Jacobian Determinants and Subdeterminants

UAB Lecture 2, Tuesday, July 17, from 10:00 - 11:00

(Hardy Space Regularity)

Let $f:\Omega \to \mathbb{R}^n$ be a mapping in the Sobolev space $\mathscr{W}^{1,n-1}_{loc}(\Omega, \mathbb{R}^n)$ whose cofactors of $Df(x)$ belong to $L^{\frac{n}{n-1}}(\Omega)$. Does the Jacobian determinant $\det Df$ lay in the Hardy space $\mathcal{H}^1(\Omega)$?

"*Coffeeholics*" easily solve such problems once they work as "*Coffeecolleagues*"

We study mappings $f = (f^1, ..., f^n) : \Omega \to \mathbb{R}^n$ in the Sobolev space $W^{1,n-1}_{loc}(\Omega, \mathbb{R}^n)$ whose differential matrix $Df = [\partial f^i / \partial x_j]$ satisfies

$$\left| D^\sharp f \right| \in L^{\frac{n}{n-1}}(\Omega) \tag{1}$$

where $D^\sharp f$ denotes the cofactor matrix of Df. The aim of this lecture is to answer a question of Müller, Qi and Yan [MQY] by proving the following theorem.

THEOREM A . The Jacobian determinant $J(x, f) = \det Df(x)$ belongs to the Hardy space $\mathcal{H}^1(\Omega)$, and we have the uniform estimate

$$\| \det Df \|_{\mathcal{H}^1(\Omega)} \le C(n) \int_\Omega \left| D^\sharp f(x) \right|^{\frac{n}{n-1}} dx \tag{2}$$

We shall use a maximal characterization of the space $\mathcal{H}^1(\Omega)$ over a domain $\Omega \subset \mathbb{R}^n$. On account of the well known fact [FeSt] that BMO$(\mathbb{R}^n)$ is the dual space to $\mathcal{H}^1(\mathbb{R}^n)$, we obtain

COROLLARY.

Under the assumptions of the theorem, with $\Omega = \mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} \varphi(x) J(x,f) \mathrm{d}x \preccurlyeq \|\varphi\|_{BMO(\mathbb{R}^n)} \int_{\mathbb{R}^n} \left|D^\sharp f(x)\right|^{\frac{n}{n-1}} dx \tag{3}$$

where $\varphi \in$ BMO$(\mathbb{R}^n)$.

This integral in the left hand side is understood by means of $\mathcal{H}^1 -$ BMO duality, see [Stb, p.142-144].

The reader will notice that the expression in the left hand side of (2) is homogeneous of degree 1 with respect to each of the coordinate functions $f^1, f^2, ..., f^n$, whereas the right hand side is lacking this type of homegeneity. It is, therefore, natural to reformulate Estimate (2) by using the $(n-1)$-forms

$$\begin{aligned}
&df^1 \wedge ... \wedge df^{k-1} \wedge df^{k+1} \wedge ... \wedge df^n \\
&= \sum_{i=1}^{n} \frac{\partial\left(f^1, ..., f^{k-1}, f^{k+1}, ..., f^n\right)}{\partial\left(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n\right)} dx_1 \wedge ... \wedge dx_{i-1} \wedge dx_{i+1} \wedge ... \wedge dx_n
\end{aligned}$$

where the coefficients are all the cofactors of Df (sub-determinants) that are missing the coordinate function f^k.

Elementary analysis of homogeneity now leads us to the following strenthening of the estimate in Theorem A

$$\|\det\, Df\|_{\mathcal{H}^1(\Omega)} \preccurlyeq \prod_{k=1}^{n} \left\| df^1 \wedge ... \wedge df^{k-1} \wedge df^{k+1} \wedge ... \wedge df^n \right\|_{L^{p_k}(\Omega)} \quad (4)$$

where $\mathbf{p} = (p_1, ..., p_n)$ is an arbitrary n-tuple of exponents $p_1, ..., p_n > 1$, such that

$$\frac{1}{p_1} + ... + \frac{1}{p_n} = n - 1$$

In a way the $\mathcal{H}^1$-regularity of Jacobians was first observed by Wente [We], though a systematic study begins with the work by Coifman, Lions, Meyer and Semmes [CLMS1] and [CLMS2]. They proved that the Jacobian determinant, denoted here by $J(x, f) = \det Df(x)$, of a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ in the Sobolev space $\mathscr{W}^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ lies in the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$; also see Müller [Mu] for the case of nonnegative Jacobians. As a matter of fact, the Jacobian operator $J : \mathscr{W}^{1,n}(\mathbb{R}^n, \mathbb{R}^n) \to \mathcal{H}^1(\mathbb{R}^n)$ is continuous. We actually have a uniform bound

$$\| \det Df - \det Dg \|_{\mathcal{H}^1(\mathbb{R}^n)} \preccurlyeq \| Df - Dg \|_n \left(\| Df \|_n + \| Dg \|_n \right)^{n-1} \quad (5)$$

Since then there has been dramatic development in the $\mathcal{H}^1$-theory of Jacobians and related nonlinear differential forms. Most notable subsequent publications are [Gr],[CoGr], [IV] and [MQY]. In [MQY] the authors considered orientation preserving mappings in $\mathscr{W}^{1,n-1}_{loc}(\Omega, \mathbb{R}^n)$ which satisfy Condition (1). The term orientation preserving pertains to the mappings whose Jacobian determinant is nonnegative. It is shown in [MQ] that the Jacobian belongs to the Zygmund class $L \log L(\Omega')$, on every compact subdomain $\Omega' \subset \Omega$. In particular, it belongs to $\mathcal{H}^1(\Omega')$. The condition (1) was motivated by a study of the existence problems in nonlinear elasticity [B] and [Sv]. On the analogy of [CLMS2] Müller, Qi and Yan raised the question as to whether the Jacobian remains in the Hardy space $\mathcal{H}^1(\Omega')$ if one allows it to change sign. Our **Theorem A** answers this question in the affirmative. Of course, as a consequence of a result by Stein [Stp], we recover that for orientation preserving mappings $\det Df \in L \log L(\Omega')$.

The main ingredient to our arguments is the use of spherical maximal inequality, as shown in [St1] and [Bu]. This seems to be the first time that spherical maximal functions have been successfully employed in the study of Jacobians.

Combining the uniform estimate (2) with the fact that the dual of $\mathcal{H}^1(\mathbb{R}^n)$ is VMO$(\mathbb{R}^n)$ we conclude with the following convergence of Jacobians in the weak star topology of $\mathcal{H}^1(\mathbb{R}^n)$.

THEOREM B .

Suppose

$$f_j \rightharpoonup f \text{ weakly in } \mathscr{W}^{1,n-1}(\mathbb{R}^n, \mathbb{R}^n) \tag{6}$$

and

$$\int_{\mathbb{R}^n} \left| D^\sharp f_j(x) \right|^{\frac{n}{n-1}} dx \le K \tag{7}$$

for $i = 1, 2, 3, ...$ Then for every $\varphi \in \mathrm{VMO}(\mathbb{R}^n)$ we have

$$\lim_{j \to \infty} \int_{\mathbb{R}^n} \varphi(x) J(x, f_j)\ dx = \int_{\mathbb{R}^n} \varphi(x) J(x, f)\ dx \tag{8}$$

Here the functions need not be L^1-integrable, so the meaning of the integrals is understood in the sense of $\mathcal{H}^1 - \mathrm{BMO}$ pairing. Jacobians need not converge in the usual weak topology of $\mathcal{H}^1(\mathbb{R}^n)$. Precisely, (8) fails for $\varphi(x) = \log |x|$.

Maximal Operators

Let Ω be an open subset of $\mathbb{R}^n$. The Hardy-Littlewood maximal operator is defined on $L^1_{loc}(\Omega)$ by the rule

$$Mh(x) = M_\Omega h(x) = \sup \left\{ \frac{1}{|B(x,t)|} \int_{B(x,t)} |h(y)| \, dy \, ; \quad 0 < t < \mathrm{dist}(x, \partial\Omega) \right\}$$

In the above definition, $B = B(x,t)$ is the ball centered at $x \in \Omega$ and radius t. Most often the dependence of M on the domain Ω will not be emphasized. We record the following local variant of the well know maximal inequality

$$\|Mh\|_{L^p(\Omega)} \preccurlyeq \|h\|_{L^p(\Omega)} \tag{9}$$

for $1 < p \leq \infty$, where the implied constant $C_p(n) \leqslant \frac{p\ C(n)}{p-1}$.

Another important maximal operator was introduced to harmonic analysis by Stein [St1]. It involves spherical averages:

$$(\mathcal{S}h)(x) = (\mathcal{S}_\Omega h)(x) = \sup\left\{ \frac{1}{|S(x,t)|} \int_{S(x,t)} |h(y)|\, dy : 0 < t < \mathsf{dist}(x, \partial\Omega) \right\}$$

Here we use the notation $S(x,r) = \partial B(x,r)$. Notice that the integral average of $|h|$ is taken with respect to the $(n-1)$-dimensional surface area. As shown by Bourgain [Bu] for $n = 2$ and Stein [St1] in higher dimensions, the spherical maximal operator is bounded in L^p-spaces for all $p > \frac{n}{n-1}$, but not for $p = \frac{n}{n-1}$. That is,

$$\|\mathcal{S}h\|_{L^p(\Omega)} \preccurlyeq \|h\|_{L^p(\Omega)}, \qquad p > \frac{n}{n-1} \tag{10}$$

Maximal Operator $\{\mathfrak{M}_\theta\}_{\theta\geq 1}$

We shall now introduce one parameter family $\{\mathfrak{M}_\theta\}_{\theta\geq 1}$ of maximal operators

$$\mathfrak{M}_\theta h(x) = \sup \left\{ \left[\frac{n}{t^n} \int_0^t r^{n-1} \left(\frac{1}{|S(x,r)|} \int_{S(x,r)} |h(y)| \right)^\theta dr \right]^{\frac{1}{\theta}} \right\} \quad \textbf{(11)}$$

where $0 < t <$ dist$(x, \partial\Omega)$.

The Hardy-Littlewood operator is none other than $\mathfrak{M}_1$, whereas the spherical operator arises by letting θ go to infinity. In consequence of the maximal inequalities at (9) and (10) we have the following result.

Theorem 1. ***The sublinear operator***

$$\mathfrak{M}_\theta : L^p(\Omega) \to L^p(\Omega)$$

is bounded for all $p > \frac{n}{n-1+\frac{1}{\theta}}$

Proof. The case $\theta = 1$ reduces to the Hardy-Littlewood maximal inequality, so we may assume that $\theta > 1$. The condition on p can be rewritten as

$$1 - \frac{p}{\theta} < p \cdot \frac{n-1}{n} \cdot \frac{\theta - 1}{\theta}$$

Now we choose $0 < \alpha < 1$, to satisfy

$$1 - \frac{p}{\theta} < \alpha < p \cdot \frac{n-1}{n} \cdot \frac{\theta - 1}{\theta} \tag{12}$$

By Hölder's inequality we estimate the spherical averages

$$\left(\frac{1}{|S(x,r)|}\int_{S(x,r)}|h|\right)^{\theta}\leqslant$$

$$\left(\frac{1}{|S(x,r)|}\int_{S(x,r)}|h|^{\frac{\alpha\theta}{\theta-1}}\right)^{\theta-1}\left(\frac{1}{|S(x,r)|}\int_{S(x,r)}|h|^{\theta-\alpha\theta}\right)$$

$$\leqslant\quad\left(\mathcal{S}\,|h|^{\frac{\alpha\theta}{\theta-1}}\right)^{\theta-1}\frac{1}{n\,\omega_n r^{n-1}}\int_{S(x,r)}|h|^{\theta-\alpha\theta}$$

Here and in the sequel all maximal functions will be evaluated at x. Substituting this inequality into formula (11) yields

$$[\,\mathfrak{M}_{\theta}h\,]^{\theta}\leqslant\left(\mathcal{S}\,|h|^{\frac{\alpha\theta}{\theta-1}}\right)^{\theta-1}M\,|h|^{\theta-\alpha\theta}$$

Next, we use Hölder's inequality

$$
\begin{aligned}
||\mathfrak{M}_\theta h||_p &\leqslant \left\| \left(\mathcal{S}\,|h|^{\frac{\alpha\theta}{\theta-1}} \right)^{1-\frac{1}{\theta}} \right\|_{\frac{p}{\alpha}} \left\| \left(M\,|h|^{\theta-\alpha\theta} \right)^{\frac{1}{\theta}} \right\|_{\frac{p}{1-\alpha}} \\
&= \left\| \mathcal{S}\ \ |h|^{\frac{\alpha\theta}{\theta-1}} \right\|_{\frac{p\theta-p}{\alpha\theta}}^{1-\frac{1}{\theta}} \left\| M\,|h|^{\theta-\alpha\theta} \right\|_{\frac{p}{\theta-\alpha\theta}}^{\frac{1}{\theta}}
\end{aligned}
$$

Notice that (12) yields $\frac{p\theta-p}{\alpha\theta} > \frac{n}{n-1}$ **and** $\frac{p}{\theta-\alpha\theta} > 1$**. This makes it**

legitimate to apply maximal inequalities at (9) and (10).

$$\begin{aligned}\|\,\mathfrak{M}_\theta h\,\|_p &\preccurlyeq \left\| |h|^{\frac{\alpha\theta}{\theta-1}} \right\|_{\frac{p\theta-p}{\alpha\theta}}^{1-\frac{1}{\theta}} \left\| \, |h|^{\theta-\alpha\theta} \right\|_{\frac{p}{\theta-\alpha\theta}}^{\frac{1}{\theta}} \\ &= \|h\|_p^{\alpha} \; \|h\|_p^{1-\alpha} \\ &= \|h\|_p\end{aligned}$$

as desired.

$\boldsymbol{REMARK.}$ Notice that $\frac{n}{n-1}$ is always larger than $\frac{n}{n-1+\frac{1}{\theta}}$, so $\mathfrak{M}_\theta$ is always bounded in $L^{\frac{n}{n-1}}(\Omega)$.

Maximal Function of a Distribution

We shall rely on one particular approximation of the identity. That is, we fix a radially symmetric function $\Phi \in C_0^\infty(\mathbb{R}^n)$ supported in the unit ball and having integral 1. For example

$$\Phi(x) = C(n) \begin{cases} \exp \frac{1}{|x|^2-1} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

where the constant $C(n)$ is chosen so that $\int \Phi(x)\ dx = 1$. For each $t > 0$, we consider one parameter approximation to the Dirac mass $\Phi_t(x) = t^{-n}\Phi\left(\frac{x}{t}\right)$. Given $h \in L^1_{loc}(\Omega)$, we recall the mollifiers

$$(h * \Phi_t)(x) = \int_\Omega \Phi_t(x-y)h(y)\ dy\ , \quad \text{whenever } 0 < t < \text{dist}(x, \partial\Omega)$$

We mimic this convolution formula to extend the mollification procedure to Schwartz distributions $h \in \mathcal{D}'(\Omega)$ as follow

$$(h * \Phi_t)(x) = h[\Phi_t(x - \cdot)]$$

where we notice that the function $y \to \Phi_t(x-y)$ belongs to $C_0^\infty(\Omega)$ for $0 < t < \mathsf{dist}(x, \partial\Omega)$.

Then the associated maximal function of h can be defined as

$$\mathcal{M}h(x) = \mathcal{M}_\Omega h(x) = \sup\Big\{|h * \Phi_t(x)| : 0 < t < \mathsf{dist}(x, \partial\Omega)\Big\}$$

This maximal function works well in connection with $\mathcal{H}^1$-spaces because it takes the possible cancellation into account.

The Hardy Space $\mathcal{H}^1(\Omega)$

The estimate $\mathcal{M}h(x) \leqslant C(n)Mh(x)$ follows easily from the inequality

$$\left|\int_\Omega \Phi_t(x-y)h(y)\,dy\right| \leqslant \frac{C(n)}{t^n}\int_{B(x,t)} |h(y)|\,dy$$

***DEFINITION.* The Hardy space $\mathcal{H}^1(\Omega)$ consists of distributions $h \in \mathcal{D}'(\Omega)$ such that**

$$||h||_{\mathcal{H}^1(\Omega)} = \int_\Omega \mathcal{M}h(x)\;dx < \infty$$

Note that $\mathcal{H}^1(\Omega) \subset L^1(\Omega)$ and $\|\cdot\|_{L^1(\Omega)} \leqslant \|\cdot\|_{\mathcal{H}^1(\Omega)}$. For $\Omega = \mathbb{R}^n$ our definition coincides with the usual definition of the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$, see [Stb].

Various bounds for $\mathcal{M}h$ can be obtained from the following elementary calculation.

LEMMA 3.

Let $B(x,r)$ denote the ball centered at $x \in \Omega$ and radius $0 < r <$ dist$(x, \partial\Omega)$. Then

$$|h * \Phi_t(x)| \leqslant \frac{C(n)}{t^{n+1}} \int_0^t \left| \int_{B(x,r)} h \right| \, dr \tag{13}$$

Note that the absolute value is carried out upon the integration of h, which is the key to subsequent estimates.

Proof. Denote by $S(x,r) = \partial B(x,r)$. By Fubini's Theorem we obtain

$$
\begin{aligned}
(h * \Phi_t)(x) &= \int_{B(x,t)} \Phi_t(x-y)h(y)\,dy \\
&= \int_0^t \left(\int_{S(x,r)} \Phi_t(x-y)h(y)\,dy \right) dr \\
&= \int_0^t \Phi_t(r) \left(\int_{S(x,r)} h(y)\,dy \right) dr \\
&= \int_0^t \Phi_t(r) \left(\frac{d}{dr} \int_{B(x,r)} h(y)\,dy \right) dr \\
&= -\int_0^t \left[\frac{d}{dr} \Phi_t(r) \right] \left[\int_{B(x,r)} h(y)\,dy \right] dr \\
&\leqslant \frac{C(n)}{t^{n+1}} \int_0^t \left| \int_{B(x,r)} h \right| dr\,, \qquad \text{as desired}
\end{aligned}
$$

Isoperimetric Inequality

Let $f \in W^{1,n-1}_{loc}(\Omega, \mathbb{R}^n)$ satisfy Condition (1). Our main device to estimating the maximal function $\mathcal{M}J$ by the subdeterminants of Df is the following isoperimetric type inequality

$$\left| \int_{B(x,r)} J(y,f)\ dy \right| \leqslant C(n) \left(\int_{S(x,r)} \left| D^\sharp f(y) \right|\ dy \right)^{\frac{n}{n-1}} \tag{14}$$

for almost every $0 < r < \mathrm{dist}(x, \partial\Omega)$. We refer to [MQY] for the proof.

Now Lemma 3 gives

$$
\begin{aligned}
\left|\left(J * \Phi_t\right)(x)\right| &\leqslant \frac{C(n)}{t^{n+1}} \int_0^t \left(\int_{S(x,r)} \left|D^\sharp f\right|\right)^{\frac{n}{n-1}} dr \\
&\leqslant \frac{C(n)}{t^{n+1}} \int_0^t r^n \left(\frac{1}{|S(x,r)|} \int_{S(x,r)} \left|D^\sharp f\right|\right)^{\frac{n}{n-1}} dr \\
&\leqslant \frac{C(n)}{t^n} \int_0^t r^{n-1} \left(\frac{1}{|S(x,r)|} \int_{S(x,r)} \left|D^\sharp f\right|\right)^{\frac{n}{n-1}} dr \\
&\leqslant C(n) \left(\mathfrak{M}_{\frac{n}{n-1}} \left|D^\sharp f\right|\right)^{\frac{n}{n-1}} (x)
\end{aligned}
$$

Taking supremum with respect to $0 < t < \operatorname{dist}(x, \partial\Omega)$, we conclude with the pointwise inequality

$$\mathcal{M}J(x) \preccurlyeq \left[\mathfrak{M}_\theta|D^\sharp f|\right]^{\frac{n}{n-1}}(x) \qquad (15)$$

for almost every $x \in \Omega$, where $\theta = \frac{n}{n-1}$. The implied constant depends only on the dimension. Let us strenghten inequality (14) slightly as follows.

LEMMA 4.

For each $f \in W^{1,n-1}_{loc}(\Omega, \mathbb{R}^n)$ satisfying (1) and $0 < r < \mathrm{dist}(x, \partial\Omega)$ we have

$$\left|\frac{1}{|B(x,r)|}\int_{B(x,r)} J(y,f)\,dy\right| \leqslant \qquad (14')$$

$$C(n)\prod_{k=1}^{n}\left(\int_{S(x,r)}\left|df^1\wedge\ldots\wedge df^{k-1}\wedge df^{k+1}\wedge\ldots\wedge df^n\right|\right)^{\frac{1}{n-1}}$$

Proof. This result is a consequence of (14). First, we have

$$\left|D^\sharp f\right| \leqslant \sum_{k=1}^{n} \left|df^1 \wedge ... \wedge df^{k-1} \wedge df^{k+1} \wedge ... \wedge df^n\right| \tag{16}$$

Fubini's Theorem tells us that for almost every r the integrals in the right hand side of (14') are finite. In what follows we consider only such radii. Fix any positive integer l. For $k = 1, 2, ..., n$ we define the positive numbers

$$\lambda_k = \frac{\frac{1}{l} + \frac{1}{|S(x,r)|}\int_{S(x,r)} \left|df^1 \wedge ... \wedge df^{k-1} \wedge df^{k+1} \wedge ... \wedge df^n\right|}{\prod_{j=1}^{n}\left[\frac{1}{l} + \frac{1}{|S(x,r)|}\int_{S(x,r)} |df^1 \wedge ... \wedge df^{j-1} \wedge df^{j+1} \wedge ... \wedge df^n|\right]^{\frac{1}{n}}}$$

Clearly $\lambda_1 \cdots \lambda_n = 1$ and we may apply (14) to the mappings $(\lambda_1 f^1, ..., \lambda_n f^n)$ in place of $(f^1, ..., f^n)$, to obtain

$$
\begin{aligned}
&\left|\frac{1}{|B(x,r)|}\int_{B(x,r)} J(y,f)\,dy\right| \leqslant \\
\leqslant\ & C(n)\left(\frac{1}{|S(x,r)|}\int_{S(x,r)}\sum_{k=1}^{n}\lambda_k^{-1}\left|df^1\wedge\ldots\wedge\widehat{df^k}\wedge\ldots\wedge df^n\right|\right)^{\frac{n}{n-1}} \\
\preccurlyeq\ & \left(\sum_{k=1}^{n}\lambda_k^{-1}\left[\frac{1}{l}+\frac{1}{|S(x,r|)}\int_{S(x,r)}\left|df^1\wedge\ldots\wedge\widehat{df^k}\wedge\ldots\wedge df^n\right|\right]\right)^{\frac{n}{n-1}} \\
\preccurlyeq\ & \prod_{j=1}^{n}\left[\frac{1}{l}+\frac{1}{|S(x,r|)}\int_{S(x,r)}\left|df^1\wedge\ldots\wedge\widehat{df^j}\wedge\ldots\wedge df^n\right|\right]^{\frac{1}{n-1}}
\end{aligned}
$$

As usual, the terms under $\widehat{\ }$ are to be omitted.

REMARK (optional)

We should observe here that the radii of the balls for which these inequalities hold may depend on the mappings $(\lambda_1 f^1, ..., \lambda_n f^n)$. However, we are dealing with only countable number of such mappings, so we may assume that the inequalities hold for almost every $0 < r <$ dist$(x, \partial\Omega)$, and with every $l = 1, 2, ...$ Now the inequality (14') is established by letting l go to infinity.

With this version of isoperimetric inequality we can obtain a better estimate of $\mathcal{M}J(x)$. Indeed, using Inequality (14') instead of (14), we obtain

$$|(J * \Phi_t)\ (x)| \leqslant$$

$$\leqslant \quad \frac{C(n)}{t^{n+1}} \int_0^t r^n \prod_{k=1}^{n} \left(\frac{1}{|S(x,r)|} \int_{S(x,r)} |df^1 \wedge ... \wedge \widehat{df^k} \wedge ... \wedge df^n| \right)^{\frac{1}{n-1}} dr$$

$$\leqslant \quad \frac{C(n)}{t^{n}} \int_0^t r^{n-1} \prod_{k=1}^{n} \left(\frac{1}{|S(x,r)|} \int_{S(x,r)} |df^1 \wedge ... \wedge \widehat{df^k} \wedge ... \wedge df^n| \right)^{\frac{1}{n-1}} dr$$

Then, by Hölder's inequality with $p_1, ..., p_n > 0$ satisfy $\frac{1}{p_1} + ... + \frac{1}{p_1} = n - 1$.

$$\leqslant \prod_{k=1}^{n} \left[\frac{C(n)}{t^n} \int_0^t r^{n-1} \left(\frac{1}{|S(x,r)|} \int_{S(x,r)} |df^1 \wedge ... \wedge \widehat{df^k} \wedge ... \wedge df^n| \right)^{p_k} dr \right]^{\frac{1}{p_k(n-1)}}$$

Taking supremum with respect to $0 < r < \text{dist}(x, \partial\Omega)$ we conclude with the pointwise inequality between maximal functions

$$\mathcal{M}J(x) \preccurlyeq \prod_{k=1}^{n} \left[\mathfrak{M}_{p_k}\big(df^1 \wedge ... \wedge df^{k-1} \wedge df^{k-1} \wedge ... \wedge df^n\big)\right]^{\frac{1}{n-1}} \tag{17}$$

where $\frac{1}{p_1} + ... + \frac{1}{p_1} = n - 1$. The implied constant depends only on the dimension.

Proof of Theorems A

We estimate the L^1-norm of $\mathcal{M}J$ by using inequality (15)

$$\begin{aligned}
\|\mathcal{M}J\|_{L^1(\Omega)} &\leqslant C(n)\,\big\|\mathfrak{M}_\theta|D^\sharp f|\big\|^{\frac{n}{n-1}}_{L^{\frac{n}{n-1}}(\Omega)} \\
&\leqslant C(n)\,\big\|D^\sharp f\big\|^{\frac{n}{n-1}}_{L^{\frac{n}{n-1}}(\Omega)} = C(n)\int_\Omega \big|D^\sharp f\big|^{\frac{n}{n-1}}
\end{aligned}$$

Here, of course, we have used boundednes of the operator $\mathfrak{M}_\theta$ in $L^{\frac{n}{n-1}}(\Omega)$. This completes the proof of Theorem A.

We now proceed to the proof of inequality (4). We integrate both side of (17) and use Hölder's inequality to obtain

$$
\begin{aligned}
&\left(\int_\Omega \mathcal{M}J(x)\,dx\right)^{n-1}\\
\leqslant\ & \prod_{k=1}^{n}\left(C(n)\int_\Omega \left[\mathfrak{M}_{p_k}(df^1\wedge \ldots \wedge \widehat{df^k}\wedge \ldots \wedge df^n)\right]^{p_k}\right)^{\frac{1}{p_k}}\\
\leq\ & C_{\mathbf{p}}(n)\left(\int_\Omega \left|df^1\wedge \ldots \wedge \widehat{df^k}\wedge \ldots \wedge df^n\right|^{p_k}\right)^{\frac{1}{p_k}}
\end{aligned}
$$

This last inequality follows from the fact that the sublinear operators $\mathfrak{M}_{p_k}$ are bouded from $L^{p_k}(\Omega)$ to $L^{p_k}(\Omega)$, provided $p_k > 1$ for $k = 1, \ldots, n$, see Theorem 1. The proof of inequality (4) is complete.

Proof of Theorem B

First observe that Inequality (7) remains valid for the limit mapping, that is:

$$\int_{\mathbb{R}^n} \left| D^\sharp f(x) \right|^{\frac{n}{n-1}} dx \leq K \tag{18}$$

This follows from the well know fact that $D^\sharp f_j \to D^\sharp f$ in $\mathcal{D}'(\Omega)$. In particular, by Theorem A, we have

$$\|\text{det } Df_j\|_{\mathcal{H}^1} + \|\text{det } Df\|_{\mathcal{H}^1} \leq C(n)\ K \tag{19}$$

for all $j = 1, 2, ...$

Next we recall that the dual of $\mathcal{H}^1(\mathbb{R}^n)$ is BMO$(\mathbb{R}^n)$ (C. Fefferman, Bulletin of AMS 1971), see also (C. Fefferman & E. Stein, Acta Math. 1972). On the other hand $\mathcal{H}^1(\mathbb{R}^n)$ is dual of VMO$(\mathbb{R}^n) \subset$ BMO$(\mathbb{R}^n)$. The $BMO-\mathcal{H}^1$ pairing is customary denoted by

$$\varphi[h] = \int_{\mathbb{R}^n} \varphi(x)h(x)\ dx$$

Here the integral just signifies the action of φ on h or h on φ, respectively. These actions coincide with the integral formula if $\varphi \in C_0^\infty(\Omega)$.

For any $\epsilon > 0$ we can find a test function $\varphi_\epsilon \in C_0^\infty(\Omega)$ such that $||\varphi - \varphi_\epsilon||_{BMO} \leqslant \epsilon$, because $C_0^\infty(\Omega)$ is dense in VMO$(\mathbb{R}^n)$. To simplify notation it will be convenient to write $J_k = J(x, f_k)$ and $J = J(x, f)$.

Now

$$\varphi[J_k] - \varphi[J] = \varphi_\epsilon[J_k] - \varphi_\epsilon[J] + (\varphi - \varphi_\epsilon)[J_k] - (\varphi - \varphi_\epsilon)[J] \qquad (20)$$

It follows from (3), (19) and $\lim_{k\to\infty} \varphi_\epsilon[J_k] = \varphi_\epsilon[J]$ that

$$\begin{aligned} \lim_{k\to\infty} \sup |\varphi[J_k] - \varphi[J]| &\leqslant \lim_{k\to\infty} \sup \|\varphi - \varphi_\epsilon\|_{BMO} \left(\|J_k\|_{\mathcal{H}^1} + \|J\|_{\mathcal{H}^1}\right) \\ &\leqslant C(n)\ K\ \epsilon \end{aligned}$$

As ϵ was arbitrary we conclude that

$$\lim_{k\to\infty} \varphi[J_k] = \varphi[J]$$

completing the proof of Theorem B.

In Theorem B, the space VMO$(\mathbb{R}^n)$ cannot be replaced by BMO$(\mathbb{R}^n)$ (optional)

In other words, the sequence of Jacobians in our example will not converge weakly in $\mathcal{H}^1(\mathbb{R}^n)$.

EXAMPLE. There is a sequence $\{f_k\}$ bounded in $W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$, converging uniformly to zero, and such that

$$\liminf_{k \to \infty} \int_{\mathbb{R}^n} \left(\log \frac{1}{|x|} \right) J(x, f_k)\, dx \geqslant \frac{1}{n} \tag{21}$$

This example is of course stronger than we need. Uniform convergence together with boundedness in $W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ implies that $f_k \rightharpoonup 0$ weakly in

$W^{1,n-1}(\mathbb{R}^n, \mathbb{R}^n)$. Moreover, we have uniform bound

$$\sup_{k\geq 1} \int_{\mathbb{R}^n} \left|D^\sharp f_k\right|^{\frac{n}{n-1}} \leq \sup_{k\geq 1} \int_{\mathbb{R}^n} |Df_k|^{\frac{n}{<}} \infty$$

Note that the function $\log|x|$ lies in BMO$(\mathbb{R}^n)$ but not in VMO$(\mathbb{R}^n)$.

Construction. Given any positive integer k we consider a radial stretching in the ball $B = \{x \in \mathbb{R}^n : |x| \leq \frac{1}{e}\}$.

$$g_k(x) = \frac{C_k\ x}{|x|\ \ |\log|x||^{\frac{1}{n}+\frac{1}{k}}}, \tag{22}$$

where C_k is a constant determined by the equation

$$C_k = \sqrt[n]{\frac{n}{k\ \omega_{n-1}}} \tag{23}$$

An elementary computation shows that

$$|Dg_k(x)| = \frac{C_k}{|x|\ |\log|x||^{\frac{1}{n}+\frac{1}{k}}} \tag{24}$$

and

$$J(x, g_k) = \frac{\left(\frac{1}{n} + \frac{1}{k}\right)\ C_k^n}{|x|^n\ |\log|x||^{2+\frac{n}{k}}} \tag{25}$$

Using polar coordinates we compute

$$\int_B |Dg_k(x)|^n \, dx = 1 \tag{26}$$

Hence, by (24) and (25) we also have

$$\int_B \left(\log \frac{1}{|x|}\right) J\left(x, g_k\right) = \left(\frac{1}{n} + \frac{1}{k}\right) \int_B |Dg_k(x)|^n \, dx \geq \frac{1}{n} \tag{27}$$

We shall now define the desired sequence $\{f_k\}$ of mappings $f_k : \mathbb{R}^n \to \mathbb{R}^n$ by extending g_k beyond the ball B as follows

$$f_k(x) = \begin{cases} g_k(x) & \text{if } 0 < |x| \leq \frac{1}{e} \\ C_k \, x \, \eta(|x|) & \text{if } \frac{1}{e} < |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Here $\eta = \eta(t)$ is an arbitrary Lipschitz function defined on the interval $[\frac{1}{e}, 1]$, such that $\eta\left(\frac{1}{e}\right) = 1$, $\eta(1) = 0$ and $0 \leq \eta(t) \leq 1$.
Notice that

$$\lim_{k\to\infty} \int_{|x|\geq\frac{1}{e}} (\log|x|)\, J\left(x, f_k\right)\; dx = A_1 \lim_{k\to\infty} C_k^n = 0 \tag{28}$$

where the constant A_1 depends only on the choice of the function η. Combinnig this fact with inequalty (27) we conclude that

$$\lim_{k\to\infty} \inf \int_{\mathbb{R}^n} \left(\log \frac{1}{|x|}\right) J(x, f_k)\; dx \geq \frac{1}{n}$$

proving the claim at Inequality (21).

Similarly, we find

$$\int_{\mathbb{R}^n} |Df_k(x)|^n \ dx = \int_B |Dg_k(x)|^n \ dx + A_2 \ C_k^n = 1 + A_2 \ C_k^n$$

with A_2 depending only on η. In particular, the sequence $\{f_k\}$ is bounded in $W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$. The last thing to show is that

$$|f_k(x)| \leq C_k$$

for all $x \in \mathbb{R}^n$ and $k = 1, 2, ...$ Thus $f_k \to 0$ uniformly in $\mathbb{R}^n$.

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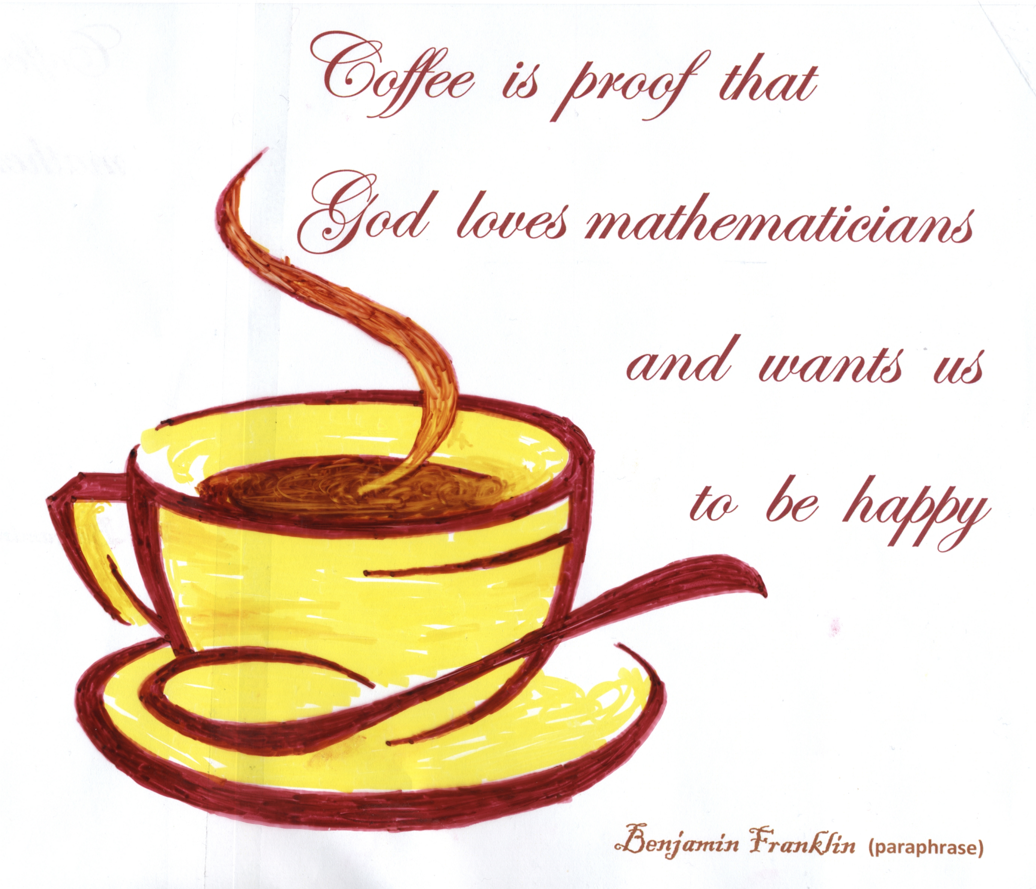

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