Mixing and Loss of Regularity
for Divergence-Free Velocity Fields (II)

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The work presented in these two talks (Alberti (I), Crippa (II)) stems from a conjecture by A. Bressan which states that, under certain assumptions, the "mixing scale" of the flow associated to a divergence-free velocity field decays at most exponentially in time. Despite the fact that this conjecture has already been proved in some relevant cases (see the work of G. Crippa and C. De Lellis) there are not so many examples of flows which actually exhibit an exponential decay, and until a couple of years ago all examples involved rather irregular velocity fields. In the first of these two companion talks we will illustrate some examples of exponential decay of the mixing scale, constructed in a recent work by G. Alberti, G. Crippa, and A. L. Mazzucato, together with several consequences for the geometry of regular Lagrangian flows. In particular we give an example of exponential decay given by a smooth velocity field ("smooth exponential mixer").

As a by-product, in a follow-up paper the same authors obtained examples of instantaneous loss of regularity (in the space variable) for the solutions of the continuity equation associated to (divergence-free) Sobolev velocity fields. It is well-known that for a linear continuity equation with a Lipschitz velocity field the classical Cauchy-Lipschitz theory ensures propagation in time of the (Lipschitz) regularity of the initial datum. For less regular velocity fields (for instance Sobolev or BV) a well-posedness theory for this equation is by now available, based on seminal results by DiPerna-Lions and Ambrosio, but the issue of the propagation in time of the regularity is much more delicate. In the second of the two talks we will explain how, using the (smooth) exponential mixers described in the previous talk, we can build examples of Sobolev velocity fields and smooth initial data in such a way that any fractional regularity of the solution is instantaneously destroyed.