$\sigma'_5 : [2 + b - t_5, 1 + b] \to [0, 2] \qquad \text{and} \quad \sigma'_5\,(t) = \frac{b - t + 1}{t_5 - 1},$$

$$\sigma'_6 : [1 + b, 2] \to [0, 2] \qquad \text{and} \quad \sigma'_6\,(t) = \frac{b - t + 1}{t_4 - 1}.$$

Note that $\{C\} = \{g\,C\}$ if and only if

$$\{\sigma_1^{-1}\,\gamma_1\}\ \{\gamma_2\}\ \{\gamma_3\,\sigma_6^{-1}\} = \{\sigma_2\}\ \{\sigma_3\}\ \{\sigma_4\}\ \{\sigma_5\}$$

where each factor is an element of $\Pi(F_2, p)$. But $\{\gamma_2\} = \{\sigma_3\} = \{\sigma_4\} = e$, the unit element of $\Pi(F_2, p)$. Therefore we must prove that

$$\{\sigma_1^{-1}\gamma_1\}\{\gamma_3\sigma_6^{-1}\} = \{\sigma_2\}\{\sigma_5\}.$$

For this, we consider the following diagrams, where in each case, h is a linear homeomorphism.

$$\begin{array}{ccc} [0, a-t_1] & \xrightarrow{\sigma_1'} [0,2] \xrightarrow{\tau} & F_2 \\ h\uparrow & \nearrow_{\gamma_1'} & \\ [0, a] & & \end{array} \qquad (1)$$

where $h(t) = \frac{t}{a}(a - t_1)$.

$$\begin{array}{ccc} [a - t_1, a] & \xrightarrow{\sigma_2'} [0,2] \xrightarrow{\tau} & F_2 \\ h\uparrow & \nearrow_{\alpha} & \\ [0, 1] & & \end{array} \qquad (2)$$

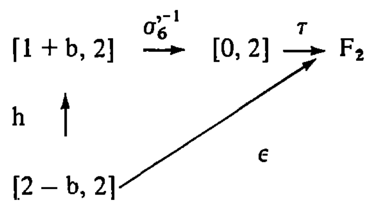
where $h(t) = \dfrac{t(1-b) + b(3+b) - 2}{b}$ and $\alpha(t) = 2 - t$.

$$\begin{array}{ccc} [2 + b - t_5, 1 + b] & \xrightarrow{\sigma_5'} [0,2] \xrightarrow{\tau} & F_2 \\ h\uparrow & \nearrow_{\beta} & \\ [1, 2] & & \end{array} \qquad (3)$$

where $h(t) = t_5(t - 2) + 3 + b - t$ and $\beta(t) = 2 - t$.

$$\begin{array}{ccc} [b, 2] & \xrightarrow{\gamma_3'} [0,2] \xrightarrow{\tau} & F_2 \\ h\uparrow & \nearrow_{\rho} & \\ [0, 2 - b] & & \end{array} \qquad (4)$$

where $h(t) = t+b$ and $\rho(t) = 2-t$.



(5)

where $h(t) = \frac{(1-b)t+4b-2}{b}$, $\epsilon(t) = 2-t$ and $\sigma_6'^{-1}(t) = \frac{t-2}{t_4-1}$.

Then we have $\{\sigma_1\} = \{\gamma_1\}$, $\{\sigma_2\} = \{\tau\,\alpha\}$, $\{\sigma_5\} = \{\tau\,\beta\}$, $\{\gamma_3\} = \{\tau\,\rho\}$, $\{\sigma_6^{-1}\} = \{\tau\,\epsilon\}$ and $\tau(\alpha\,\beta) = \tau\,(\rho\,\epsilon)$. Therefore $\{\sigma_1^{-1}\,\gamma_1\} = e$, and

$$\{\sigma_2\}\ \{\sigma_5\} = \{\tau\,\alpha\}\ \{\tau\,\beta\} = \{\tau\,(\alpha\,\beta)\} = \{\tau\,(\rho\,\epsilon)\} = \{\tau\,\rho\}\ \{\tau\,\epsilon\} = \{\gamma_3\}\ \{\sigma_6^{-1}\},$$

This completes the proof that $\{C\} = \{g\,C\}$.

Now, note that g has four fixed points, and that $\tau(a)$ and $\tau(b)$ are g-equivalent. Therefore $N(g) < 4$. Since $N(f) = N(g)$ and $m(f) = 4$, we have an example with $N(f) < m(f)$.

§ 5. PROOF OF THEOREM D.

We shall use the following lemma, which is proved in [1] p. 127-128.

Lemma 5. *Let $f\colon F_n \to F_n$ be a continuous map and let $x \in X_i$, $x \neq p$ an isolated fixed point. Suppose that U is a neighborhood of x such that $f(U) \subset X_i - \{p\}$. Let $U - \{x\}$ consist of components U_1 and U_2. Then we have:*

(i) If $f(U_1) \subset U_2$ and $f(U_2) \subset U_1$, then $i(F_n, f, x) = 1$.

(ii) If $f(U_1) \subset U_1$ and $f(U_2) \subset U_2$, then $i(F_n, f, x) = -1$.

If $f\colon F_n \to F_n$ is a continuous map, then we denote by $L(f)$ the Lefschetz number of f (for a definition see [1] p. 25).

We say that a continuous map $f\colon F_n \to F_n$ satisfies condition F if and only if there exists $1 \leqslant i \leqslant n$ such that $a_i = 1$ and $a_j \leqslant 0$ for all $1 \leqslant j \leqslant n$, $j \neq i$; where $f_*\{\tau_i\} = \{\gamma^{-1}\,\tau_i^{a_i}\,\gamma\}$ for all $1 \leqslant i \leqslant n$. From the proof of (iii) of Theorem 1, it follows that there exists a continuous map $g\colon F_n \to F_n$ homotopic to f such that

$g(p) = p$, $g(X_i) \subset X_i$ for all $1 \leqslant i \leqslant n$, and g has exactly M (g) fixed points. For these maps f and g, we have the following three lemmas.

Lemma 6. The map f satisfies condition F if and only if $i(F_n, g, p) = 0$.

Proof. It is known that $L(g) = 1 - \mathrm{Tr}(f^*)$ where f^* is the morphism induced by f on the first homology group. Therefore

$$L(g) = 1 - \sum_{j=1}^{n} a_j. \tag{1}$$

On the other hand, we recall that

$$L(g) = \sum_{k=1}^{M(g)} i(F_n, g, p_k),$$

where p_k, $1 \leqslant k \leqslant M(g)$, are the fixed points of g (for more details see [1] p. 52). Then, by Lemma 5, we obtain

$$L(g) = \sum_{a_j < 0} |a_j| - \sum_{a_j \geqslant 2} |2 - a_j| + i(F_n, g, p). \tag{2}$$

From (1) and (2) it follows that $i(F_n, g, p) = 0$ if and only if

$$1 - \sum_{j=1}^{n} a_j = \sum_{a_j < 0} |a_j| - \sum_{a_j \geqslant 2} |2 - a_j|.$$

This condition is equivalent to

$$1 - \sum_{a_j = 1} a_j = 2m,$$

where m is the number of $a_j \geqslant 2$. But this last condition is the same that condition F. This proves the lemma.

Lemma 7. Two arbitrary fixed points of g are not g-equivalent.

Proof. Let x and y be two fixed points of g and suppose that they are g-equivalent, i.e. there is a path $C: [0, 1] \to F_n$ such that $C(0) = x$, $C(1) = y$, and for the path gC: $[0, 1] \to F_n$ we have $\{gC\} = \{C\}$. We separate the proof into two cases.

Case 1. $\{x, y\} \subset X_i$, for some $1 \leqslant i \leqslant n$.

Let g_i be the restriction of g on X_i. Since g satisfies $g(X_i) \subset X_i$, we have that g_i is a continuous map of the circle into itself such that $\deg g_i = a_i$. By Theorem B, we obtain that $a_i \neq 0, 1$ because g_i has at least two fixed points.

From $\{gC\} = \{C\}$ it follows that $\{r_i gC\} = \{r_i C\}$. Since $r_i gC = gr_i C$, we have that $\{gr_i C\} = \{r_i C\}$. Then x and y are g_i-equivalent. But the number of fixed points of g_i is exactly $N(g_i) = |1 - a_i|$ (by Theorem B), and this is a contradiction.

Case 2. $x \in X_i$, $y \in X_j$, $i \neq j$.

Let $H: [0, 1] \times [0, 1] \to F_n$ be the homotopy map between gC and C. Then $r_i H$ is a homotopy map between $r_i gC$ and $r_i C$. This implies that x and p are g-equivalent. By case 1, this is a contradiction. This completes the proof of Lemma 7.

Lemma 8. *The following hold.*

(i) If f satisfies condition F, then $M(g) = m(f) + 1 = N(g) + 1$.

(ii) If f does not satisfy condition F, then $M(g) = m(f) = N(g)$.

Proof. We consider n continuous maps $h_j: F_n \to F_n$ homotopic to f such that $h_j(p) \in X_j$ and $h_j(p) \neq p$ for all $1 \leqslant j \leqslant n$. Then we have

$$M(h_j) = \sum_{\substack{i=1\\ i\neq j}}^{n} |a_i| + |1 - a_j| ,$$

$$M(g) = \sum_{a_i < 0} |a_i| + \sum_{a_i \geqslant 2} |2 - a_i| + 1$$

It is clear that $m(f) = \min \{M(h_1), \ldots, M(h_n), M(g)\}$.

We prove statement (i). Let $a_i = 1$. From Table II we obtain

$$M(g) = M(h_i) + 1 \quad \text{and} \quad M(h_j) = M(h_i) + 2 \quad \text{if} \quad j \neq i.$$

Hence, $M(g) = m(f) + 1$. From Lemmas 5, 6 and 7, it follows that

$$M(g) = N(g) + 1.$$

Note that we have construct Table II from the definition of $M(f)$.

Table II

	a_i	$M_i(g)$	$M_i(h_j)$
$i \neq j$	<0	$\lvert a_i \rvert$	$\lvert a_i \rvert$
	0	0	0
	1	0	1
	2	0	2
	>2	$\lvert 2 - a_i \rvert$	$\lvert a_i \rvert$
$i = j$	<0	$\lvert a_i \rvert$	$\lvert 1 - a_i \rvert$
	0	0	1
	1	0	0
	2	0	1
	>2	$\lvert 2 - a_i \rvert$	$\lvert 1 - a_i \rvert$

Now, we prove statement (ii). We separate the proof into three cases.

Case 1. There exists $1 \leqslant i \leqslant n$ such that $a_i \geqslant 2$.

From Table II, we obtain $M(g) < M(h_j)$ for all $1 \leqslant j \leqslant n$, $j \neq i$, and

$$M(g) \leqslant M(h_i).$$

Case 2. There exist $1 \leqslant i, j \leqslant n$ such that $a_i = a_j = 1$ and $a_k \leqslant 1$ for all $1 \leqslant k \leqslant n$, $k \neq i, j$.

From Table II, we have $M(g) \leqslant M(h_j)$, for all $1 < j < n$.

Case 3. For all $1 \leqslant i \leqslant n$, we have $a_i \leqslant 0$.

Again from Table II, $M(g) = M(h_i)$, for all $1 \leqslant i \leqslant n$.

In the three cases we obtain $M(g) = m(f)$. From Lemmas 5, 6 and 7, it follows that $N(g) = M(g)$. This completes the proof of Lemma 8.

By Lemmas 8 and 2, Theorem D follows.

§ 6. PROOF OF THEOREM E.

The 0-simpleces and 1-simpleces of a graph are also known as vertices and edges, respectively. A tree is defined to be a simply connected graph. A maximal tree is a tree which contains all of the vertices of the graph.

It is known that if K is a finite connected graph, and if T is a maximal tree in K, then the fundamental group of K is isomorphic to a free group on n-generators in one-to-one correspondence with the edges of K - T (see [3] p. 141). It is also known that K is homotopy equivalent to F_n for some positive integer n (see [4] p. 95). Hence, there exist two continuous maps $g: K \to F_n$ and $h: F_n \to K$ such that $h \cdot g$ and $g \cdot h$ are homotopic to the identity map of K and F_n, respectively. We define $X'_j = h(X_j)$.

From the proof of Theorem 1, it follows that a continuous map $f: F_n \to F_n$ has at least $M_j(f)$ fixed points in X_j without to count the point p, if p is a fixed point. In a similar way for a continuous map $f: K \to K$ we should prove that f has at least $M_j(f)$ fixed points on X'_j. But now we cannot claim that a fixed point in X'_i is different of a fixed point in X'_j if $i \neq j$, because in general the intersection of X'_i with X'_j is not a single point. In short, if we define $M'(f) = \min_{1 \leqslant j \leqslant n} \{M_j(f)\}$, Theorem E follows.

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