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On the structure of the set of periodic points of a continuous map of the interval with finitely many periodic points

By

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1. Introduction. Let I denote a closed interval on the real line and let $C^0(I, I)$ denote the space of continuous maps from I into itself. For $f \in C^0(I, I)$ let $P(f)$ denote the set of positive integers k such that f has a periodic point of period k (see section 2 for definitions).

From Šarkovskii's theorem we know: (i) if $P(f)$ is finite then $P(f) = \{1, 2, 4, \ldots, 2^n\}$ for some integer $n \geqq 0$ (see [3], [4] or [5]), (ii) if P is a periodic orbit of f of period 2^m, then f has a periodic orbit of period 2^k, which is contained in [min P, max P] for each $k = 0, 1, \ldots, m - 1$ (see [5]). In this paper we study the relation between these orbits.

Our main result is the following.

Theorem A. *Let $f \in C^0(I, I)$ and suppose $P(f) = \{1, 2, 4, \ldots, 2^n\}$. Then for any periodic orbit of period 2^m, with $m \leqq n$, there exist $m + 1$ periodic orbits of periods $1, 2, 4, \ldots, 2^m$ such that the 2^k periodic points of period 2^k are separated by the $1 + 2 + 4 + \cdots + 2^{k-1} = 2^k - 1$ fixed points of $f^{2^{k-1}}$, for any $k = 1, 2, \ldots, m$ (see Fig. 1 for $m = 3$).*

This theorem will be proved in section 3.

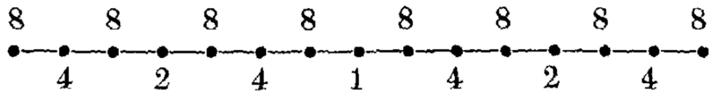

Figure 1.

Let $f \in C^0(I, I)$ and suppose $P(f) = \{1, 2, 4, \ldots, 2^n\}$. If f has a unique periodic orbit of period 2^k for any $k = 0, 1, \ldots, n$ (for instance the map given by Block in [1]), then Theorem A give us the complete structure of the set of periodic points of f.

2. Preliminary definitions and results. Let $f \in C^0(I, I)$. For any positive integer n, we define f^n inductively by $f^1 = f$ and $f^n = f \circ f^{n-1}$. We let f^0 denote the identity map of I.

Let $p \in I$. We say p is a fixed point of f if $f(p) = p$. If p is a fixed point of f^n, for some $n \in N$ (the set of positive integers), we say p is a periodic point of f. In this case, the smallest element of $\{n \in N : f^n(p) = p\}$ is called the period of p.

We define the orbit of p to be $\{f^n(p) \quad n = 0, 1, 2, \quad \}$. If p is a periodic point of f of period n, we say the orbit of p is a periodic orbit of period n. In this case the orbit of p contains exactly n points each of which is a periodic point of period n.

Let $A = \{I_1, \quad, I_k\}$ be a partition of I into subintervals, that is, a family of closed intervals such that $I_1 \cup \quad \cup I_k$ is I, and if $i \neq j$ then $I_i \cap I_j$ consists of at most one point.

We say that an interval I f-covers J if there exists a closed subinterval K of I such that $f(K) = J$ We say that I f-covers J n times if there exist n closed subintervals $K_1, \quad, K_n$ of I with pairwise disjoint interiors such that $f(K_i) = J$ for $i = 1, 2, \quad, n$.

An A-graph of f is an oriented generalized (i.e., possibly with several arrows joining the same vertices) graph with vertices $I_1, \quad, I_k$ and such that if I_i f-covers I_j n times, but no $n+1$ times, then there are n (but no $n+1$ arrows from I_i to I_j In this paper a loop in the A-graph of f will be a loop without proper subloops. We state the following lemma which will be used in the next section.

Lemma 1. (Lemma 1.4 of [3]). *If $J_0 \to J_1 \to \quad \to J_{n-1} \to J_0$ is a loop in the A-graph of f then there exists a fixed point x of f^n such that $f^i(x) \in J_i$ for $i = 0, 1, \quad, n \quad 1$.*

Let $P = \{p_1, \quad, p_n\}$ a periodic orbit of $f \in C^0(I, I)$ of period $n = 2^m$ We define a simple periodic orbit inductively If $m = 1$ then P is simple. Suppose $m > 1$, then we say P is simple if the two subsets $\{p_1, \quad, p_{n/2}\}$ and $\{p_{1+n/2}, \quad, p_n\}$ of P are simple periodic orbits of f^2 and we have $f\{p_1, \quad, p_{n/2}\} = \{p_{1+n/2}, \quad, p_n\}$. The definition of simple periodic orbit has given by Block in [2]. We conclude this section by stating the following lemma.

Lemma 2. (See Theorem A of [2]). *Let $f \in C^0(I, I)$ and suppose $P(f) = \{1, 2, 4, \quad, 2^n\}$. Then any periodic orbit of f is simple.*

3. Proof of Theorem A. In this section we assume that $f \in C^0(I, I)$ and $P(f) = \{1, 2, 4, \quad, 2^n\}$. Also we suppose that f has a periodic orbit $P = \{p_1, \quad, p_{2^n}\}$ of period 2^n where $p_1 < p_2 < \quad < p_{2^n}$. We let $I_k = [p_k, p_{k+1}]$ for $k = 1, 2, \quad, 2^n \quad 1$, and set $A = \{I_1, \quad, I_{2^n-1}\}$ a partition of $J = [p_1, p_{2^n}]$. Finally, we define a map $g \in C^0(J, J)$ by $g(p_i) = f(p_i)$, for $i = 1, 2, \quad, 2^n$ and on each interval I_k, g is linear.

Lemma 3. *If I_i g^2-covers I_j, with $i < 2^{n-1}$, $j < 2^{n-1}$, then there exists an interval I_k, with $k > 2^{n-1}$, such that I_i g-covers I_k and I_k g-covers I_j, i.e. $I_i \to I_k \to I_j$*

Proof. Let K be a closed subinterval of I_i such that $g^2(K) = I_j$ Then there exist $a, b \in g(K) \subset [p_{2^n - 1_{+1}}, p_{2^n}]$ such that $g(a) = p_j$, and $g(b) = p_{j+1}$ It is easy to see that we can choose a, b such that $(\min(a, b), \max(a, b)) \cap P = \emptyset$. Therefore, there exists $I_k \subset f(I_i)$ with $a, b \in I_k$ and so $I_i \to I_k \to I_j$ □

Since g has a periodic orbit of period 2^n it follows, from Šarkovskii's theorem, that g has periodic points of periods 2^k for $k = 0, 1, \quad, n \quad 1$

Lemma 4 (Existence). *For each periodic point q of period 2^k of g, there is a loop*

$$I_{a(0)} \to I_{a(1)} \to \quad \to I_{a(2^k)} \to I_{a(0)}$$

in the A-graph of g, with $f^i(q) \in I_{a(i)}$, $i = 0, 1, \quad, n$.

Proof. By induction on k. As g is linear, the lemma is clearly true for $k = 0$. Suppose it is true for certain $k > 0$. Let q be a periodic point of period 2^{k+1}. By lemma 2, the orbit of q is simple. Therefore the sets $\{q, g^2(q), \quad, g^{2^{k+1}-2}(q)\}$ and $\{g(q), g^3(q), \quad, g^{2^{k+1}-1}(q)\}$ are periodic orbits of period 2^k of g^2. By induction, there are two loops

$$I_{b(0)} \to I_{b(1)} \to \quad \to I_{b(2^k)} \to I_{b(0)} \quad \text{and}$$

$$I_{c(0)} \to I_{c(1)} \to \quad \to I_{c(2^k)} \to I_{c(0)}$$

in the A-graph of g^2 with $g^{2i}(q) \in I_{b(i)}$ and $g^{2i+1}(q) \in I_{c(i)}$ for $i = 0, 1, \quad, 2^k \quad 1$. From Lemma 3, we have the loop

$$I_{b(0)} \to I_{c(0)} \to \quad \to I_{b(2^k)} \to I_{c(2^k)} \to I_{b(0)}$$

in the A-graph of g, and this proves the lemma. □

We can assume $n \geqq 2$, because for $n = 1$ Theorem A is immediate. We say that a loop in the A-graph of g is an L_k-loop, for $k = 1, \quad, n \quad 1$, if its 2^{n-k} vertices are the intervals $I_{a(j,k)}$ where $a(j,k) = 2^{k-1} + j2^k$ for $j = 0, 1, \quad, 2^{n-k} \quad 1$, and the loop has length 2^{n-k} Note that $a(j,k)$ take each value of the set $\{1, 2, 3, \quad, 2^n \quad 1\}$ $\{2^{n-1}\}$ only once.

Lemma 5 (Uniqueness). *If $I_{a(j,k)}$ is a vertex of a loop in the A-graph of g, then this loop is an L_k-loop, for all $j = 0, 1, \quad, 2^{n-k} \quad 1$ and all $k = 1, 2, \quad, n \quad 1$.*

Proof. We prove this lemma inductively on n. If $n = 2$, then $P(g) = \{1, 2, 4\}$ and $a(j,k) \in \{1, 3\}$. Since the two unique simple periodic orbits of period 4 are (by Lemma 2) $g(p_1) = p_3$, $g(p_2) = p_4$, $g(p_3) = p_2$, $g(p_4) = p_1$ and $g(p_1) = p_4$, $g(p_2) = p_3$, $g(p_3) = p_1$ $g(p_4) = p_2$, we have that $I_1 \to I_3 \to I_1$ is the only loop in the A-graph of g which contains I_1 or I_3 (note that the unique interval I_j that g-covers I_2 is I_2). Therefore, Lemma 4 follows for $n = 2$.

Let $n > 2$ and suppose that Lemma 4 is true for $n \quad 1$. Let L be a loop in the A-graph of g and suppose $I_{a(j,k)}$ is a vertex of L. Without loss of generality, we may assume that $a(j,k) < 2^{n-1}$. From Lemma 1 and since $P(f)$ is finite, the length of L is even. Then $I_{a(j,k)}$ is a vertex of a loop M_1 in the A_1-graph of g^2, where $A_1 = \{I_1, \quad, I_{2^{n} \quad 1-1}\}$ and $A_2 = \{I_{2^{n} \quad 1+1}, \quad, I_{2^n}\}$ because P is simple. By induction, the loop M_1 is an L_k-loop in the A_1-graph of g^2.

Let $I_{k_1} \to I_{k_2} \to \quad \to I_{k_{2^{n-k-1}}} \to I_{k_1}$ the loop M_1, where $k_i < 2^{n-1}$ for each $i = 1, 2, \quad, 2^{n-k-1}$. By Lemma 3, we have the following loop M in the A-graph of g·

$$I_{k_1} \to I_{h_1} \to I_{k_2} \to I_{h_2} \to \quad \to I_{k_{2^{n-k-1}}} \to I_{h_{2^{n-k-1}}} \to I_{k_1}$$

where $h_i > 2^{n-1}$ for each $i = 1, 2, \quad, 2^{n-k-1}$. Note that this is a loop in the A-graph of g, because if $I_{h_i} = I_{h_j}$ for some $1 \leqq i < j \leqq 2^{n-k-1}$, then the loop

$$I_{k_{i+1}} \to I_{k_{i+2}} \to \quad \to I_{k_j} \to I_{k_{i+1}}$$

in the A_1-graph of g^2 would have length smaller than 2^{n-k-1}, and this is impossible since M_1 is an L_k-loop. Therefore, by induction, the loop

$$I_{h_1} \to I_{h_2} \to \quad \to I_{h_{2^{n-k-1}}} \to I_{h_1}$$

in the A_2-graph of g^2 is an L_k-loop denoted by M_2 Hence, the vertices of M_1 are the intervals $I_{a(j,k)}$, where $A(j,k) = 2^{k-1} + j2^k$, for $j = 0, 1, \quad, 2^{n-k-1} \quad 1$ and the vertices of M_2 are the intervals $I_{a(j,k)}$ where $a(j,k) = 2^{n-1} + 2^{k-1} + j2^k$, for $j = 0, 1, \quad, 2^{n-k-1} \quad 1$. So the vertices of the loop M are the intervals $I_{a(j,k)}$ where $a(j,k) = 2^{k-1} + j2^k$, for $j = 0, 1, \quad, 2^{n-k} \quad 1$. Then M is an L_k-loop in the A-graph of g. □

Proof of Theorem A. From lemma 4, there are loops of length 2^{n-k} for $k = 1, 2, \quad, n \quad 1$, and from Lemma 5 these loops are the L_k-loops. Since an L_k-loop has 2^{n-k} vertices, from lemma 1, g has a periodic orbit of period 2^{n-k} Each periodic point of this periodic orbit is in a unique $I_{a(j,k)}$ for some $j = 0, 1, \quad, 2^{n-k} \quad 1$. Furthermore, since the orbit P is simple (by lemma 2), there is a fixed point in $I_{2^n \quad 1}$. Hence, Theorem A is proved for the map g.

Since the A-graph of g is a subgraph of the A-graph of f, Theorem A follows. □

Remark. The referee has pointed out that we have proved in fact the following slightly stronger theorem

Theorem B. *Let $f \in C^0(I, I)$ and suppose that f has a simple periodic orbit of period 2^n Then there exist $n + 1$ simple periodic orbits of periods $1, 2, 4, \quad, 2^n$ such that the 2^k periodic points of period 2^k are separated by the $1 + 2 + \quad + 2^{k-1} = 2^k \quad 1$ fixed points of $f^{2^{k-1}}$ for any $k = 1, 2, \quad, n$.*

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