Diffeomorphic Approximation of Sobolev Mappings

UAB Lecture 4, Thursday, July 19, 10:00 - 11:00

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Let no one ignorant of geometry enter here

ΑΓΕΩΜΕΤΡΗΤΟΣ ΜΗΔΕΙΣ ΕΙΣΙΤΩ

This Greek phrase was allegedly inscribed over the entrance to Plato's Academy in Athens, circa 387 BC

Diffeomorphic Approximation of Sobolev Homeomorphisms

T. Iwaniec, L. Kovalev, J. Onninen ,
Arch. Ration. Mech. Anal. (2011)

Every homeomorphism $h : \mathbb{X} \to \mathbb{Y}$ between planar open sets that belongs to the Sobolev class $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})\,, 1 < p < \infty$, can be approximated uniformly and in the Sobolev norm with $\mathscr{C}^\infty$-smooth diffeomorphisms. [1]

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[1] Earlier, related partial results have been handled by **Mora-Corral (2009)**. An interesting borderline case $p = 1$, has been solved by **S. Hencl and A. Pratelli**. However, diffeomorphic approximation of a Sobolev homeomorphism $h \in \mathscr{W}^{1,1}(\mathbb{X}, \mathbb{Y})$ cannot be used for the existence of energy-minimal deformations; the Direct Method fails.

Precise Statement

Let $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ be a homeomorphism in the Sobolev space $\mathscr{W}^{1,p}_{\text{loc}}(\mathbb{X}, \mathbb{R}^2)$, $1 < p < \infty$, between planar open sets. Then there exists a sequence of $\mathscr{C}^\infty$-smooth diffeomorphisms $h_j\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ such that

$$(h_j - h) \in \mathscr{W}^{1,p}_{\circ}(\mathbb{X}, \mathbb{R}^2)\ , \qquad j = 1, 2, \dots$$

$$(h_j - h) \longrightarrow 0 \text{ uniformly on } \mathbb{X}$$

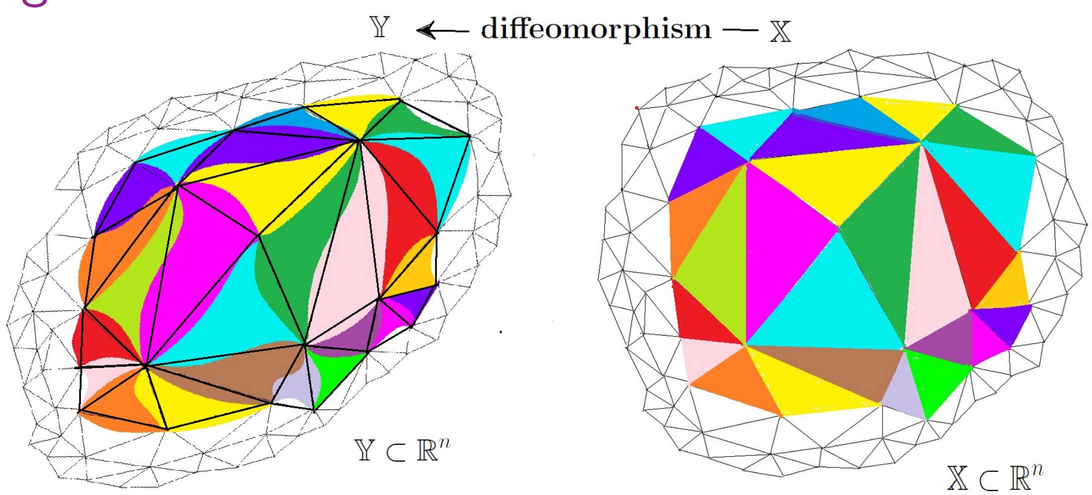
$$(Dh_j - Dh) \longrightarrow 0 \text{ strongly in } \mathscr{L}^p(\mathbb{X}, \mathbb{R}^{2\times 2})$$

$$\|Dh_j\|_{\mathscr{L}^p(\mathbb{X})} \leqslant \|Dh\|_{\mathscr{L}^p(\mathbb{X})} \quad \big(\text{provided the latter term is finite}\big)^2$$

[2] Analogous approximation of **Monotone Sobolev Mappings** (limits of homeomorphisms) will be discussed

Triangulation of Diffeomorphisms

Can one aproximate given diffeomorphism $f\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ between domains $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^n$ with piecewise affine homeomorphisms $f_j\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$, uniformly together with the first order derivatives? [3] .



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[3] This was a question addressed by students during my "Retreat Lectures at the Centre for Doctoral Training" of the University of Oxford, April 20-23, 2015. The answer is yes. However, it was indeed a thankless task to patch holes in the existing literature. Rigorous proof reveals the complexity of this question, especially in higher dimensions. Here is the result in the greatest generality possible

Lagrange Interpolation of Diffeomorphisms

THEOREM [T. Iwaniec & J. Onninen, Mathematische Annalen 2017]
Let $f\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ be a $\mathscr{C}^1$-smooth diffeomorphism of domains in $\mathbb{R}^n$, $n \geqslant 1$, and $g\colon \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ denote its inverse. Suppose we are given real continuous functions: $\varepsilon = \varepsilon(x) > 0$ defined on $\mathbb{X}$ and $\delta = \delta(y) > 0$ defined on $\mathbb{Y}$. Then there exists a piecewise affine homeomorphism $f_\circ\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ (actually Lagrange interpolation of f) which satisfies the inequality:

$$\| Df(x) - Df_\circ(x) \| \leqslant \varepsilon(x)\,, \qquad \text{for almost every } x \in \mathbb{X}\,.$$

Its inverse, denoted by $g_\circ\colon \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$, satisfies analogous inequality:

$$\| Dg(y) - Dg_\circ(y) \| \leqslant \delta(y)\,, \qquad \text{for almost every } y \in \mathbb{Y}\,.$$

Novelty lies in selfsimilar-isotropic triangulation. **Quasiregular Mappings** come into play.

J. Ball - C. Evans Conjecture [4] (now theorem)

Let $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ be a homeomorphism in the Sobolev space $\mathscr{W}^{1,p}_{\text{loc}}(\mathbb{X}, \mathbb{R}^2)$, $1 < p < \infty$, between planar open sets. Then there exists a sequence of piecewise affine homeomorphisms $h_j\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ such that

$$h_j - h \in \mathscr{W}^{1,p}_{\circ}(\mathbb{X}, \mathbb{R}^2)\ , \qquad j = 1, 2, \ldots$$

$(h_j - h) \longrightarrow 0$ uniformly on $\mathbb{X}$

$(Dh_j - Dh) \longrightarrow 0$ strongly in $\mathscr{L}^p(\mathbb{X}, \mathbb{R}^2)$

If h is piece-wise affine near $\partial\mathbb{X}$, then $h_j \equiv h$ near $\partial\mathbb{X}$.

[4] Actually J. Ball attributed this conjecture to L.C. Evans. It is still unclear whether h_j can be constructed as Lagrange interpolations of h. And this is where the difficulties were hidden.

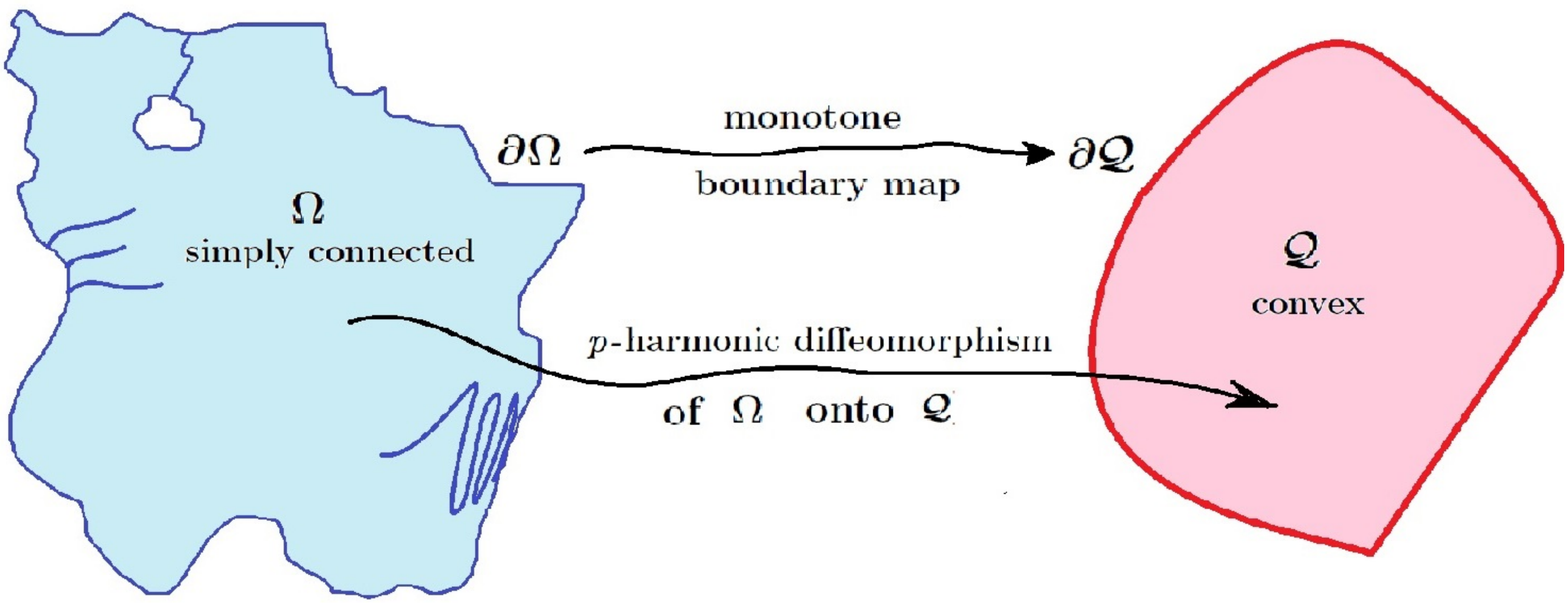
Radó-Kneser-Choquet Theorem

THEOREM Let $h = u + iv : \partial\Omega \overset{\text{onto}}{\longrightarrow} \partial\mathcal{Q}$ be a homeomorphism of the boundary of a (bounded) Jordan domain $\Omega \subset \mathbb{R}^2$ onto the boundary of a convex domain $\mathcal{Q} \subset \mathbb{R}^2$. Then its continuous harmonic extension $H = U + iV$ is a $\mathscr{C}^\infty$-diffeomorphism of Ω onto $\mathcal{Q}$.[5]

[5] This theorem (RKC) was conjectured in 1926 by Radó. It was proved the same year by Kneser. Choquet, apparently unaware of Kneser's work, gave his own proof in 1945. The extension of RKC-theorem to linear elliptic systems is due to P. Bauman, A. Marini and V. Nesi (2001). Further extension to the anisotropic (uncoupled) p-Laplace system was given by G. Alessandrini and M. Sigalotti , 2001. The history goes on, including generalization to simply connected domains.

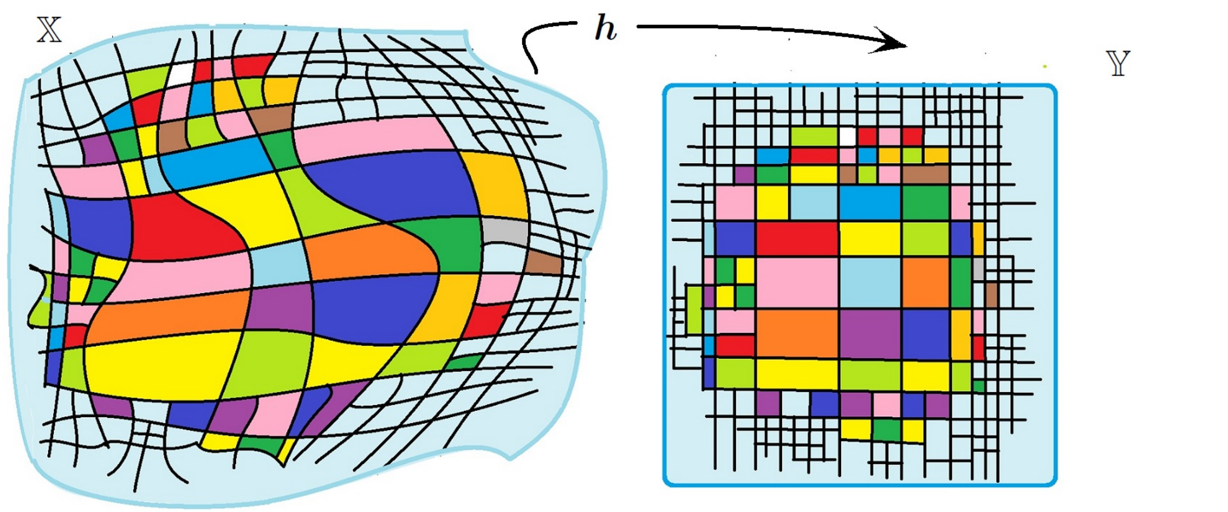
Radó-Kneser-Choquet Theorem for Simply Connected Domains

(T. Iwaniec and J. Onninen, Trans. AMS 201?)



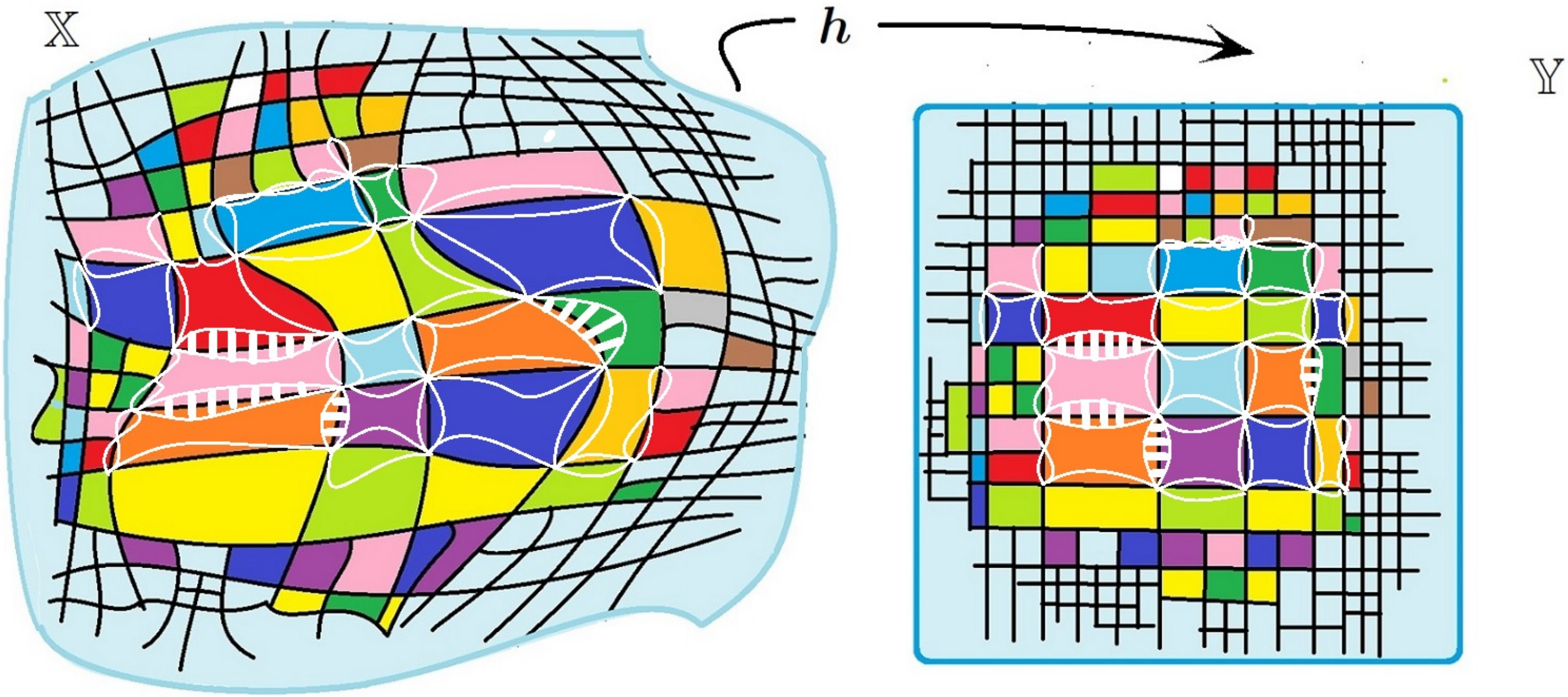
Sketch of the approximation with diffeomorphisms

Step 1. (Partition of $\mathbb{Y}$ into rectangular cells and p-harmonic replacements in pre-cells in $\mathbb{X}$)

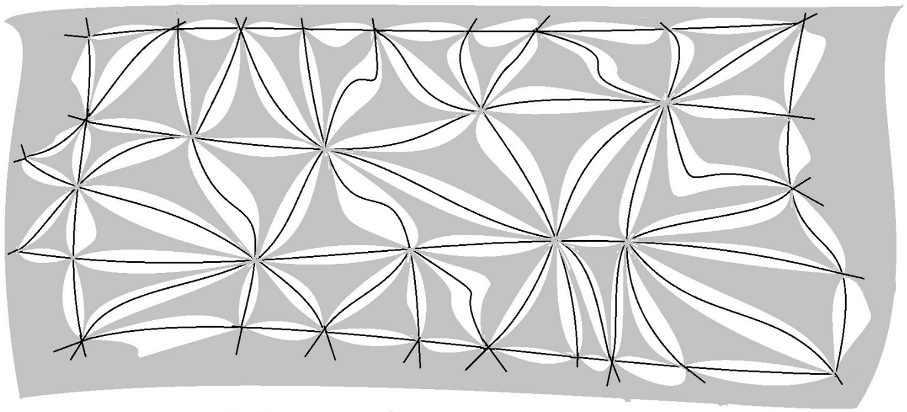


Picasso rectangulation of $\mathbb{X}$ Euclidean rectangulation of $\mathbb{Y}$

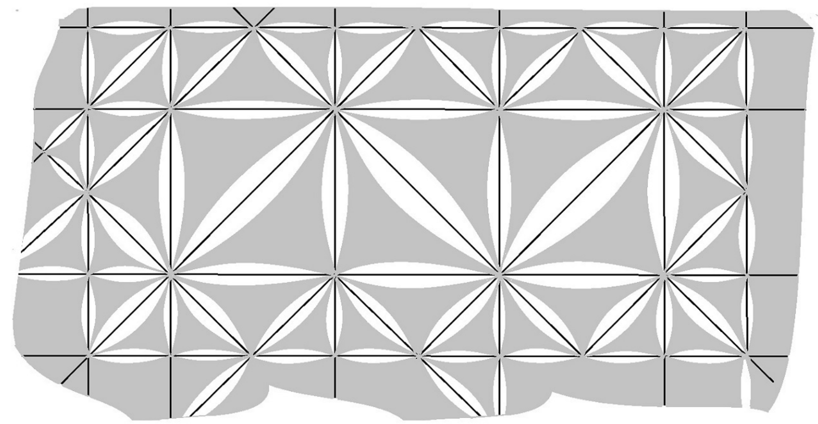
Step 2. (p-harmonic replacements in lenses).



Simplified illustration



curved lenses in $\mathbb{X}$



circular lenses in $\mathbb{Y}$

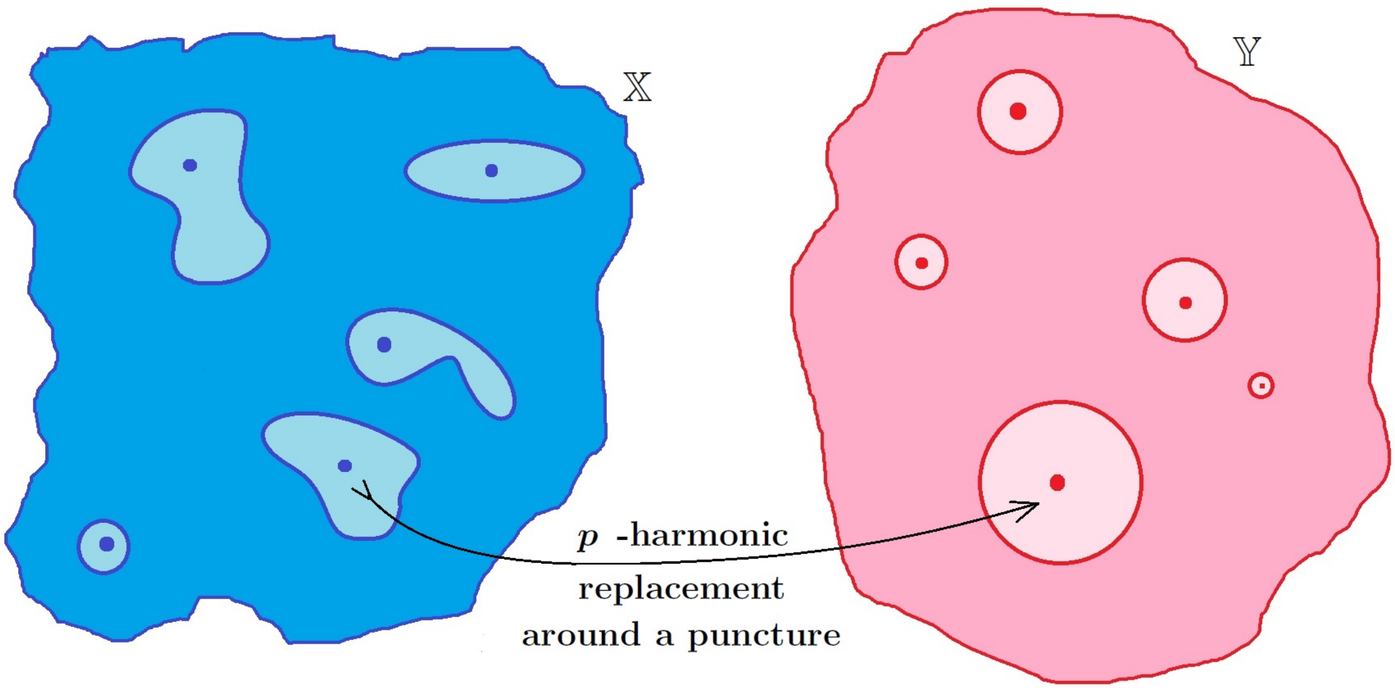
Step 3. (Smoothing along the faces of the lenses **idea of J. Munkres (1960)**).

Our ingredient is that an arbitrarily small energy is needed to smooth around the faces. [6]

[6] Iwaniec and Onninen "*Smoothing defected welds and hairline cracks*", SIAM J. Math. Anal. (2016)

Step 4. (Smoothing around punctures)

We are now reduced to smoothing a Sobolev homeomorphism $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ which is already a $\mathscr{C}^\infty$-diffeomorphism outside the isolated points.



This special case has been settled by C. Mora-Corral in (2009). However, even in this particular case, the p-harmonic replacements around punctures are superior to piecewise affine approximation. The point is that the Lipschitz or piecewise affine mappings with positive Jacobian determinant need not be injective.

The above approach to diffeomorphic approximation of Sobolev mappings manifests the enormous potential of nonlinear PDEs in Geometric Function Theory.

Injectivity up to the boundary (optional)

Let $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ be a homeomorphisms of class $\mathscr{W}^{1,2}(\mathbb{X}, \mathbb{R}^2)$ between bounded Lipschitz domains. It should be noted that such h extends as a continuous (monotone) map between the closures of the domains, still denoted by $h\colon \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$.

Now suppose we are willing to sacrifice the boundary condition $h_j = h : \partial\mathbb{X} \xrightarrow{\text{onto}} \partial\mathbb{Y}$, for diffeomorphisms h_j constructed above to approximate h. Then one can ensure that the continuous extensions $h_j\colon \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ are also homeomorphisms, even if h admits no homeomorphic extension up to the boundary.

These are exactly traction free problems. Slipping along the boundaries are allowed (like in the Riemann's conformal mapping problem).

Non-Interpenetration of Matter

In general, when passing to a weak limit of an energy-minimizing sequence of homeomorphisms the injectivity is lost. We say *interpenetration of matter occurs*. John Ball imposed requirements on the hyperelastic deformations to make it energetically impossible to compress part of the body to zero volume. Also, it is unrealistic that a part of the body changes orientation (like folding). Jacobian determinant has to be positive.

However, from mathematical point of view, one quickly runs into a serious difficulty when passing to the (weak) limit of an energy-minimizing sequence of homeomorphisms. We propose a **complementary vision of J. Ball**; simply by accepting and exploring the limits of Sobolev homeomorphisms as realistic deformations. These are none other than 2D - **Monotone Mappings** (allowing for squeezing but not folding), and 3D -**Cellular Mappings**.

Monotone Mappings, C.B. Morrey (1935)

Definition. A continuous map $h : \mathbf{X} \overset{\text{onto}}{\longrightarrow} \mathbf{Y}$ between compact metric spaces is monotone if every fiber $h^{-1}(y)$ of a point $y \in \mathbf{Y}$ is connected in $\mathbf{X}$.[7]

Theorem (Kuratowski-Lacher, 1968) If $\mathbf{Y}$ is locally connected, then the space of all monotone mappings from $\mathbf{X}$ onto $\mathbf{Y}$ is closed under uniform convergence.

[7] Whyburn, G. T. , 1942

Limits of Homeomorphisms (in topological category)

(Theorem of Youngs, 1948) Let $\mathbb{X}$ and $\mathbb{Y}$ be compact 2-manifolds (with or without boundary) of the same topological type. Then every monotone mapping $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ can be approximated uniformly with homeomorphisms.

Diffeomorphic Approximation of Monotone Sobolev Mappings

THEOREM (**T. Iwaniec and J. Onninen** , ARMA 2016) **Let $\mathbb{X}$, $\mathbb{Y} \subset \mathbb{R}^2$ be Jordan domains (surfaces) of the same topological type, $\mathbb{Y}$ being Lipschitz. Then for every monotone (continuous) map $h : \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ of Sobolev class $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$, $1 < p < \infty$, there exists a sequence of homeomorphisms $h_j : \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ converging to h uniformly and in the norm topology of $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{R}^2)$. Actually, the mappings $h_j : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ are $\mathscr{C}^\infty$-diffeomorphisms.** [8]

[8] Concerning the proof, like in case of approximation of homeomorphisms, we perform p-harmonic replacements. However, this time, the pre-images of rectangular cells in $\mathbb{Y}$ need not be Jordan domains in $\mathbb{X}$. Also other topological complications occur near $\partial\mathbb{X}$.

Weak and Strong Limits are Equal

THEOREM. **(IO, JEMS (2017)) Let** $\mathbb{X}, \mathbb{Y} \in \mathbb{R}^2$ **be bounded multiply connected Lipschitz domains and** $h_j \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ **homeomorphisms converging weakly in** $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$ **to** $h \in \mathscr{W}^{1,p}(\mathbb{X}, \mathbb{R}^2)$**,** $p \geqslant 2$. **Then** h **is monotone and, therefore, there exists a sequence of** $\mathscr{C}^\infty$**-diffeomorphisms**

$h^*_j \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$, $h^*_j \in h + \mathscr{W}^{1,p}_\circ(\mathbb{X}, \mathbb{Y})$ **, converging to** h

uniformly on $\overline{\mathbb{X}}$ **and strongly in** $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{R}^2)$**.** [9]

[9] Sequencial weak closure equals strong closure

Existence of Energy-Minimal Deformations
(IO, ARMA (2016))

Let $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^2$ be bounded Lipschitz domains of the same topological type. Among all monotone mappings $h : \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ of Sobolev class $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$, $p \geqslant 2$, there exists $h_\circ : \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ of smallest energy. Moreover,

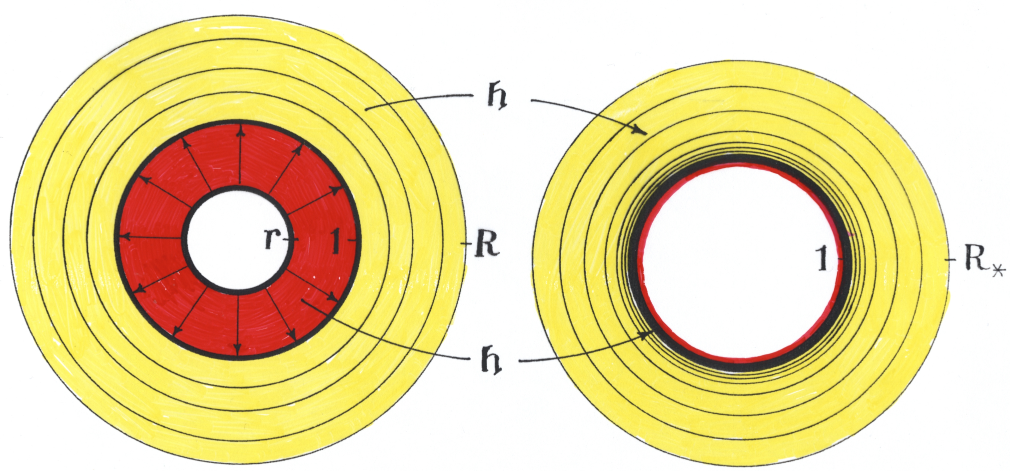
$$\mathscr{E}[h_\circ] = \inf \left\{ \mathscr{E}[h]; \ \ h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y} \text{ - a homeomorphism} \right\}$$

A monotone energy-minimal deformation may squeeze part of the body.

In spite of the impressive progress in squeezing phenomena, such occurrence is not fully resolved. Nonetheless, energy-minimal monotone map that fails to be invertible tells us when to stop the minimizing sequence of homeomorphisms prior to the conditions favorable to the formation of cracks.

Theoretical prediction of failure of bodies caused by cracks is a good motivation that should appeal to MATHEMATICAL ANALYSTS and researchers in the ENGINEERING FIELDS.

Squeezing Phenomenon (and Nitsche Conjecture, IO and L. Kovalev, JAMS (2011)) $R_* < \frac{1}{2}\left(\frac{R}{r} + \frac{r}{R}\right)$



$$\mathfrak{h}(z) = \begin{cases} \frac{z}{|z|}, & r < |z| < 1 \quad \left(\begin{matrix}\text{squeezing into}\\ \text{concave boundary}\end{matrix}\right) \\ \frac{1}{2}\left(z + \frac{1}{\bar{z}}\right), & 1 < |z| < R \quad \left(\begin{matrix}\text{critical harmonic}\\ \text{Nitsche map}\end{matrix}\right) \end{cases}$$

This energy-minimal monotone map is $\mathscr{C}^{1,1}$ -smooth.

Inner Variation of the Dirichlet Energy

Hopf-Laplace Equation

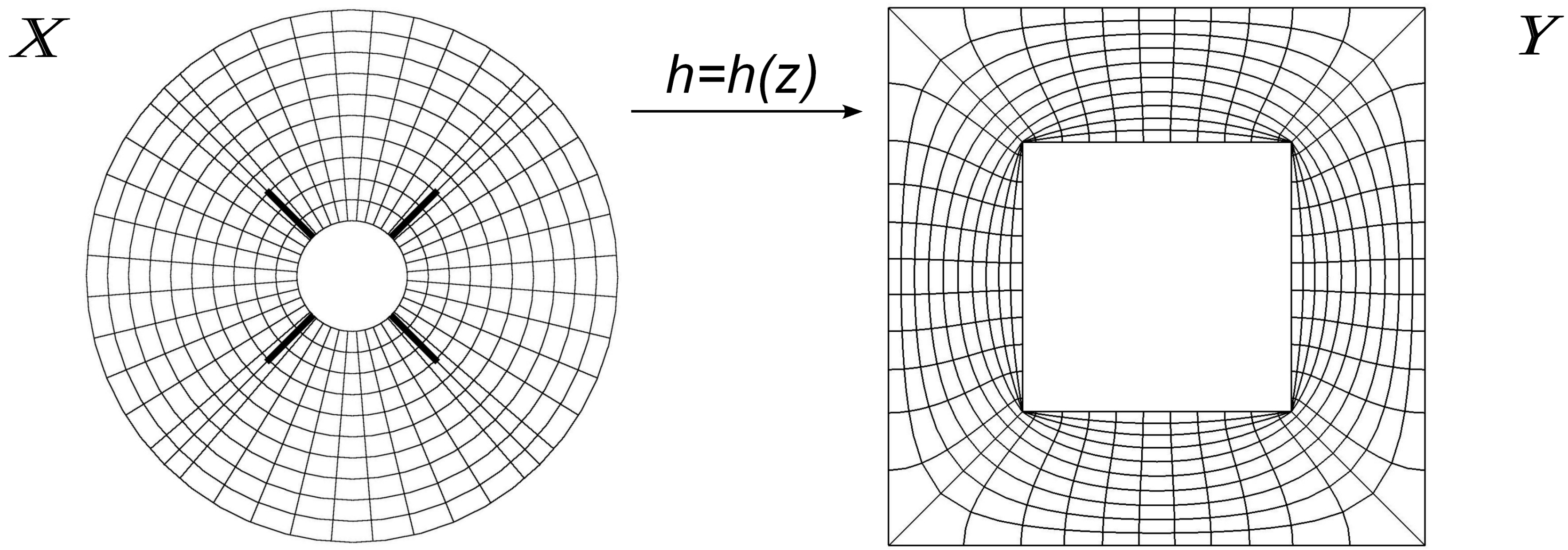
$$\frac{\partial}{\partial \bar z}\left(h_z \overline{h_{\bar z}}\right) = 0$$

Hopf Quadratic Differential

$$h_z \overline{h_{\bar z}} \; \mathrm{d}z \otimes \mathrm{d}z$$

Injectivity of an energy-minimal map is lost exactly in a neighborhood where it fails to be harmonic.

Round and Rectangular Annuli (IO, ARMA (2013))



The round annulus is too fat. Consequently, fractures are inevitable along vertical trajectories of the Hopf quadratic differential.

Cracks Need Not Occur

THEOREM (**IO, Koh, Kovalev, Invent. Math. 2011**) ***Among all homeomorphisms*** $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ ***between bounded doubly connected domains such that***

$$\operatorname{Mod}\mathbb{X} \leqslant \operatorname{Mod}\mathbb{Y}$$

there exists one of smallest Dirichlet energy. This is a harmonic diffeomorphism, unique up to conformal automorphisms of $\mathbb{X}$***.***

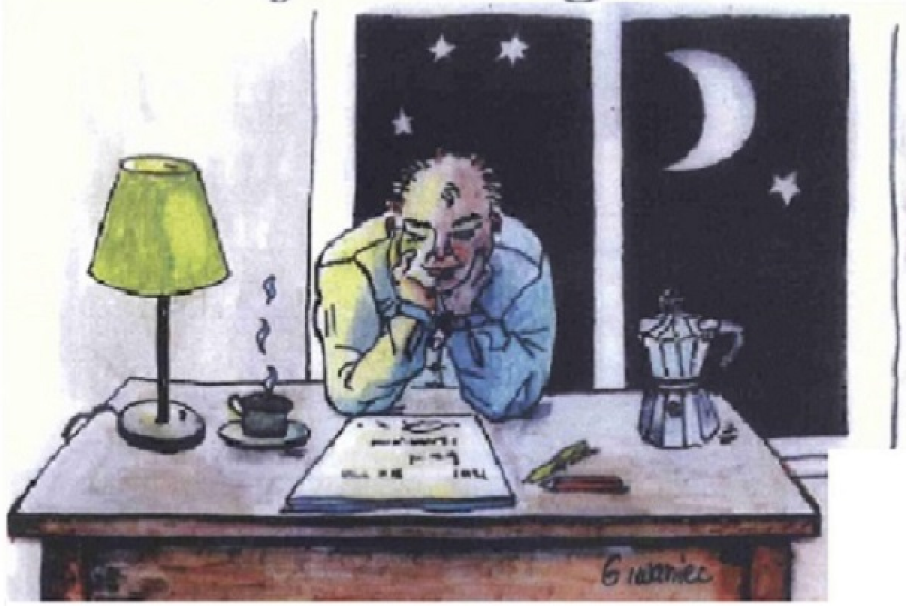
Three blacksmiths are hammering a piece of hot iron to create a new shape.

Consecutive strikes define an energy-minimizing sequence of Sobolev homeomorphisms. The limit satisfies the inner variational

Hopf-Laplace equation

Hopf quadratic differential and its trajectories come into play.

Afterthought



Should I tell the blacksmiths when to stop hammering prior to the permanent damage caused by cracks?

Time for Coffeeholics and Coffee Colleagues