An Interesting Property of the Evolute

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1. INTRODUCTION

The starting point of this note is the following inequality: if $C = \partial K$ is the boundary of a compact, convex set K of area $\mathbf{F}$ in $\mathbb{R}^2$, then

$$\int_C \frac{1}{k} ds \geqslant 2\mathbf{F}, \tag{1}$$

where $k = k(s)(> 0)$ is the curvature function of C and ds signifies arclength measure on C. Equality holds if and only if C is a circle.

In this note we give a very short new proof of (1), which has the advantage of providing a geometric interpretation of the difference $2\mathbf{F} - \int_C k^{-1} ds$. To be precise, we prove that

$$\int_C \frac{1}{k} ds = 2(\mathbf{F} - \mathbf{F_e}),$$

where $\mathbf{F_e}(\leq 0)$ is the (algebraic) area of the domain bounded by the evolute of C.

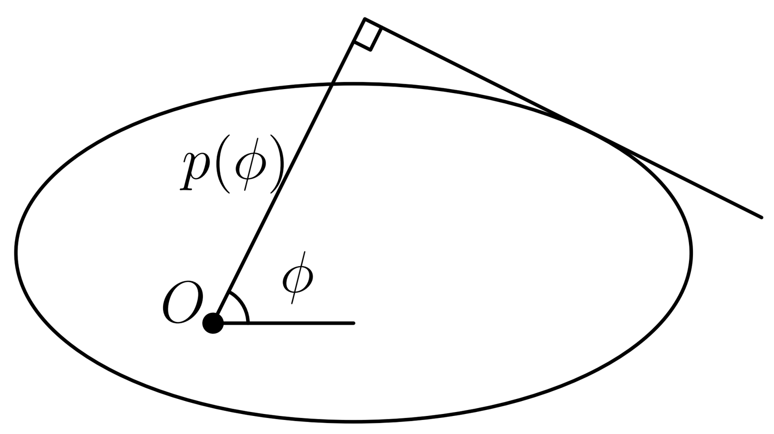
Inequality (1) is the two-dimensional analogue of Heintze and Karcher's inequality:

$$\int_S \frac{1}{H} dA \geqslant 3\mathbf{V},$$

where $H(> 0)$ is the mean curvature of a compact embedded surface S in $\mathbb{R}^3$ bounding a domain D of volume $\mathbf{V}$. Equality holds if and only if S is a standard sphere. This raises the obvious question: Is there a geometric interpretation of the difference $3\mathbf{V} - \int_C H^{-1} dA$?

2. CONVEX SETS AND SUPPORT FUNCTIONS

The boundary of a convex set K is the envelope of its tangents.



In terms of the support function $p(\phi)$ (distance between the tangents and a fixed point) ∂K is given by

$$\begin{pmatrix} x(\phi) \\ y(\phi) \end{pmatrix} = R_{-\phi} \begin{pmatrix} p(\phi) \\ p'(\phi) \end{pmatrix}$$

where $R_{-\phi}$ is the rotation of angle $-\phi$. Convexity implies $p + p'' > 0$.

The length and area of the convex set K is expressed in terms of the support function by

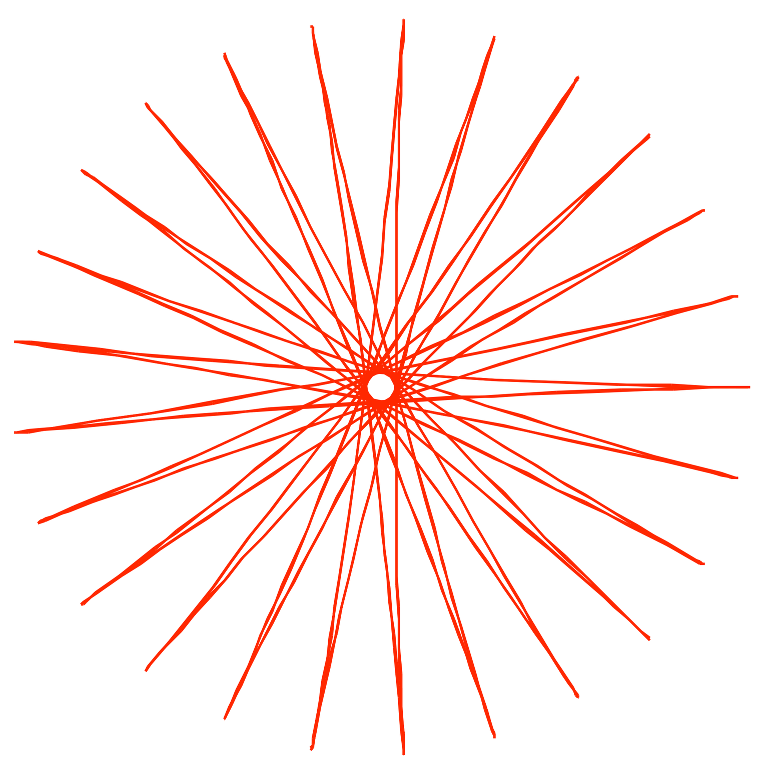
$$\mathbf{L} = \int_0^{2\pi} p \, d\phi. \qquad \mathbf{F} = \frac{1}{2}\int_{\partial K} p ds = \frac{1}{2}\int_0^{2\pi} p(p + p'') \, d\phi$$

3. HEDGEHOGS

If $h : \mathbb{R} \to \mathbb{R}$ is a C^1-function of period 2π, the *hedgehog* γ_h corresponding to h is defined by

$$\begin{pmatrix} x(\phi) \\ y(\phi) \end{pmatrix} = R_{-\phi} \begin{pmatrix} p(\phi) \\ p'(\phi) \end{pmatrix}$$

For instance, if $h(\phi) = \cos(25\phi)$, this envelope actually looks like a hedgehog (Figure 2).

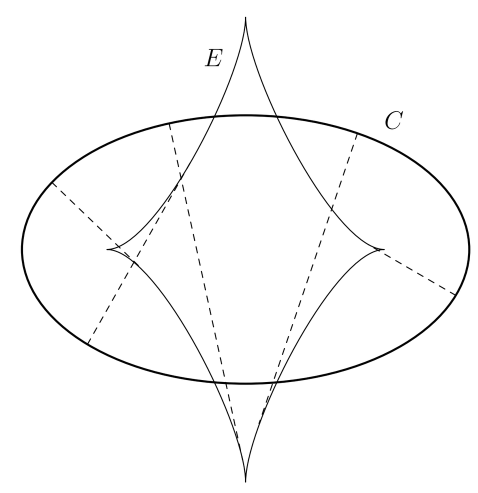


The (algebraic) area $\mathbf{F_h}$ of the hedgehog corresponding to h is given by

$$\mathbf{F_h} = \frac{1}{2}\int_0^{2\pi} h(h + h'') d\phi = \frac{1}{2}\int_0^{2\pi} (h^2 - h'^2) d\phi.$$

4. THE EVOLUTE OF A CONVEX CURVE

The evolute of a curve is the envelope of its normals.



Let C be the boundary of a convex set K with support function $p = p(\phi)$.

It can be seen that *The evolute of C is the hedgehog of $-p'(\phi + \pi/2)$*

Hence the (algebraic) area $\mathbf{F_e}$ of the evolute of the convex curve supported by $p(\phi)$ is equal to the (algebraic) area $\mathbf{F_H}$ of the hedgehog corresponding to $H(\phi) = -p'(\phi + \pi/2)$.

$$\mathbf{F_e} = \frac{1}{2}\int_0^{2\pi} H(H + H'') d\phi = \frac{1}{2}\int_0^{2\pi} p'(p' + p''') d\phi.$$

We have proved:

Theorem 1 *The integral with respect to arclength of the radius of curvature of a plane convex curve is twice the area of the domain it bounds minus the (algebraic) area of the domain bounded by its evolute:*

$$\int_C \rho \, ds = 2(\mathbf{F} - \mathbf{F_e}).$$

As a consequence of the Wirtinger inequality we have

Corollary 1 *If the boundary $C = \partial K$ of a convex set K in the plane is a C^2-curve, then*

$$\int_C \rho \, ds \geq 2\mathbf{F},$$

where ds is arclength measure on C, $\rho = \rho(s)$ is the radius of curvature of C, and $\mathbf{F}$ is the area of K. Equality holds if and only if C is a circle.

Remark 1 (Geometric interpretation of Wirtinger inequality) We prove that Wirtinger inequality is equivalent to the following statement:

$$\boxed{\forall y \in K, \ \exists x \in C \quad \text{such that} \quad d(x, y) < \rho(x)}$$

where $\rho(x)$ is the curvature radius of C at x.

5. FOCAL SETS IN SPACE FORMS

Let X_c^2 be the 2-dimensional complete and simply connected riemannian manifold of constant curvature c, i.e. the sphere $\mathbb{S}_c^2$ of radius $R = \frac{1}{\sqrt{c}}$ for $c > 0$, or the hyperbolic plane $\mathbb{H}_c^2$ for $c < 0$ (the imaginary sphere of radius Ri). We obtain the following result, which coincides, for $c = 0$, with Theorem 1.

Theorem 2 *Let K be a strongly convex set in X_c^2, if $c \geq 0$, or strongly h-convex set if $c < 0$, with smooth regular boundary $M = \partial K$. Then*

$$\int_M \tan_c\left(\frac{\rho(s)}{2}\right) ds = \mathbf{F} - \mathbf{F_e},$$

where ds signifies arclength measure on M, $\mathbf{F}$ is the area of K and $\mathbf{F_e}$ is the (algebraic) area enclosed by the focal set $F(M)$ of M.

6. HEINZE-KARCHER IN SPACE FORMS

Theorem 3 (with E. Gallego and E. Teufel) *Let K be a strongly convex set in X_c^3 (strongly h-convex if $c < 0$) with smooth boundary $M = \partial K$ and volume $\mathbf{V}$. Then*

$$\mathbf{V} \leq \int_M \frac{V(\rho_H)}{F(\rho_H)} dM_x \tag{2}$$

where $V(\rho_H)$ and $F(\rho_H)$ are the volume and area of the sphere of radius $\rho_H(x)$, the mean curvature radius of M at x.

For $c = 0$ is the classical Heinze and Karcher inequality.

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