Free Lagrangians, Definitions, Examples and Applications

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The origin of what is termed free-Lagrangians lies in the study of traction free problems; that is, energy-minimal deformations $f : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ (usually homeomorphisms, or limits of homeomorphisms) with no boundary values prescribed up front. We say, tangential slipping along $\partial\mathbb{X}$ is allowed. This is physically realised by deforming an incompressible material confined in a given domain. In a particular way of speaking, free-Lagrangians can be described as follows.

For a pair of domains $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^n$ consider a class of Sobolev homeomorphisms $f : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ of finite energy

$$\mathfrak{L}[f] := \int_{\mathbb{X}} \mathbf{L}(x, f, Df)\,\mathrm{d}x < \infty \tag{1}$$

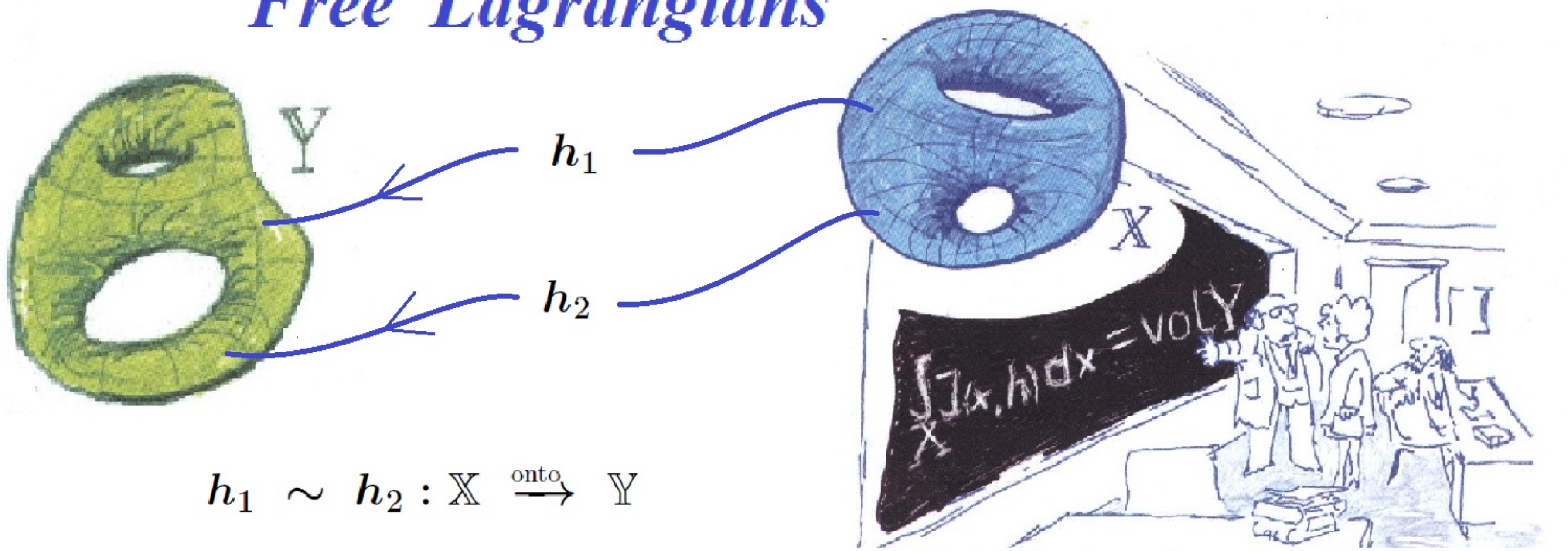
where $\mathbf{L} : \mathbb{X} \times \mathbb{Y} \times \mathbb{R}^{n \times n} \to \mathbb{R}$. The relevant conditions on $\mathbf{L}$ and the class of admissible homeomorphisms, can be specified as needed.

Definition

<u>**DEFINITION**</u> (free-Lagrangian) The term free-Lagrangian refers to a differential n-form $\mathbf{L}(x, f, Df)\mathrm{d}x$ whose integral mean $\mathfrak{L}[f]$ is constant within any homotopy class of Sobolev homeomorphisms $f : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$.

These are special null-Lagrangians.

Free Lagrangians



$$h_1 \sim h_2 : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$$

$$\int_{\mathbb{X}} \mathbb{E}(x, Dh_1)\, \mathrm{d}x = \int_{\mathbb{X}} \mathbb{E}(x, Dh_2)\, \mathrm{d}x$$

But this is only a simplified version for the general public

A trivial example of a free Lagrangian is

$$\mathbf{L}(x, f, Df)\mathsf{d}x = F(x)\,\mathsf{d}x \quad \text{where} \;\; F \in \mathscr{L}^1(\mathbb{X}) \tag{2}$$

Less trivial, though still simple, is the dual example obtained by pulling back a volume form on $\mathbb{Y}$ given by $\omega = G(y)\,\mathsf{d}y$, with $G \in \mathscr{L}^1(\mathbb{Y})$, via homeomorphisms $f \in \mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$

$$\mathbf{L}(x, f, Df)\mathsf{d}x = G(y)\, J(x, h)\mathsf{d}x \tag{3}$$

$$\int_{\mathbb{X}} \mathbf{L}(x, f, Df)\mathsf{d}x \;=\; \pm \int_{\mathbb{Y}} G(y)\,\mathsf{d}y \;, \quad \text{independent of } f$$

The idea of pulling back a differential form on $\mathbb{Y}$ by a homeomorphisms $h : \mathbb{X} \overset{\text{onto}}{\longrightarrow} \mathbb{Y}$ is effective in constructing other free-Lagrangians. Here we confine ourselves to concentric annuli $\mathbb{A} \subset \mathbb{R}^n$ and $\mathbb{A}^* \subset \mathbb{R}^n$ and homeomorphisms

$h : \mathbb{A} \overset{\text{onto}}{\longrightarrow} \mathbb{A}_*$ preserving the order of boundary components; that is, $|h(x)| = r_*$ for $|x| = r$ and $|h(x)| = R_*$ for $|x| = R$.

For mappings of class $\mathscr{W}^{1,n-1}(\mathbb{A}, \mathbb{A}^*)$ we have well defined tangential energy of $h = (h^1, ..., h^n)$ by the rule

$$\mathbf{L}_T(x, h, Dh)\,\mathsf{d}x = \sum_{i=1}^{n} \frac{h^i dh^1 \wedge ... \wedge dh^{i-1} \wedge d|x| \wedge dh^{i+1} \wedge ... \wedge dh^n}{|x|\,|h|^n} \tag{4}$$

$$\mathscr{E}_T[h] = \int_{\mathbb{A}} \mathbf{L}(x, h, Dh)\,\mathsf{d}x = \omega_{n-1} \log \frac{R}{r} = \omega_{n-1} \mathsf{Mod}\,\mathbb{A} \tag{5}$$

A dual example is furnished by the normal energy of h of the Sobolev class $\mathscr{W}^{1,1}(\mathbb{A}, \mathbb{A}^*)$

$$\mathbf{L}_N(x, h, Dh)\, \mathrm{d}x = \sum_{i=1}^{n} \frac{x^i dx^1 \wedge \dots \wedge dx^{i-1} \wedge d|h| \wedge dx^{i+1} \wedge \dots \wedge dx^n}{|h|\, |x|^n} \tag{6}$$

$$\mathscr{E}_N[h] = \int_{\mathbb{A}} \mathbf{L}_N(x, h, Dh)\, \mathrm{d}x = \omega_{n-1} \log \frac{R_*}{r_*} = \omega_{n-1} \mathsf{Mod}\, \mathbb{A}^* \tag{7}$$

The duality between tangential and normal free-Lagrangians is emphasized by the following identity $\mathscr{E}_T[h] = \mathscr{E}_N[f]$, where $f : \mathbb{A} \overset{\text{onto}}{\longrightarrow} \mathbb{A}$ is the inverse of h.

Schottky's Theorem

Via free-Lagrangians

The study of conformal deformations of annuli and more general multiply connected domains goes back to the doctoral dissertation of F. H. Schottky, a student of Weierstrss in Berlin.
By the Riemann mapping theorem, annuli are the first place one meets nontrivial conformal invariants such as moduli-obstructions to the existence of conformal mappings.

THEOREM (F.H. Schottky, 1877)
An annulus $\mathbb{A} = A(r, R)$ can be mapped conformally onto the annulus $\mathbb{A}^* = A(r_*, R_*)$ if an only if

$$\text{Mod}\,\mathbb{A} := \log\frac{R}{r} \; = \; \log\frac{R_*}{r_*} =: \text{Mod}\,\mathbb{A}_*$$

Moreover, every conformal map $h : \mathbb{A} \overset{\text{onto}}{\longrightarrow} \mathbb{A}^*$ takes the form

$$h(z) = \lambda z^{\pm 1}, \quad \text{where} \quad \lambda \in \mathbb{C}$$

$$|\lambda| = \frac{r_*}{r} \quad \text{or} \quad |\lambda| = r_* R \;, \; \text{ as the case may be.}$$

The underlying ideas differ very much from those used in the extremal length method. Other approaches can be based around the reflection principle given knowledge of the conformal authomorphisms of $\mathbb{C}$, something one meets in any first course in complex analysis, usually as an application of Schwarz lemma. We prove the theorem without any reference to analytic functions and we do not appeal to any significant part of the theory of PDEs. In fact the Cauchy-Riemann equations are used only as to define conformality of the map $h : \mathbb{A} \overset{\text{onto}}{\longrightarrow} \mathbb{A}^*$. Yet we curry out these arguments under minimal regularity hypotheses, that is in the Sobolev class $\mathscr{W}^{1,1}_{\text{loc}}(\mathbb{A}, \mathbb{A}^*)$. The reader satisfied with $\mathscr{C}^1$-deformations of annuli may skip adjustments necessary to fit the computation to Sobolev mappings.

Free Lagrangians for a pair of planar annuli

Let $\mathbb{A} = A(r,R)$ and $\mathbb{A}^* = A(r_*,R_*)$ be two circular annuli in $\mathbb{C}$. We shall work with one particular homotopy class of $\mathscr{W}^{1,2}$-homeomorphisms $h : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$. Let $\mathscr{F}(\mathbb{A},\mathbb{A}^*)$ be the class of orientation preserving homeomorphisms $h : \mathbb{A} \to \mathbb{A}^*$ in the Sobolev space $\mathscr{W}^{1,2}_{\mathrm{loc}}(\mathbb{A},\mathbb{A}^*)$ which also preserve the order of the boundary components; that is, $|h(z)| = r_*$ for $|z| = r$ and $|h(z)| = R_*$ for $|z| = R$.

In this context a free Lagrangian refers to a differential 2-form $L(x,h,Dh)\,\mathrm{d}x$, whose integral mean over $\mathbb{A}$ does not depend on a particular choice of the mapping $h \in \mathscr{F}(\mathbb{A},\mathbb{A}^*)$. Naturally, polar coordinates

$$x = \rho e^{i\theta}\,, \quad r < \rho < R \quad \text{and} \quad 0 \leqslant \theta < 2\pi \tag{8}$$

are best suited for dealing with mappings of planar annuli. The radial and tangential derivatives of $h : \mathbb{A} \to \mathbb{A}^*$ are defined by

$$h_N(x) = h_\rho(x) = \frac{\partial h(\rho e^{i\theta})}{\partial \rho}, \qquad \rho = |x| \tag{9}$$

and

$$h_T(x) = \frac{h_\theta}{\rho} = \frac{1}{\rho}\frac{\partial h(\rho e^{i\theta})}{\partial \theta}, \qquad \rho = |x| \tag{10}$$

For a general Sobolev mapping we have the formula

$$J(x,h) = \mathrm{Im}\,(h_T \overline{h_N}) \leqslant |h_T|\,|h_N| = \frac{|h_\theta|\,|h_N|}{\rho} \tag{11}$$

We shall make use of three free Lagrangians.

I. Pullback of a form in $\mathbb{A}^*$ via a given mapping $h \in \mathscr{F}(\mathbb{A}, \mathbb{A}^*)$;

$$L(x, h, Dh)\, \mathsf{d}x = N(|h|)\, J(x, h)\, \mathsf{d}x\,, \quad \text{where } N \in \mathscr{L}^1(r_*, R_*) \tag{12}$$

Thus, for all $h \in \mathscr{F}(\mathbb{A}, \mathbb{A}^*)$ we have

$$\int_{\mathbb{A}} L(x, h, Dh)\, \mathsf{d}x = \int_{\mathbb{A}^*} N(|y|)\, \mathsf{d}y = 2\pi \int_{r_*}^{R_*} N(\tau)\, \tau\, \mathsf{d}\tau \tag{13}$$

II. A radial free Lagrangian

$$L(x, h, Dh)\, \mathsf{d}x = A\big(|h|\big) \frac{|h|_N}{|x|}\, \mathsf{d}x\,, \quad \text{where } \; A \in \mathscr{L}^1(r_*, R_*) \tag{14}$$

Thus, for all $h \in \mathscr{F}(\mathbb{A}, \mathbb{A}^*)$ we have

$$\int_{\mathbb{A}} L(x,h,Dh)\,\mathrm{d}x = 2\pi \int_{r_*}^{R_*} A(\tau)\,\mathrm{d}\tau \tag{15}$$

III. A tangential free Lagrangian

$$L(x,h,Dh) = B\big(|x|\big)\,\mathrm{Im}\frac{h_T}{h}\,, \quad \text{where } B \in \mathscr{L}^1(r,R) \tag{16}$$

Thus, for all $h \in \mathscr{F}(\mathbb{A}, \mathbb{A}^*)$ we have

$$\int_{\mathbb{A}} L(x,h,Dh)\,\mathrm{d}x = \int_r^R \frac{B(t)}{t}\left(\int_{|x|=t} \frac{\partial \mathrm{Arg}\, h}{\partial \theta}\,\mathrm{d}\theta\right) \underset{\cdot}{\mathrm{t}} = 2\pi \int_r^R \frac{B(t)}{t}\,\mathrm{d}t \tag{17}$$

Proof of Schottky's theorem

The analytic description of conformality goes via the Cauchy-Riemann equations which we may state in polar coordinates

$$\frac{1}{\rho}\frac{\partial h}{\partial \theta} \;=\; i\,\frac{\partial h}{\partial \rho}, \;\text{ equivalently }\; h_T(z) \;=\; i\,h_N(z) \;\text{ for a.e. } z = \rho\, e^{i\theta}. \qquad (18)$$

Therefore, via (11), we have

$$J(z,h) = |h_N|^2 = |h_T|^2. \qquad (19)$$

Suppose that h belongs to $\mathscr{F}(\mathbb{A},\mathbb{A}^*)$ and satisfies (19). Choosing $N(t) = t^{-2}$, $A(t) = t^{-1}$ and $B(t) = 1$ for $t > 0$ in (i)-(iii) we have

$$2\pi \log \frac{R}{r} \cdot 2\pi \log \frac{R_*}{r_*} = \int\limits_{\mathbb{A}} \frac{\mathrm{d}z}{|z|^2} \cdot \int\limits_{\mathbb{A}} \frac{J(z,h)\,\mathrm{d}z}{|h(z)|^2} \geqslant \left(\int\limits_{\mathbb{A}} \frac{\sqrt{J(z,h)}\,\mathrm{d}z}{|z|\,|h(z)|} \right)^2$$

$$= \left\{ \begin{array}{l} \left(\int\limits_{\mathbb{A}} \left| \frac{h_N}{\rho\, h} \right| \right)^2 \\ \\ \left(\int\limits_{\mathbb{A}} \left| \frac{h_T}{\rho\, h} \right| \right)^2 \end{array} \right. \geqslant \left\{ \begin{array}{l} \left(\int\limits_{\mathbb{A}} \frac{|h|_N}{\rho\, |h|} \right)^2 \\ \\ \left(\int\limits_{\mathbb{A}} \operatorname{Im} \frac{h_T}{\rho\, h} \right)^2 \end{array} \right. = \left\{ \begin{array}{l} \left(2\pi \log \frac{R_*}{r_*} \right)^2 \\ \\ \left(2\pi \log \frac{R}{r} \right)^2 \end{array} \right.$$

XXXXXXXXXXXXXXXXXXXXXXXXXX (**OPTIONAL**) Hence a necessary condition for the existence of a conformal map $h : \mathbb{A} \to \mathbb{A}^*$ is that $\operatorname{Mod} \mathbb{A} = \operatorname{Mod} \mathbb{A}^*$. Once this condition is satisfied every conformal map $h : \mathbb{A} \to \mathbb{A}^*$ must give equality in every step of the above computation. A close inspection of these inequalities reveals that

$$\frac{J(z,h)}{|h(z)|^2} = \frac{m^2}{|z|^2}, \quad \text{and} \quad \begin{cases} |\frac{h_N}{f}| = \pm \frac{h_N}{h} \\ \\ |\frac{h_T}{h}| = \pm i \, \frac{h_T}{h} \end{cases}$$

where m is a real number. The sign in each equation remains at our choice but must be the same for all points in $\mathbb{A}$. This can easily be summarized in two differential equations

$$\begin{cases} \frac{\partial h}{\partial \rho} = \frac{m}{\rho} h \\ \frac{\partial h}{\partial \theta} = i\, m h \end{cases} \qquad \text{for some constant } \; m \in \mathbb{R} \tag{20}$$

Solving these equations poses no difficulty. First the real constant m can be identified from the second equation via the argument principle as follows

$$m = \frac{\partial \operatorname{Arg} h}{\partial \theta} = \pm 1$$

The plus sign applies when h preserves the order of the boundary components and the minus sign otherwise. Now the general solution takes the form $h(z) = \lambda z^{\pm 1}$, where λ is a complex number whose modulus is uniquely determined by requiring that $|\lambda| r = r_* \text{ or } R_*$, respectively. XXX

The Art of Integrating Free Lagrangians

Given an energy functional

$$\mathscr{E}[h] = \int_{\mathbb{X}} \mathbf{E}(x, h, Dh)\,\mathrm{d}x$$

we look for the greatest lower bound of the stored energy function by means of a free Lagrangian

$$\mathbf{E}(x, h, Dh) \geqslant L(x, h, Dh) \; , \quad \text{for } \; h \in \mathscr{F}(\mathbb{X}, \mathbb{Y})$$

Equality must occur for a map $h = h_\circ$ that is intended to be the energy-minimal. Obviously

$$\mathscr{E}[h_\circ] = \inf\{\mathscr{E}[h] \, ; \, h \in \mathscr{F}(\mathbb{X}, \mathbb{Y})\,\}$$

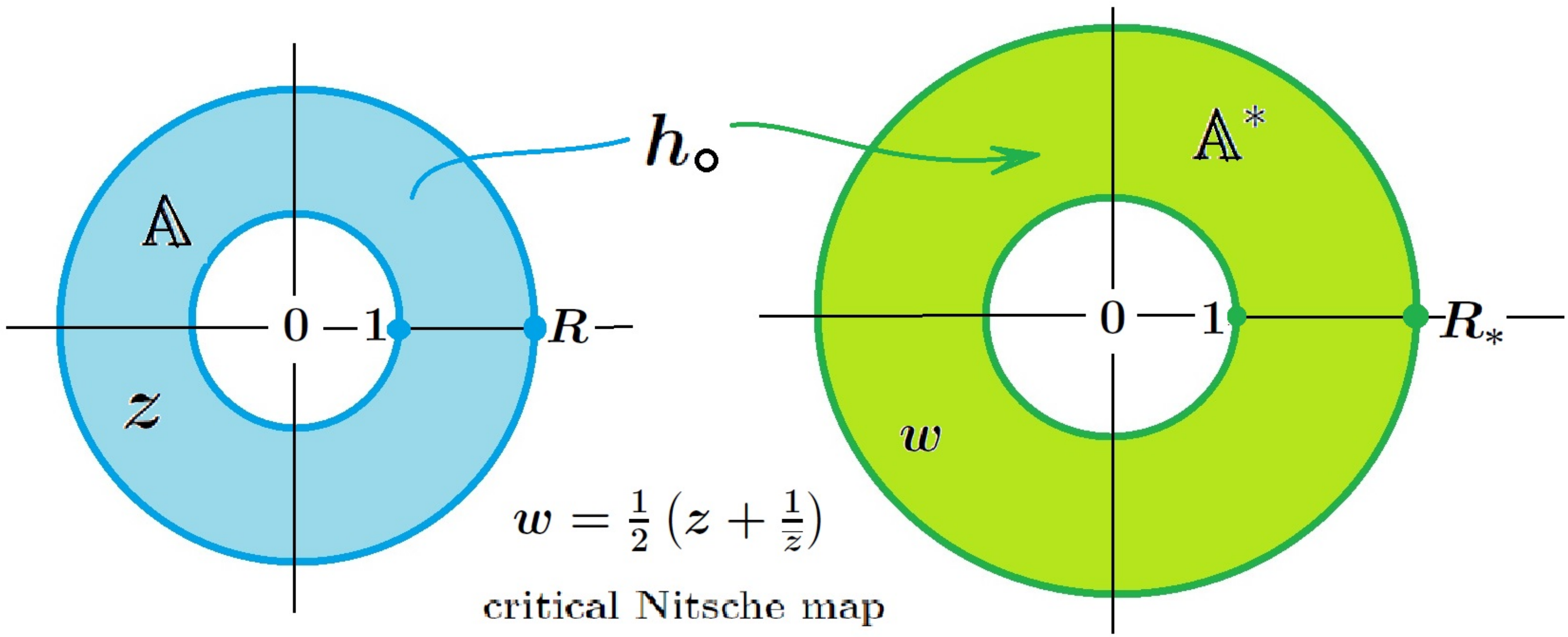
Indeed,

$$\int_{\mathbb{X}} \mathbf{E}(x,h,Dh)\,\mathsf{d}x \geqslant \int_{\mathbb{X}} L(x,h,Dh)\,\mathsf{d}x =$$

$$\int_{\mathbb{X}} L(x,h_\circ,Dh_\circ)\,\mathsf{d}x = \int_{\mathbb{X}} \mathbf{E}(x,h_\circ,Dh_\circ)$$

However, finding such a free Lagrangian $L(x,h,Dh)\,\mathsf{d}x$, for a given pair of domains $\mathbb{X},\mathbb{Y} \subset \mathbb{R}^n$ is truly a work of art.

Critical Nitsche Configuration



$$h_\circ(z) = \tfrac{1}{2}\left(z + \tfrac{1}{\overline{z}}\right) = H(\rho)e^{i\theta} \,,$$
$$H(\rho) = \tfrac{1}{2}\left(\rho + \frac{1}{\rho}\right) \,,\ 1 \leqslant \rho \leqslant R \,,$$
$$H(1) = 1 \,,$$
$$H(R) = R_* = \tfrac{1}{2}\left(R + \tfrac{1}{R}\right),$$

Proposition For every homeomorphism $h : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ of class $\mathscr{F}(\mathbb{A}, \mathbb{A}^*)$, we have

$$\int_{\mathbb{A}} |Dh|^2 \geqslant \int_{\mathbb{A}} |Dh_\circ|^2 = \frac{\pi}{2}\left(R^2 - R^{-2}\right) \qquad (21)$$

Equality occurs if and only if $h(z) = h_\circ(\lambda z)$ for some $\lambda \in \mathbb{S}^1$

Proof. Consider a point-wise inequality with parameters $0 \leqslant a \leqslant 1$ and $A \geqslant 0$

$$|Dh|^2 =$$

$$|h_\rho|^2 + \rho^{-2}|h_\theta|^2 \; = \left(|h_\rho| - \tfrac{a}{\rho}\,|h_\theta| \right)^2 + \tfrac{1-a^2}{\rho^{\,2}}|h_\theta|^2 \; + \; \tfrac{2a}{\rho}|h_\rho|\,|h_\theta|$$

$$\geqslant \tfrac{1-a^2}{\rho^{\,2}}|h_\theta|^2 \; + \; \tfrac{2a}{\rho}|h_\rho|\,|h_\theta| \; = (1-a^2)\left(\tfrac{|h_\theta|}{\rho} \; - \; A \right)^2 \; +$$

$$\tfrac{2(1-a^2)}{\rho}A|h_\theta| \; - \; (1-a^2)A^2 \; + \; \tfrac{2a}{\rho}|h_\rho|\,|h_\theta|$$

$$\geqslant \tfrac{2(1-a^2)}{\rho}A|h_\theta| \; - \; (1-a^2)A^2 \; + \; \tfrac{2a}{\rho}|h_\rho|\,|h_\theta|$$

Equality occurs if $|h_\rho| = \frac{a}{\rho}\,|h_\theta|$ and $\frac{|h_\theta|}{\rho} = A$. We want this to happen for the Nitsche map $h_\circ(\rho e^{i\theta} = H(\rho) e^{i\theta}$, which determines the parameters

$$a = \frac{\rho^2 - 1}{\rho^2 + 1} \quad \text{and} \quad A = \frac{\rho^2 + 1}{2\rho^2}$$

To figure out how to choose the parameters a and A for a general homeomorphism h, we express the above formulas in terms of $|h_\circ|$ and ρ; namely,

$$a = \frac{\sqrt{|h_\circ|^2 - 1}}{|h_\circ|} \quad \text{and} \quad A = \frac{|h_\circ|}{\rho}$$

Accordingly, for a given homeomorphism $h : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}_*$ we define the parameters by the rules:

$$a = \frac{\sqrt{|h|^2 - 1}}{|h|} \quad \text{and} \quad A = \frac{|h|}{\rho}$$

which yields:

$$|Dh|^2 \geqslant \frac{2|h_\theta|}{\rho^2\,|h|} \; - \; \frac{1}{\rho^2} \; + \; 2\frac{\sqrt{|h|^2 - 1}}{|h|}\,\frac{|h_\rho|\,|h_\theta|}{\rho}$$

We now estimate each term (from below) by a free Lagrangian

$$|Dh|^2 \geqslant \frac{2}{\rho^2}\,\mathsf{Im}\left(\frac{h_\theta}{h}\right) \;-\; \frac{1}{\rho^2} \;+\; 2\frac{\sqrt{|h|^2-1}}{|h|}\,J_h$$

Equality still holds for the Nitsche map. Upon integration we conclude that

$$\int_{\mathbb{A}} |Dh|^2 \geqslant$$

$$\int_1^R \tfrac{2}{\rho^2}\left(\int_0^{2\pi} \mathsf{Im}\tfrac{h_\theta}{h}\mathsf{d}\theta\right)\,\rho\,\mathsf{d}\rho \;-\; 2\pi\int_1^R \tfrac{\mathsf{d}\rho}{\rho} \;+\; 2\int_{\mathbb{A}^*} \frac{\sqrt{|w|^2-1}}{|w|}\,\mathsf{d}w \;=$$

$4\pi \log R \; - \; 2\pi \log R \; + 4\pi \int_1^{\mathbb{R}_*} \sqrt{s^2 - 1}\, \mathrm{d}s = \frac{\pi}{2}(R^2 - R^{-2})$ [1], as desired.

<u>REMARK</u>. Similar arguments apply in case "**above the critical Nitsche configuration**"; that is, when

$$R_* > \frac{1}{2}\left(R + R^{-1}\right)$$

In this case the energy-minimal map $h_0 : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ is a harmonic homeomorphism (unique up to a rotation) of the form $h_0 = \alpha z + \beta \overline{z}^{-1}$.

[1] make the substitution $s = \frac{1}{2}(\tau + \tau^{-1})$, $1 \leqslant \tau \leqslant R$

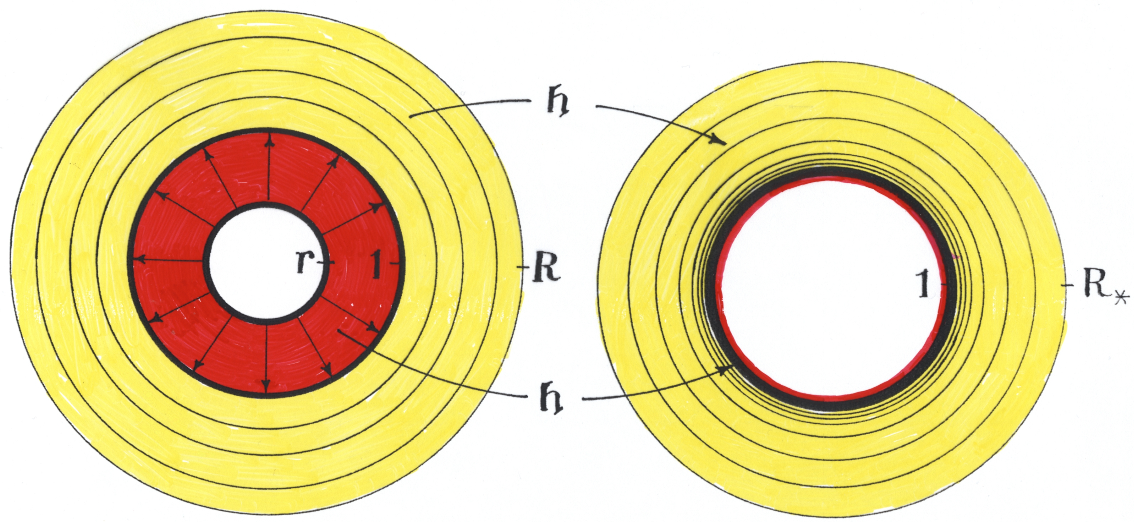
Below the critical Nitsche configuration;

However, when

$$R_* < \frac{1}{2}\left(R + R^{-1}\right)$$

, we observe a squeezing phenomenon.

Squeezing Phenomenon , $R_* > \frac{1}{2}\left(\frac{R}{r} + \frac{r}{R}\right)$



$$\mathfrak{h}(z) = \begin{cases} \frac{z}{|z|}, & r < |z| < 1 \quad \text{(squeezing into concave boundary)} \\ \frac{1}{2}\left(z + \frac{1}{\bar{z}}\right), & 1 < |z| < R \quad \text{(critical harmonic Nitsche map)} \end{cases}$$

This energy-minimal monotone map is $\mathscr{C}^{1,1}$ -smooth and monotone.

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The art of free Lagrangians is
not just integration of nonlinear differential forms
but the correct choice of such forms.

See you tomorrow at 10:00 am

PS, Early to bed, early to rise, makes a man, healthy , wealthy and wise.