Sobolev Homeomorphisms

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Let $\Omega \subset \mathbb{R}^n$ be a domain and let Sobolev mapping $f \in W^{1,p}(\Omega, \mathbb{R}^n)$ be a homeomorphism, i.e. continuous, f^{-1} exists and it is also continuous. Mappings from this class for $n = 2, 3$ are natural models in Nonlinear Elasticity as considered by J. Ball, V. Šverák, S. Mūller and many others.

Our domain Ω is considered to be a body in space and f is considered to be the deformation of the body into its deformed configuration $f(\Omega)$. As the deformation f is a minimizer of the elastic energy we naturally obtain that f belongs to some Sobolev space $W^{1,p}$. It is also natural to expect that the material does not break under reasonable deformation and that no cavities are created during the deformation. This corresponds to the assumption that f is continuous. Moreover, the assumption of ïnterpenetration of the matter"tells us that our deformation should be one to one. Therefore we consider the class of homeomorphisms.

During the course we will try to answer some of the following questions about Sobolev homeomorphism f:

1. Is f classically differentiable a.e?

2. Does the mapping preserve its orientation or can it turn over? Can the Jacobian $J_f(x) = \det Df(x)$ change sign?

3. Is there a set of zero measure $|N| = 0$ which is mapped to a set of positive measure $|f(N)| > 0$? This would mean that the new material is created from "nothing" and is not natural for physical deformation. The validity of this condition is closely related to the validity of change of variables formula.

4. Can the Jacobian vanish on a set of positive measure, i.e. is there a set $|A| > 0$ such that $|f(A)| = 0$? This would mean that some material is "lost" during the deformation.

5. Is the inverse deformation f^{-1} also weakly differentiable? This corresponds to the property that the deformation back to its original state is also nice.

6. Can we approximate f by a sequence of diffeomorphisms? It is well-known that there are smooth mappings f_k such that $f_k \to f$ in the Sobolev norm, but the usual convolution approximation does not produce an injective mappings f_k.