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On some relations between the perimeter, the area and the visual angle of a convex set

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Abstract: We establish some relations between the perimeter, the area and the visual angle of a planar compact convex set. Our first result states that Crofton's formula is the unique universal formula relating the visual angle, the length and the area. After that we give a characterization of convex sets of constant width by means of the behavior of their isotopic sets at infinity. Also for this class of convex sets we prove that the existence of an isotopic circle is enough to ensure that the considered set is a disc.

Keywords: Convex set, visual angle, isotopic set.

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1 Introduction

In this paper we establish some relations between geometric quantities associated to a planar compact convex set K with boundary of class $\mathcal{C}^2$. More precisely we consider the perimeter L, the area F of K and the visual angle $w = w(P)$ of K from a point $P \notin K$. Remember that $w(P)$ is the angle between the two tangents to the boundary of K from the point P.

The starting point is the classical Crofton formula

$$\int_{P \notin K} (w - \sin w) dP = \frac{1}{2} L^2 - \pi F. \tag{1}$$

This equality easily follows from standard arguments of Integral Geometry, see for instance [5]. Another approach to (1) is given in [1]. The natural question arises whether separate formulas for L^2, F alone exist, or equivalently, whether replacing $w - \sin w$ by some other function $f(w)$ one gets a different linear combination of L^2, F. For specific domains this is indeed possible, for instance when K is a disc one has several formulas of this type, like

$$L^2 = \int_{P \notin K} \frac{4}{3} \sin^3 w \, dP, \qquad 4\pi F = \int_{P \notin K} \frac{\sin^3 w}{\cos^2(w/2)} \, dP,$$

see [5], p. 59.

Our first result is

Theorem 1. *Let $f : [0, \pi] \to \mathbb{R}$ be a differentiable function with $f(w) = O(w^3)$ for $w \to 0$ such that for every compact convex set K one has*

$$\int_{P \notin K} f(w(P)) dP = aL^2 + bF, \tag{2}$$

where $a, b \in \mathbb{R}$ are some constants not depending on K, and $w(P)$ is the visual angle of K from the point P. Then f is, up to a constant factor λ, the Crofton function $f(\omega) = \omega - \sin \omega$. In this case $a = \lambda/2$, $b = -\pi\lambda$.

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A consequence of this result is that no integral formula of this type exists giving, say, the area F in terms of the visual angle alone. We believe, though, that a formula like

$$F = \int_{P \notin K} f(w(P))g(P)\, dP,$$

with f, g universal functions not depending on K might exist.

By the result in [3], K is completely determined by the visual angle $w(P)$ outside a big ball containing K. It is then natural to ask how specific properties of K can be read from this knowledge.

In our next results, we do so in terms of the asymptotic behavior of $w(P)$ as P goes to infinity. Now, it is easy to see that, denoting by $w(R, \theta)$ the visual angle of K from the point with polar coordinates (R, θ), the quantity $R\, w(R, \theta)$ remains bounded and

$$\lim_{R\to\infty} R\, w(R, \theta) = a(\theta), \tag{3}$$

where $a(\theta)$ is the width of K in the direction θ, meaning that the knowledge of the asymptotic behavior of w at infinity amounts to know the width function. Let us recall that $a(\theta)$ is the distance between two parallel lines of slope θ tangent to the boundary of K.

Whence, asymptotic statements can just involve quantities that depend only on the width function. For instance, the perimeter of the convex set, which is related to the width by the Formula $2L = \int_0^{2\pi} a(\theta)\, d\theta$, can be known from the behavior of the visual angle at infinity as the following result states:

Theorem 2. *Let K be a compact convex set of perimeter L and denote by $w(R, \varphi)$ the visual angle of K from the point $P(R, \varphi)$. Then*

$$2L = \lim_{R\to\infty} R \int_0^{2\pi} w(R, \varphi)\, d\varphi. \tag{4}$$

For centrally symmetric convex sets, that is convex sets that are symmetric with respect to some point, the area is determined by the width, and this enables us to obtain, in the spirit of (4), the following formula

$$\lim_{R\to\infty} \int_0^{2\pi} R^2[w(R, \theta)^2 - w_\theta(R, \theta)^2]\, d\theta = 8F, \tag{5}$$

where w_θ means partial derivative of w with respect to θ. In fact this equality characterizes centrally symmetric convex sets (see Corollary 1 and Remark 1).

After that we deal with compact convex sets of constant width for which we obtain two results. To state the first one we note that formula (3) says that the convex set K has constant width a if and only if the visual angle w behaves like a/R at infinity, which in turn roughly says that the isotopic sets $C(\alpha) = \{P : w(P) = \alpha\}$ behave like the circles of radius $R = a/\alpha$. Our result provides a precise quantitative formulation of this fact by establishing that $C(\alpha)$ tends to a circle in the sense that it tends to satisfy the equality in the isoperimetric inequality. More precisely, we say that $C(\alpha)$ tends to a circle as $\alpha \to 0$, if

$$\lim_{\alpha\to 0} \frac{L(\alpha)^2}{4\pi F(\alpha)} = 1,$$

where $L(\alpha)$, $F(\alpha)$ denote respectively the length and the area of $C(\alpha)$.

The result we obtain is the following one.

Theorem 4. *Let K be a compact convex set. Then the isotopic sets C_α of K tend to a circle as $\alpha \to 0$, if and only if K is of constant width.*

The second result deals with isotopic sets that are actually circles, not just asymptotically. Green [2] proved that if K has an isotopic circle C_α, with α an irrational multiple of π, or $\pi - \alpha = (m/n)\pi$, m even, m and n relatively prime, then K is a disc. In general K can have an isotopic circle without being a disc. Later on Nitsche [4] proved that if K has two concentric isotopic circles, then K is a disc. We prove that if K has constant width, then one isotopic circle is enough:

Theorem 5. *If a compact convex set K of constant width has an isotopic circle, then K is a disc.*

At the end we give the relationship between the area F and the perimeter L of a convex set and the radius of an isotopic circle, see (17) and (19). As a consequence we obtain the inequalities

$$F \leq \pi R^2 \sin^2(\frac{\alpha}{2}), \quad L \leq 2\pi R \sin(\frac{\alpha}{2}),$$

where R is the radius of the isotopic circle and α is the constant visual angle on this circle.

2 On Crofton's Function

As mentioned in the Introduction, Crofton's formula (1) is proven with standard arguments of integral geometry, but it is also a direct consequence of the general formula for integrating functions of the visual angle given in [1]. Concretely, Equation (16) in [1] says that for a differentiable function $f : [0, \pi] \to \mathbb{R}$ satisfying $f(w) = O(w^3)$ for $w \to 0$, one has

$$\int_{P \notin K} f(\omega) dP = -f(\pi)F + \frac{L^2}{2\pi} M(f) + \pi \sum_{k \geq 2, \text{ even}} c_k^2 \Big(M(f) + 2 \sum_{j=1, \text{ odd}}^{k-1} \alpha_j \Big) + \pi \sum_{k \geq 3, \text{ odd}} c_k^2 \Big(-2 \sum_{j=2, \text{ even}}^{k-1} \alpha_j \Big), \tag{6}$$

with

$$\alpha_j = \int_0^\pi f'(\omega) j \cos(j\omega) d\omega, \qquad M(f) = \int_0^\pi \frac{f'(\omega)}{1 - \cos\omega} d\omega,$$

and $c_k^2 = a_k^2 + b_k^2$, where a_k, b_k are the Fourier coeficients of the support function $p(\varphi)$ of the compact convex set K. Recall that

$$p(\varphi) = \sup\{\langle x, u \rangle : x \in K\} \quad \text{for } u = (\cos\varphi, \sin\varphi).$$

Note that up to a constant, α_j is the j-th Fourier coefficient of f' in the basis $\{\cos jw\}$. The above equality for $f(w) = w - \sin w$ gives immediately Crofton's formula.

We shall prove now that the function $w - \sin w$ is the only one that can provide a Crofton type formula.

Theorem 1. *Let $f : [0, \pi] \to \mathbb{R}$ be a differentiable function with $f(w) = O(w^3)$ for $w \to 0$, such that for every compact convex set K one has*

$$\int_{P \notin K} f(w(P)) dP = aL^2 + bF, \tag{7}$$

where $a, b \in \mathbb{R}$ are some constants not depending on K, and $w(P)$ is the visual angle of K from the point P. Then f is, up to a constant factor λ, the Crofton function $f(\omega) = \omega - \sin\omega$. In this case $a = \lambda/2$, $b = -\pi\lambda$.

Proof. We consider the family of convex sets given by the support functions

$$p(\varphi) = 1 + t\cos(m\varphi), \qquad 0 \leq \varphi \leq 2\pi, \tag{8}$$

for $m \in \mathbb{N}$. The condition of convexity $p + p'' > 0$ (see [5]) is satisfied if $0 < t < \frac{1}{m^2-1}$. Then, the perimeter L and the area F of these convex sets are

$$L = \int_0^{2\pi} p \, d\varphi = 2\pi, \qquad F = \frac{1}{2} \int_0^{2\pi} (p^2 - p'^2) \, d\varphi = \pi - \frac{\pi}{2}(m^2 - 1)t^2.$$

Now we combine (6) and (7) to obtain, with m even,

$$a4\pi^2 + b(\pi - \frac{\pi}{2}(m^2 - 1)t^2) = -f(\pi)(\pi - \frac{\pi}{2}(m^2 - 1)t^2) + 2\pi M(f) + \pi M(f)t^2 + 2\pi(\alpha_1 + \alpha_3 + \cdots + \alpha_{m-1})t^2.$$

Equating the coefficients of t^2 we obtain

$$(m^2 - 1)(b + f(\pi)) + 2M(f) + 4(\alpha_1 + \alpha_3 + \cdots + \alpha_{m-1}) = 0.$$

Replacing m by $m + 2$ gives

$$((m+2)^2 - 1)(b + f(\pi)) + 2M(f) + 4(a_1 + a_3 + \cdots + a_{m-1} + a_{m+1}) = 0.$$

Subtracting these last two equalities we get for $m \geq 2$, m even,

$$b + f(\pi) = -\frac{a_{m+1}}{m+1} = -\int_0^\pi f'(w)\cos((m+1)w)\,dw.$$

From the Riemann–Lebesgue lemma it follows that $b + f(\pi) = 0$, and $a_j = 0$ for j odd, $j \geq 3$.

For odd m, we have from (6)

$$a4\pi^2 + b(\pi - \frac{\pi}{2}(m^2-1)t^2) = -f(\pi)(\pi - \frac{\pi}{2}(m^2-1)t^2) + 2\pi M(f) - 2\pi(a_2 + a_4 + \cdots + a_{m-1})t^2.$$

Equating the coefficients of t^2 and using that $b + f(\pi) = 0$ we obtain

$$a_2 + a_4 + \cdots + a_{m-1} = 0, \quad m \geq 3$$

and so $a_j = 0$ for j even, $j \geq 2$.

Hence

$$f'(\omega) = a_0 + a_1\cos(\omega) \quad \text{and} \quad f(\omega) = a_0\,\omega + a_1\sin(\omega) + c$$

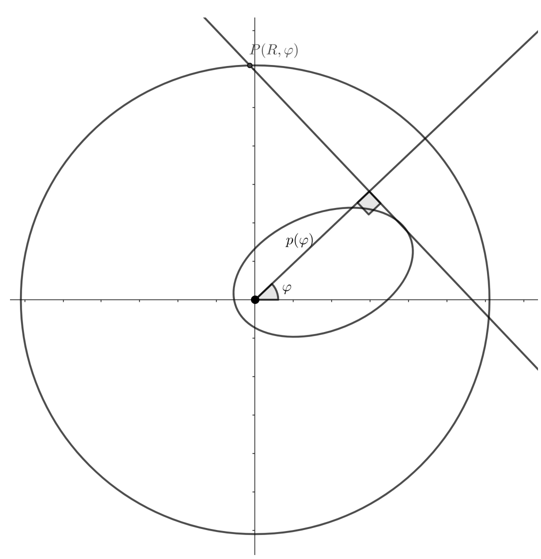
for some constants a_0, a_1, c. Since $f(w) = O(w^3)$ for $w \to 0$, we get $f(0) = f'(0) = 0$ and so $c = 0$, $a_0 + a_1 = 0$. This proves the theorem with $\lambda = a_0$. □

3 Behavior of the visual angle at infinity

The goal of this section is to obtain information about a convex set by observing it from a point that goes to infinity.

First of all we will see that the perimeter of a convex set can be evaluated by integrating the visual angle on circles of increasing radius.

The circle C_R centered at the origin with radius R can be parametrized by means of the support function $p(\varphi)$ of K in the following way. To each value of φ one associates the point $P(R,\varphi)$ given by the intersection of C_R with the half straight line, taken in the direct sense, of slope $\varphi + \pi/2$, and at distance $p(\varphi)$ of the origin.



Then we have

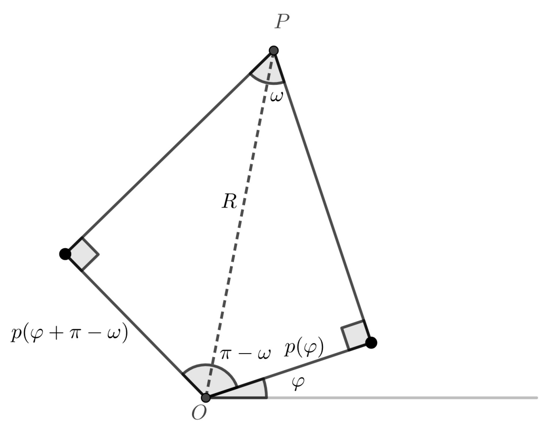
Theorem 2. *Let K be a compact convex set of perimeter L and denote by $w(R,\varphi)$ the visual angle of K from the point $P(R,\varphi)$. Then*

$$2L = \lim_{R\to\infty} R \int_0^{2\pi} w(R,\varphi)\, d\varphi.$$

Proof. The visual angle $w = w(R,\varphi)$ satisfies the fundamental relation

$$\arccos\frac{p(\varphi)}{R} + \arccos\frac{p(\varphi+\pi-w)}{R} = \pi - w(R,\varphi), \quad 0 \le \varphi \le 2\pi,$$

for every $R > 0$ such that C_R contains K, where $p(\varphi)$ is the support function of K (see the picture).



From this equation it follows that

$$p^2 + p_1^2 + 2pp_1\cos(w) = R^2\sin^2(w), \tag{9}$$

where $p = p(\varphi)$, $p_1 = p(\varphi+\pi-w)$. Then from (9) we have that $\lim_{R\to\infty} w(R,\varphi) = 0$ and

$$\lim_{R\to\infty} R\, w(R,\varphi) = \lim_{R\to\infty} R\sin(w(R,\varphi)) = a(\varphi), \quad 0 \le \varphi \le 2\pi, \tag{10}$$

where $a(\varphi) = p(\varphi) + p(\varphi+\pi)$ is the width of K in the direction φ.

The limit in (10) is uniform in φ. In fact, $w(R,\varphi) \le \tilde{w}(R)$ where $\tilde{w}(R)$ is the visual angle of the smallest circle centered at the origin and containing K from a point at distance R from the origin. So $w(R,\varphi)$ tends to zero uniformly in φ when $R \to \infty$ and we deduce, from (9) and the uniform continuity of $p(\varphi)$, that the convergence in (10) is uniform. Then the result follows by integration in (10). □

Motivated by Theorem 2 we can ask if there is an analogous result involving the area of K. We can answer this question for centrally symmetric compact convex sets. The basic result is

Theorem 3. *Let K be a compact convex set and let $w = w(R,\varphi)$ be the visual angle of K at the point $P(R,\varphi)$. Denote by $a(\varphi) = p(\varphi) + p(\varphi+\pi)$, where $p(\varphi)$ is the support function of K, the width of K in the direction φ. Then*

$$\lim_{R\to\infty} \int_0^{2\pi} R^2(w(R,\varphi)^2 - w_\varphi(R,\varphi)^2)\, d\varphi = \int_0^{2\pi} (a(\varphi)^2 - a'(\varphi)^2)\, d\varphi,$$

where w_φ denotes the derivative with respect to φ.

Proof. We begin by proving that

$$\lim_{R\to\infty} R\, w_\varphi(R,\varphi) = a'(\varphi) \tag{11}$$

uniformly on φ. In fact, differentiation of equation (9) with respect to φ gives

$$Rw_\varphi = \frac{R(pp' + p_1p_1' + (p'p_1 + pp_1')\cos(w))}{R^2\sin(w)\cos(w) + p_1p_1' + pp_1' + pp_1}.$$

Taking limits and according (10) we have

$$\lim_{R\to\infty} Rw_\varphi(R,\varphi) = \lim_{R\to\infty} \frac{Ra(\varphi)a'(\varphi)}{Ra(\varphi) + p'(\varphi+\pi)a(\varphi) + p(\varphi)p(\varphi+\pi)} = a'(\varphi).$$

Since the convergence in (10) is uniform and the functions $p(\varphi)$ and $p'(\varphi)$ are uniformly continuous, the convergence in (11) is also uniform.

As a consequence,

$$\lim_{R\to\infty} R^2 \int_0^{2\pi} (w(R,\varphi)^2 - w_\varphi(R,\varphi)^2)\,d\varphi = \int_0^{2\pi} (a(\varphi)^2 - a'(\varphi)^2)\,d\varphi,$$

as we wanted to prove. □

We note that the integral on the right-hand side of the above equality is two times the area of the convex set having $a(\varphi)$ as its support function.

We can state the following relation between the width and the area of a convex set.

Proposition 1. *Let K be a compact convex set of area F. Then*

$$8F = \int_0^{2\pi} (a(\varphi)^2 - a'(\varphi)^2)\,d\varphi$$

if and only if K is centrally symmetric.

Proof. Assume first that K is centrally symmetric with respect to the origin. Then one has $p(\varphi) = p(\varphi+\pi)$, $0 \le \varphi \le 2\pi$, and so $a(\varphi) = 2p(\varphi)$ and $a^2 - a'^2 = 4(p^2 - p'^2)$ that, integrating with respect to φ gives the desired equality.

For the converse let us observe that the hypothesis is equivalent to

$$\int_0^{2\pi} (p(\varphi)p(\varphi+\pi) - p'(\varphi)p'(\varphi+\pi))\,d\varphi = 2F. \tag{12}$$

Consider the Fourier series of the support function

$$p(\varphi) = a_0 + \sum_{k\ge1} a_k\cos(k\varphi) + b_k\sin(k\varphi)$$

and recall that

$$F = \frac{L^2}{4\pi} - \frac{\pi}{2}\sum_{k\ge2}(k^2-1)c_k^2, \tag{13}$$

with $c_k^2 = a_k^2 + b_k^2$. From (12) and (13) we have

$$2\pi a_0^2 + \pi\sum_{k\ge2}(-1)^k(1-k^2)c_k^2 = \frac{L^2}{2\pi} - \pi\sum_{k\ge2}(k^2-1)c_k^2$$

and since $L = 2\pi a_0$ it follows that $c_k = 0$ for k odd, $k > 1$. Changing the origin, the support function can be written as

$$p(\varphi) = a_0 + \sum_{k \text{ even}} a_k\cos(k\varphi) + b_k\sin(k\varphi),$$

which implies $p(\varphi) = p(\varphi+\pi)$ and so K is centrally symmetric. □

Corollary 1. *Let K be a compact convex set of area F. Then, with the notation in Theorem 3, one has*

$$F = \frac{1}{8} \lim_{R\to\infty} \int_0^{2\pi} R^2(w(R,\varphi)^2 - w_\varphi(R,\varphi)^2)\, d\varphi$$

if and only if K is centrally symmetric.

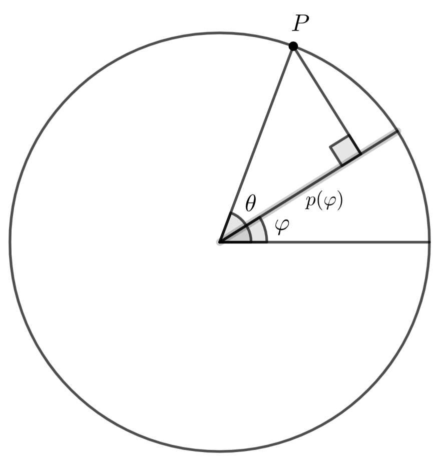
Remark 1. If we denote by $w(R,\theta)$ the visual angle of K from the point with polar coordinates (R,θ) we can consider

$$\int_0^{2\pi} Rw(R,\theta)\, d\theta$$

which is in general diferent from

$$\int_0^{2\pi} Rw(R,\varphi)\, d\varphi.$$

The relation between θ and φ is $\theta = \varphi + \arccos \frac{p(\varphi)}{R}$.



So one has

$$\int_0^{2\pi} R\omega(R,\theta)\, d\theta = \int_0^{2\pi} Rw(R,\varphi)(1 - \frac{p'(\varphi)}{\sqrt{R^2 - p(\varphi)^2}})d\varphi.$$

As a consequence, Theorem 2 gives

$$2L = \lim_{R\to\infty} \int_0^{2\pi} Rw(R,\theta)d\theta,$$

as stated in the Introduction, and analogously Corollary 1 gives

$$\lim_{R\to\infty} \int_0^{2\pi} R^2[w(R,\theta)^2 - w_\theta(R,\theta)^2]\, d\theta = 8F,$$

so that in these results we can use both polar coordinates (R,θ) and the coordinates (R,φ) associated to the convex set K.

4 A characterization of convex sets of constant width by means of isotopic sets

Given a compact convex set K we denote by C_α the (isotopic) set of points in the plane from which K is seen with angle α. In view of the isoperimetric inequality we will say that the sets C_α tend to a circle as $\alpha \to 0$, if

$$\lim_{\alpha\to 0} \frac{L(\alpha)^2}{4\pi F(\alpha)} = 1,$$

where $L(\alpha)$ is the length of C_α and $F(\alpha)$ is the area enclosed by C_α.

Theorem 4. *Let K be a compact convex set. Then the isotopic sets C_α of K tend to a circle as $\alpha \to 0$, if and only if K is of constant width.*

Proof. It is known, see for instance [1], that the points (X, Y) in C_α can be parametrized by φ as follows:

$$X = -\frac{1}{\sin\alpha}(p\sin(\varphi-\alpha) + p_1\sin\varphi)$$
$$Y = \frac{1}{\sin\alpha}(p\cos(\varphi-\alpha) + p_1\cos\varphi),$$

where $p = p(\varphi)$ is the support function of K, and $p_1 = p(\varphi + \pi - \alpha)$. Hence, the length $L(\alpha)$ of C_α is given by

$$L(\alpha) = \int_0^{2\pi} \sqrt{X'^2 + Y'^2}\, d\varphi.$$

A direct computation shows that

$$L(\alpha) = \frac{1}{\sin(\alpha)} \int_0^{2\pi} \sqrt{\Delta(\varphi, \alpha)}\, d\varphi \tag{14}$$

with

$$\Delta(\varphi, \alpha) = p^2 + p_1^2 + p'^2 + p_1'^2 + 2(pp_1 + p'p_1')\cos(\alpha) + 2(pp_1' - p'p_1)\sin(\alpha).$$

So we have

$$\lim_{\alpha\to 0} L(\alpha)\sin(\alpha) = \int_0^{2\pi} \sqrt{(p(\varphi)+p(\varphi+\pi))^2 + (p'(\varphi)+p'(\varphi+\pi))^2)}\, d\varphi = \int_0^{2\pi} \sqrt{a(\varphi)^2 + a'(\varphi)^2}\, d\varphi,$$

where $a(\varphi)$ is the width of K in the direction φ. Furthermore, the area $F(\alpha)$ enclosed by C_α satisfies (see [1])

$$\lim_{\alpha\to 0}(F(\alpha)\sin^2\alpha) = \frac{L^2}{\pi} + 2\pi \sum_{k\geq 2,\ \text{even}} c_k^2.$$

Thus

$$\lim_{\alpha\to 0} \frac{L(\alpha)^2}{4\pi F(\alpha)} = \lim_{\alpha\to 0} \frac{L(\alpha)^2\sin^2(\alpha)}{4\pi F(\alpha)\sin^2(\alpha)} = \frac{\left[\int_0^{2\pi} \sqrt{a(\varphi)^2 + a'(\varphi)^2}\, d\varphi\right]^2}{4L^2 + 8\pi^2 \sum_{k\ \text{even}} c_k^2}.$$

If K is a convex set of constant width a, $L = \pi a$ and $c_k = 0$ for k even, then

$$\lim_{\alpha\to 0} \frac{L(\alpha)^2}{4\pi F(\alpha)} = 1,$$

which proves one of the implications of the theorem.

Before looking at the converse, we check that the width $a(\varphi)$ satisfies $2\pi\int_0^{2\pi} a(\varphi)^2\, d\varphi = 4L^2 + 8\pi^2 \sum_{k\ \text{even}} c_k^2$. Indeed,

$$2\pi \int_0^{2\pi} a(\varphi)^2\, d\varphi = 4\pi \int_0^{2\pi} p(\varphi)^2\, d\varphi + 4\pi \int_0^{2\pi} p(\varphi)p(\varphi+\pi)\, d\varphi$$
$$= 4\pi\Big(2\pi a_0^2 + \pi\sum_k c_k^2\Big) + 4\pi(2\pi a_0^2 + \pi\sum_k (-1)^k c_k^2) = 4L^2 + 8\pi^2 \sum_{k\ \text{even}} c_k^2,$$

hence

$$\lim_{\alpha\to 0} \frac{L(\alpha)^2}{4\pi F(\alpha)} = \frac{[\int_0^{2\pi} \sqrt{a^2 + a'^2}\, d\varphi]^2}{2\pi \int_0^{2\pi} a^2\, d\varphi}.$$

Now assuming that the above limit is equal to 1 we have

$$\Big[\int_0^{2\pi} \sqrt{a^2 + a'^2}\, d\varphi\Big]^2 = 2\pi \int_0^{2\pi} a^2\, d\varphi. \tag{15}$$

But this equality implies that $a(\varphi)$ is constant. In fact (15) says that equality holds in the isoperimetric inequality applied to the curve given in polar coordinates by $r = a(\varphi)$. □

5 Isotopic circles

In this section we consider the particular case in which the isotopic set C_α of a compact convex set K is a circle. We will say that C_α is an isotopic circle of K.

It is known that if a compact convex set K has two concentric isotopic circles, then K is a disc, see [4]. The existence of only one isotopic circle is not enought to conclude that K is a disc, for instance all the ellipses have an isotopic circle with $\alpha = \pi/2$, see [2].

In fact we can provide a family of compact convex sets having an isotopic circle with visual angle $\alpha = \pi/2$ and different from discs or ellipses. The examples given in [2] do not have this property. To construct this family we remark that from (9) it follows that K has an isotopic circle $C_{\pi/2}$ of radius R if $p(\varphi)^2 + p(\varphi + \pi/2) = R^2$. If we write the Fourier series for the function $p(\varphi)^2$ as $p(\varphi)^2 = \sum_{-\infty}^{\infty} c_k e^{ik\varphi}$, it follows that

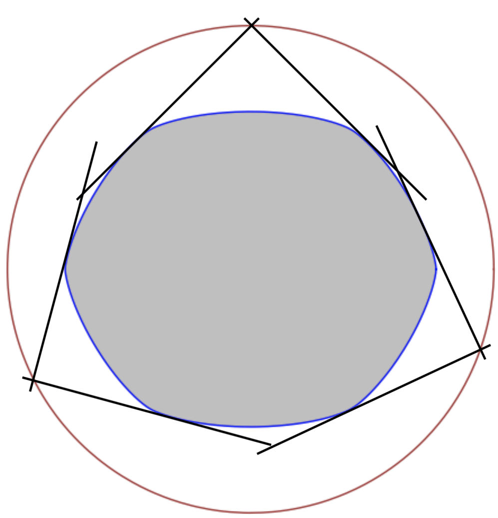
$$p(\varphi)^2 + p(\varphi + \pi/2)^2 = \sum_{-\infty}^{\infty} c_n e^{in\varphi}(1 + e^{in\pi/2}),$$

so that this quantity is constant if and only if

$$c_k = 0, \quad k \neq 2 + 4m, \quad m \text{ integer}.$$

Then, any positive 2π-periodic function $p(\varphi)$ with $p + p'' > 0$, such that the Fourier series of $p(\varphi)^2$ has only coefficients c_k with k congruent to 2 module 4, will give rise to a convex set seen from angle $\pi/2$ from a circle. For instance we can take

$$p(\varphi) = \sqrt{15 + 9\cos^2(\varphi) + 4\sin^2(\varphi) + \cos(6\varphi)}.$$



5.1 Convex sets of constant width with an isotopic circle

For compact convex sets of constant width the above quoted result in [4], about convex sets with two isotopic circles, can be improved. Concretely we have

Theorem 5. *If a compact convex set K of constant width has an isotopic circle, then K is a disc.*

Proof. Assume that K has an isotopic circle of radius R with visual angle α. Then equation (9) applied with angle $\varphi + \pi$ instead of φ gives

$$p(\varphi+\pi)^2 + p(\varphi-\alpha)^2 + 2p(\varphi+\pi)p(\varphi-\alpha)\cos(\alpha) = C,$$

where C is some constant. By the condition of constant width, $p(\varphi) + p(\varphi+\pi) = a$, one has

$$(a - p(\varphi))^2 + p(\varphi-\alpha)^2 + 2(a - p(\varphi))p(\varphi-\alpha)\cos(\alpha) = C.$$

Changing the constant one can write

$$p(\varphi)^2 - 2ap(\varphi) + p(\varphi-\alpha)^2 + 2ap(\varphi-\alpha)\cos(\alpha) - 2p(\varphi)p(\varphi-\alpha)\cos(\alpha) = C.$$

Replacing φ by $\varphi - \alpha$ and taking into account that $p(\varphi)$ is 2α-periodic (see [2]) it follows that

$$p(\varphi-\alpha)^2 - 2ap(\varphi-\alpha) + p(\varphi)^2 + 2ap(\varphi)\cos(\alpha) - 2p(\varphi-\alpha)p(\varphi)\cos(\alpha) = C.$$

Subtracting the last two equalities we obtain $2a\big(p(\varphi-\alpha) - p(\varphi) + (p(\varphi-\alpha) - p(\varphi))\cos(\alpha)\big) = 0$ and so $p(\varphi) = p(\varphi-\alpha)$, that is, $p(\varphi)$ is α-periodic.

Then equation (9) reads

$$p(\varphi)^2 + p(\varphi+\pi)^2 + 2p(\varphi)p(\varphi+\pi)\cos(\alpha) = C,$$

which together with $p(\varphi) + p(\varphi+\pi) = a$ gives

$$p(\varphi)^2 + p(\varphi)^2 - 2ap(\varphi) + 2ap(\varphi)\cos(\alpha) - 2p(\varphi)p(\varphi)\cos(\alpha) = C,$$

or

$$(2p(\varphi)^2 - 2ap(\varphi))(1 - \cos(\alpha)) = C.$$

In conclusion $p(\varphi)$ is, for $0 \le \varphi \le 2\pi$, a solution of a second degree equation $x^2 + mx + n = 0$ with $m, n \in \mathbb{R}$, and hence it is constant and K is a disc. □

In fact, this result can be considered as a consequence of Nitsche's result that assumes the existence of two isotopic circles, because in the case of constant width one of the isotopic circles is given at the infinity by Theorem 4.

5.2 Relationship between the area of a convex set and the radius of an isotopic circle

We compare the area of the convex set K with the area enclosed by an isotopic circle of K.

Theorem 6. *Let K be a compact convex set of area F that has an isotopic circle C_α of radius R. Then*

$$F \le F_R \sin^2\Big(\frac{\alpha}{2}\Big),$$

with $F_R = \pi R^2$.

Proof. In [1] it is proved that one has the equality

$$F(\alpha)\sin^2\Big(\frac{\alpha}{2}\Big) = F + \frac{\pi}{4\cos^2(\frac{\alpha}{2})}\sum_{k\ge 2}\Big(2(k^2+1)\cos^2\Big(\frac{\alpha}{2}\Big) + g_k(\alpha)\Big)c_k^2 \tag{16}$$

expressing the area $F(\alpha)$ enclosed by the isotopic set C_α of a compact convex set K in terms of the area F of K, the Fourier coefficients a_k, b_k of the support function of K ($c_k^2 = a_k^2 + b_k^2$), and Hurwitz's functions $g_k(\alpha)$ given by

$$g_k(\alpha) = 1 + \frac{(-1)^k}{2}((k+1)\cos(k-1)\alpha - (k-1)\cos(k+1)\alpha).$$

It is known from [2] that when C_α is a circle and K is not a disc then $\alpha = \pi - \frac{m}{n}\pi$ with $(m, n) = 1$ and m odd.

We will assume from now on that K is not a disk. In this case $p(\varphi)$ is $(2\pi/n)$-periodic and so $k = \mu n$, $\mu \in \mathbb{N}$, and $g_k(\alpha) = 1 + (-1)^\mu \cos(\alpha)$. So, equality (16) says

$$F_R \sin^2\Big(\frac{\alpha}{2}\Big) = F + \frac{\pi}{4\cos^2\frac{\alpha}{2}} \sum_\mu \Big(2(\mu^2 n^2 - 1)\cos^2\Big(\frac{\alpha}{2}\Big) + 1 + (-1)^\mu \cos(\alpha)\Big) c_{n\mu}^2 \tag{17}$$

and since the coefficient of $c_{n\mu}^2$ is positive for each μ, we obtain the desired inequality. □

5.3 Relationship between the perimeter of a convex set and the radius of an isotopic circle

Now we compare the perimeter of a convex set with the length of one of its isotopic circles.

Theorem 7. *Let K be a compact convex set of perimeter L that has an isotopic circle C_α of radius R. Then*

$$L \le L_R \sin\Big(\frac{\alpha}{2}\Big).$$

with $L_R = 2\pi R$.

Proof. Let us consider the Fourier series of the functions $p = p(\varphi)$ and $p_1 = p(\varphi + \pi - \alpha)$ given by

$$p = a_0 + \sum_{k\ge1} a_k \cos(k\varphi) + b_k \sin(k\varphi)$$
$$p_1 = a_0 + \sum_{k\ge1} A_k \cos(k\varphi) + B_k \sin(k\varphi)$$

with

$$A_k = (-1)^{k+1}(-a_k \cos(k\alpha) + b_k \sin(k\alpha))$$
$$B_k = (-1)^{k+1}(-a_k \sin(k\alpha) - b_k \cos(k\alpha)).$$

Then

$$\int_0^{2\pi} pp_1\, d\varphi = \int_0^{2\pi} \Big(a_0^2 + \sum_{k\ge1}(a_k A_k \cos^2(k\varphi) + b_k B_k \sin^2(k\varphi))\Big)\, d\varphi,$$

and substituting the given values of A_k, B_k, it follows

$$\int_0^{2\pi} pp_1\, d\varphi = \frac{L^2}{2\pi} - \pi \sum_{k\ge1} (-1)^{k+1} c_k^2 \cos(k\alpha). \tag{18}$$

Assuming that C_α is an isotopic circle of radius R, integrating the equality (9) on this circle and taking into account (18), we obtain

$$2L^2(1+\cos(\alpha)) + 4\pi^2 \sum_{k\ge1}(1 + (-1)^k \cos(\alpha)\cos(k\alpha))c_k^2 = L_R^2 \sin^2(\alpha),$$

where $L_R = 2\pi R$. Considering, as in the previous section, that $\alpha = \pi - \frac{m}{n}\pi$ with $(m, n) = 1$ and m odd, the above equation reads

$$L^2 + 2\pi^2 \sum_{\mu,\ \text{even}} c_{n\mu}^2 + 2\pi^2 \tan^2\Big(\frac{\alpha}{2}\Big) \sum_{\mu,\ \text{odd}} c_{n\mu}^2 = L_R^2 \sin^2\Big(\frac{\alpha}{2}\Big). \tag{19}$$

In particular we have the inequality

$$L \le L_R \sin\Big(\frac{\alpha}{2}\Big).$$ □

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