Vortex dynamic

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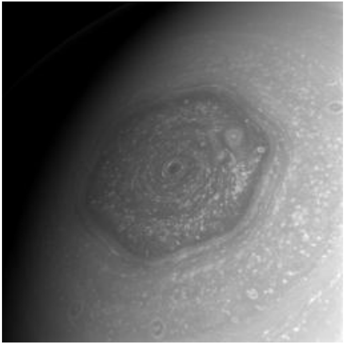
- Saturn's hexagon



- Wingtip vortices

Plan

1. Well-posedness problem for Euler equations.
2. Vortex patch problem.
3. Generalities on relative equilibria.
4. Elements of bifurcation theory.
5. Rotating patches : simply connected cases.
6. Smooth rotating patches.

Euler equations 1755

$$
(E) \begin{cases} \partial_t v + (v \cdot \nabla) v + \nabla p = 0, & x \in \mathbb{R}^d,\ t \geq 0 \\ \text{div}\, v = 0, \\ v_{|t=0} = v_0. \end{cases}
$$

- Velocity field : $(t, x) \in [0, T] \times \mathbb{R}^d \mapsto v = (v^1, .., v^d) \in \mathbb{R}^d$
- The operator $v \cdot \nabla$ is defined by

$$
v \cdot \nabla = \sum_{j=1}^{d} v^j \partial_j .
$$

- The pressure p is a scalar satisfying the elliptic equation

$$
\Delta p = -\text{div}\,(v \cdot \nabla v).
$$

- Kato : For $v_0 \in H^s, s > \frac{d}{2} + 1$ there is a unique maximal solution $v \in C([0, T^\star), H^s)$.

Vorticity formulation in $2d$

- The vorticity $\omega = \partial_1 v^2 - \partial_2 v^1$ satisfies

$$(E)\begin{cases} \partial_t \omega + v \cdot \nabla \omega = 0, \quad t \geq 0, \, x \in \mathbb{R}^2 \\ v = \nabla^{\perp} \Delta^{-1} \omega \\ \omega_{|t=0} = \omega_0 \end{cases}$$

- Biot-Savart law

$$v(t,x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^{\perp}}{|x-y|^2} \omega(t,y) dy, \quad x^{\perp} = e^{i\frac{\pi}{2}} x$$

Global existence in $2d$

- Characteristic method

$$\omega(t,x)=\omega_0(\phi^{-1}(t,x))$$

with ϕ being the flow map :

$$\begin{cases}\partial_t\phi(t,x)=v(t,\phi(t,x))\\ \phi(0,x)=x.\end{cases}$$

- Conservation laws : since $\phi(t)$ preserves Lebesgue measure, then

$$\forall p\in[1,\infty],\ \forall t\geq 0\quad \|\omega(t)\|_{L^p}=\|\omega_0\|_{L^p}$$

- Classical solutions are global.

Yudovich solutions

- Yudovich (1963) : If $\omega_0 \in L^1 \cap L^\infty$ then (E) has a unique global solution $\omega \in L^\infty(\mathbb{R}_+; L^1 \cap L^\infty)$ and

$$\omega(t,x) = \omega_0(\phi^{-1}(t,x))$$

- Note that the velocity is not Lipschitz in general but only log-Lipschitz.

$$\omega_0 = \mathbf{1}_\square \Longrightarrow v_0 \notin \text{Lip}$$

However

$$\omega_0 = \mathbf{1}_\bigcirc \Longrightarrow v_0 \in \text{Lip}$$

- The flow ϕ still exists and it is unique and continuous in (t,x). For each t, $\phi(t)$ is a homeomorphism preserving Lebesgue measure. It is a diffeomorphism for classical solutions.

Vortex patch problem

- A patch is $\omega_0 = \mathbf{1}_D$, with D a bounded domain.

$$\omega(t) = \mathbf{1}_{D_t}, \qquad D_t = \phi(t, D).$$

- What about the regularity of the boundary ?
- Contour dynamics equation (Deem Zabusky 1978) :
  Let $s \in [0, 2\pi] \mapsto \gamma_t(s)$ be a parametrization of ∂D_t,

$$\big(\partial_t\gamma_t(s) - v(t, \gamma_t(s))\big) \cdot \ \vec{n}(\gamma_t(s)) = 0$$

  Lagrangian parametrization is given by : $\qquad \partial_t\gamma_t = v(t, \gamma_t)$

$$\partial_t\gamma_t(s) \quad = \quad -\frac{1}{2\pi}\int_{\partial D_t} \log\big|\gamma_t(s) - z\big|\, dz.$$

- Persistance regularity : Chemin(1993),

$$\partial D \in C^{1+\varepsilon} \Longrightarrow \forall t \geq 0 \quad \partial D_t \in C^{1+\varepsilon}.$$

- The cases C^1 and Lip are open even locally in time.

Relative equilibria

Relative equilibria are vortices that do not change their shapes in time.

1. vortices
2. Translating vortices
3. Rotating vortices

Stationary vortices

- A stationary solution is such that $\omega(t,x)=\omega_0(x)(\in L^1\cap L^\infty)$

$$v_0\cdot\nabla\omega_0=\operatorname{div}(v_0\omega_0)=0\quad(\text{in}\quad\mathcal{D}'(\mathbb{R}^2)),\quad v_0(x)=\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(x-y)^\perp}{|x-y|^2}\omega_0(y)dy$$

Examples : radial solutions

$$\omega_0(x)=f_0(|x|),$$

with f_0 be a compactly supported bounded function : Rankine vortices : disc, annulus..

:

- A translating solution is such that

$$\omega(t,x) = \omega_0\big(x - U\,t\big), \quad U \in \mathbb{R}^2$$

One can check that $v(t,x) = v_0\big(x - U\,t\big)$ and

$$\big(v_0(x) - U\big) \cdot \nabla\omega_0(x) = 0$$

- If ω_0 is compactly supported then we have the conservation law :

$$\int_{\mathbb{R}^2} x\omega(t,x)dx = \int_{\mathbb{R}^2} x\omega_0(x)dx.$$

Hence change of variables give

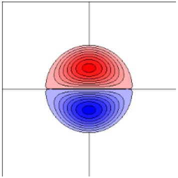
$$\int_{\mathbb{R}^2} x\omega(t,x)dx = \int_{\mathbb{R}^2} x\omega_0(x)dx + U\,t \int_{\mathbb{R}^2} \omega_0(x)dx$$

and thus the circulation vanishes $\int_{\mathbb{R}^2} \omega_0(x)dx = 0$

- Consequence : Vortices in the patch form never translate.

Nontrivial example :

- Dipolar Chaplygin-Lamb vortex(around 1900).



The construction is explicit and based on the resolution in the disc of

$$\Delta\psi = \kappa^2\psi, |x| \leq 1, \quad \omega_0(x) = 0, |x| > 1$$

- Counter-rotating pairs of patches can be constructed implicitly.

Rotating vortice

- Rotating vortice with the angular velocity Ω are solutions in the form :

$$\omega(t,x) = \omega_0\big(e^{-i\Omega t}x\big)$$

- The equation of ω_0 is given by

$$\big(v_0(x) - \Omega x^{\perp}\big)\cdot\nabla\omega_0(x) = 0,$$

with

$$v_0(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(x-y)^{\perp}}{|x-y|^2}\omega_0(y)dy$$

- Examples :
  - Radial solutions (they rotate with any angular velocity).
  - Kirchhoff ellipses (1876). An elliptic patch rotates uniformly about its centre.

Rotating patche

Rotating patche

- We shall restrict the discussion to rotating patche with the angular velocity Ω :

$$D_t = e^{i\,\Omega t} D.$$

- The boundary equation is given by

$$\big(v(x) - \Omega x^{\perp}\big) \cdot n(x) = 0, \quad \forall\, x \in \partial D.$$

where n is a normal vector to the boundary. By Green-Stokes theorem

$$\begin{aligned}\overline{v(z)} &= \frac{1}{2i\pi} \int_D \frac{dA(w)}{z - w} \\ &= \frac{1}{4\pi} \int_{\partial D} \frac{\overline{z} - \overline{\xi}}{z - \xi} d\xi\end{aligned}$$

Hence

$$\mathrm{Re}\Big\{\Big(\frac{1}{2i\pi} \int_{\partial D} \frac{\overline{z} - \overline{\xi}}{z - \xi} d\xi + 2\Omega \overline{z}\Big) \vec{\tau}(z)\Big\}, \quad \forall z \in \partial D$$

Kirchhoff ellipses (1876)

Any ellipse with semi-axes a and b rotates about its center of mass with

$$\Omega = \frac{ab}{(a+b)2}.$$

Proof : we use the conformal parametrization of the ellipse

$$w \in \mathbb{T} \mapsto \phi(w) = \frac{a+b}{2}\Big(w + Q\overline{w}\Big), \quad Q := \frac{a-b}{a+b}$$

Note that for $z = \phi(w), \xi = \phi(\tau)$ we have

$$\frac{\overline{z}-\overline{\xi}}{z-\xi} = \frac{Q\tau - \overline{w}}{\tau - Qw}$$

Thus

$$\frac{1}{2i\pi}\int_{\partial D}\frac{\overline{z}-\overline{\xi}}{z-\xi}d\xi = \frac{a+b}{2}\frac{1}{2i\pi}\int_{\mathbb{T}}\frac{Q\tau-\overline{w}}{\tau-Qw}(1-Q\overline{\tau}^2)d\tau$$

We use residue theorem.

- There are many ways to formulate the problem :

1. Variational formulation. Kelvin's variational principle
2. Potential formulation ($\Omega \leq 0$)
   - Elliptic tools : moving plane method.
3. Free boundary problem.
4. Formulation with Faber polynomials.
   - Suitable for numerical approximation.
5. Conformal mapping formulation ($\Omega > 0$).
   - Bifurcation theory
6. Riemann-Hilbert problem.
   - Global bifurcation theory

Kelvin's variational principle

- Rotating solutions $(\omega(t,z) = \omega_0(e^{-it\Omega}z))$ are the critical points of

$$H - \Omega I,$$

with Ω being a Lagrange multiplier with respect to area preserving displacements.

$$\begin{aligned} H(\omega) &= -\frac{1}{2}\int_{\mathbb{R}^2} \omega(x)\psi(x)dx \quad (\neq \frac{1}{2}\|v\|_{L^2}^2) \qquad \text{[Kinetic energy]} \\ &= -\frac{1}{4\pi}\iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} \log|x-y|\ \omega(x)\omega(y)dxdy. \\ I(\omega) &= \int_{\mathbb{R}^2} |x|^2\omega(x)dx, \qquad \text{[Angular impulse]} \end{aligned}$$

- This is the starting-point for variational approaches.

Variational characterization of circular vortice

- Set $H(\omega) = -\frac{1}{2}\int_{\mathbb{R}^2} \omega(x)\psi(x)dx$ and

$$\mathcal{M}_\kappa = \Big\{ w \in L^1, 0 \leq \omega \leq \kappa, \int_{\mathbb{R}^2} \omega(x)dx = 1 \Big\}$$

Then $\max \Big\{ H(\omega), \omega \in \mathcal{M}_\kappa \Big\}$ is given by the circular patch $\omega_\kappa \equiv \kappa \, \mathbf{1}_{\mathbf{D(0,R)}}$ with $R = \sqrt{\frac{1}{\pi\kappa}}$ (modulo translations) and

$$\kappa \to \infty, \quad \omega_\kappa \rightharpoonup \delta_0.$$

Potential formulation

- Recall that the boundary equation is given by the strong formulation

$$(v(x) - \Omega x^{\perp}) \cdot n(x) = 0, \quad \forall\, x \in \partial D.$$

- Note that $v = \nabla^{\perp}\psi$ with ψ the stream function

$$\Delta\psi = \omega = \mathbf{1}_D, \quad \psi(x) = \frac{1}{2\pi}\int_D \log|x-y|dA(y)$$

- Integrating we get the weak formulation

$$\frac{1}{2}\Omega|x|^2 - \frac{1}{2\pi}\int_D \log|x-y|dy - \mu = 0, \quad \forall x \in \partial D.$$

Free boundary formulation

- Set
$$\varphi(x) = \frac{1}{2}\Omega|x|^2 - \frac{1}{2\pi}\int_D \log|x-y|dy - \mu$$
- Then φ satisfies the elliptic equation
$$\Delta\varphi(x) = \begin{cases} 2\Omega - 1, & x \in D \\ 2\Omega, & x \in \mathbb{C}\backslash\overline{D} \end{cases}$$
supplemented with the boundary condition : $\varphi(x) = 0, \quad \forall x \in \partial D$.
- The fact that $\varphi \in C^{2-\epsilon}(\mathbb{C})$ introduces a rigidity on the boundary !
- Free boundary problem for elliptic equations was discussed by : Brezis, Caffarelli, Kinderlehrer, Nirenberg, Schaeffer,...

Trivial solutions (simply connected domains)

1. Fraenkel (2000) : let D be a solution with $\Omega = 0$ then D must be a disc.
2. H. (2014) : let D be a convex solution with $\Omega < 0$ then D must be a disc.
3. Let let D be a solution with $\Omega = \frac{1}{2}$ then D must be a disc.

Case $\Omega = \frac{1}{2}$

- Set $\varphi(x) = Cte + \frac{1}{2}\Omega|x|^2 - \psi(x)$ then

$$\Delta\varphi(x) = 2\Omega - \mathbf{1}_D, \quad \varphi(x) = 0, x \in \partial D.$$

- For $\Omega = \frac{1}{2}$ we find that φ is harmonic in D and thus

$$\psi(x) = Cte + \frac{1}{4}|x|^2, \quad \forall\, x \in D.$$

It follows that

$$\partial_z\psi = \frac{1}{4\pi}\int_D \frac{1}{z-y}dA(y) = \frac{1}{4}\overline{z}, \quad \forall z \in D$$

By holomorphy we get

$$z\partial_z\psi = Cte, \forall z \in D^c$$

Case $\Omega \leq 0$

Using the maximum principle

$$\mathbf{1}_D = H(\varphi), \quad H \quad \text{Heaviside function}$$

Thus φ satisfies the integral equation

$$\varphi(x) = Cte + \frac{1}{2}\Omega|x|^2 - \frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y| H(\varphi(y))dA(y).$$

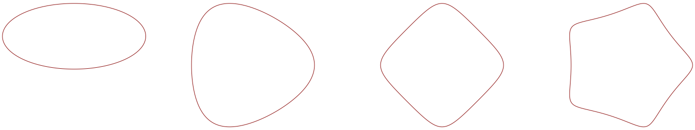
The moving plane method shows that φ up to a translation is a strictly monotonic radial function.

Nontrivial solutions

1. Kirchhoff vortex (1876). Any *ellipse* with semi-axes a and b rotates with

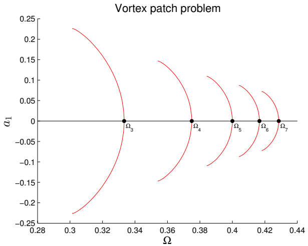
$$\Omega = \frac{ab}{(a+b)2}.$$

2. Numerical observation Deem-Zabusky 1978 : existence of $m-$fold V-states (same symmetry of regular polygon with m sides).



Burbea's re (1982)

- There exists a family of rotating patches $(V_m)_{m\geq 2}$ bifurcating from the disc at the spectrum $\Omega \in \{\frac{m-1}{2m}, m \geq 2\}$. Each point of V_m describes a V-state with m-fold symmetry .



- The case $m = 2$ corresponds to Kirchhoff ellipses.

Elements of bifurcation theory

Consider the finite-dimensional dynamical system

$$\dot{x} = f(x, \Omega), x \in \mathbb{R}^d, \Omega \in \mathbb{R}$$

- The phase portrait is the set of all the disjoint orbits.
- We say that there is a bifurcation at some value Ω_0, if there is a topological accident in the phase portrait.

Examples

1. Let $f(x, \Omega) = \Omega x - x^3$ in $d = 1$, then there a pitchfork bifurcation at $\Omega = 0$
2. Poincaré-Andronov-Hopf bifurcation $d = 2$:

$$f(x, y, \Omega) = \begin{pmatrix} \Omega x - y - x(x^2 + y^2) \\ x + \Omega y - y(x^2 + y^2) \end{pmatrix}. \tag{1}$$

Emergence of periodic orbits (limit cycles) for $\Omega > 0$

Assume that

1. $f : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ is C^3,
2. $\forall \Omega \in \mathbb{R}, \quad f(0, \Omega) = 0$,
3. The matrix $\partial_x f(0, \Omega)$ admits two complex eigenvalues
$$\alpha(\Omega) \pm i\beta(\Omega), \alpha(0) = 0, \beta(0) \neq 0,$$
4. Transversality assumption $\alpha'(0) = 0$

Then there is a parametrization $s \in (-a, a) \mapsto (x(s), \Omega(s))$ of periodic solutions.

Stationary bifurcation in infinite dimension

- Consider two Banach spaces X, Y and

$$F : \mathbb{R} \times X \to Y$$

a smooth function such that

$$F(\Omega, 0) = 0, \quad \forall \Omega \in \mathbb{R}$$

- If $\partial_x F(\Omega, 0) \in \mathrm{Isom}(X, Y)$ then by the implicit function theorem, there is no bifurcation at Ω.
- Bifurcation may occur when 0 is an eigenvalue for $\partial_x F(\Omega, 0))$

Fredholm operators

Let X, Y be two Banach spaces, a continuous operator $T : X \mapsto Y$ is said Fredholm if

1. $\mathrm{Ker}\, T$ is finite dimenional.
2. The range $\mathrm{Im}\, T$ is closed and of finite co-dimension

The index of T is

$$ind(T) = \mathrm{dimKer}\, T - \mathrm{codim}\ \mathrm{Im}\, T$$

Let T be Fredholm and K a compact operator then

1. $T + K$ is Fredholm,
2. $ind(T + K) = ind(T)$

Example : Let $X = \left\{f \in C^2([0,1];\mathbb{R}), f(0) = f(1) = 0\right\}$, $Y = C([0,1];\mathbb{R})$, $\phi \in Y$ and define $T : X \to Y$

$$Tf = f'' - \phi f$$

Then T is Fedholm of zero index. Moreover, if $\phi \geq 0$ then T is an isomorphism.

Crandall-Rabinowitz theorem

Let X, Y be two Banach spaces and

$$F : \mathbb{R} \times X \to Y$$

be a smooth function such that

1. $F(\Omega, 0) = 0, \quad \forall \Omega \in \mathbb{R}$
2. The kernel Ker $\partial_x F(0,0) = \langle x_0 \rangle$ is one-dimensional and the range $R(\partial_x F(0,0)$ is closed and of co-dimension one.
3. Transversality assumption :

$$\partial_\Omega \partial_x F(0,0) x_0 \notin R(\partial_x F(0,0))$$

Then there is a curve of non trivial solutions $s \in (-a, a) \mapsto (\Omega(s), x(s))$ with

$$\forall s \in (-a, a), \quad F(\Omega(s), x(s)) = 0$$

General approach

- The boundary is subject to the equation

$$\mathrm{Re}\left\{\left(2\Omega\,\overline{z}+\frac{1}{2i\pi}\int_{\partial D}\frac{\overline{\xi}-\overline{z}}{\xi-z}d\xi\right)\vec{\tau}(z)\right\}=0,\quad\forall\, z\in\partial D.$$

- Let $\Phi:\mathbb{T}\to\partial D$ be the *conformal parametrization*

$$\Phi(w)=w+\sum_{n\geq 0}\frac{a_n}{w^n},\quad a_n\in\mathbb{R}.$$

We have assumed that the real axis is an axis of symmetry of D.

- Then for any $w\in\mathbb{T}$

$$\begin{aligned}F(\Omega,\Phi(w)) &\equiv \mathrm{Im}\left\{\left(2\Omega\,\overline{\Phi(w)}+\frac{1}{2i\pi}\int_{\mathbb{T}}\frac{\overline{\Phi(\xi)}-\overline{\Phi(w)}}{\Phi(\xi)-\Phi(w)}\Phi'(\xi)d\xi\right)w\,\Phi'(w)\right\}\\ &= 0.\end{aligned}$$

- Rankine vortices : $\forall\Omega\in\mathbb{R},\quad F(\Omega,w)=\mathrm{Im}\left\{\left((2\Omega-1)\overline{w}\right)w\right\}=0$

- Recall that

$$F(\Omega,\Phi(w)) \quad \equiv \quad \mathrm{Im}\Big\{\Big(2\Omega\,\overline{\Phi(w)}+\frac{1}{2i\pi}\int_{\mathbb{T}}\frac{\overline{\Phi(\xi)}-\overline{\Phi(w)}}{\Phi(\xi)-\Phi(w)}\Phi'(\xi)d\xi\Big)\,w\,\Phi'(w)\Big\}=0.$$

- We look for solutions which are small perturbation of the disc :

$$\Phi=\mathrm{Id}+f,\, f(w)=\sum_{n\geq 0}a_n w^{-n},\, a_n\in\mathbb{R}$$

We still denote $F(\Omega,f)=F(\Omega,\Phi)$.

- Function spaces :

$$X=\Big\{f\in C^{1+\alpha}(\mathbb{T})\Big\},\quad Y=\Big\{g(w)=\sum_{n\geq 1}b_n\mathrm{Im}(w^n)\in C^{\alpha}(\mathbb{T}),\, b_n\in\mathbb{R}\Big\}$$

- The coefficient associated to $n=0$ vanishes since the Fourier coefficients of $F(\Omega,f)$ are real !
- For small r, $F:(-1,1)\times B(0,r)\to Y$ is well-defined and smooth.

1. Straightforward computations yield : for $h(w)=\sum_{n\geq 0} a_n w^{-n}\in X$

$$\begin{aligned}\partial_f F(\Omega,0)h(w) &= \frac{d}{dt}F(\Omega,th(w))_{|t=0}\\ &= \mathrm{Im}\Big\{2\Omega\Big(w\overline{h(w)}+h'(w)\Big)-h'(w)\Big\}\\ &= \sum_{n\geq 1} n\Big(2\Omega-\frac{n-1}{n}\Big)a_{n-1}\mathrm{Im}(w^n)\end{aligned}$$

2. $\Big\{\Omega,\quad \mathrm{Ker}\ \partial_f F(\Omega,0)\neq 0\Big\}=\Big\{\Omega_m:=\frac{m-1}{2m}, m\geq 1\Big\}$ and

$$\mathrm{Ker}\ \partial_f F(\Omega,0)=\langle v_m\rangle,\quad v_m(w)=\overline{w}^{m-1}$$

3. Transversality condition

$$\begin{aligned}\partial_\Omega\partial_f F(\Omega_m,0)v_m &= 2m\,\mathrm{Im}(w^m)\\ &= \notin R(\partial_f F(\Omega_m,0))\end{aligned}$$

Boundary regularity

- H.-Mateu-Verdera [2013]. Close to the circle the V-states are C^∞ and convex.
- Castro, Córdoba, Gómez-Serrano [2015] : Analyticity of the boundaries.

Euler in the unit disc

- Recall the vorticity equation

$$\partial_t\omega + v\cdot\nabla\omega = 0, \quad v = \nabla^\perp\psi \quad \text{in} \quad \mathbb{D}$$

$$\psi(x) = \frac{1}{2\pi}\int_{\mathbb{D}} \log\frac{|x-y|}{|1-x\overline{y}|}\omega(y)dy$$

- V-states equation : recall that a rotating patch is a solution s. t.

$$\omega(t) = \mathbf{1}_{D_t}, \quad D_t = e^{it\Omega}D,$$

then

$$\mathrm{Re}\Big\{\Big(2\Omega\,\overline{z} + \frac{1}{2i\pi}\int_{\partial D}\frac{\overline{z}-\overline{\xi}}{z-\xi}d\xi - \frac{1}{2i\pi}\int_{\partial D}\frac{|\xi|^2}{1-z\xi}d\xi\Big)\vec{\tau}(z)\Big\} = 0, \quad z\in\partial D.$$

- Trivial solutions

$$D_t = \mathbb{D}_b := b\mathbb{D}, \quad 0 < b < 1$$

Bifurcation from the trivial solutions

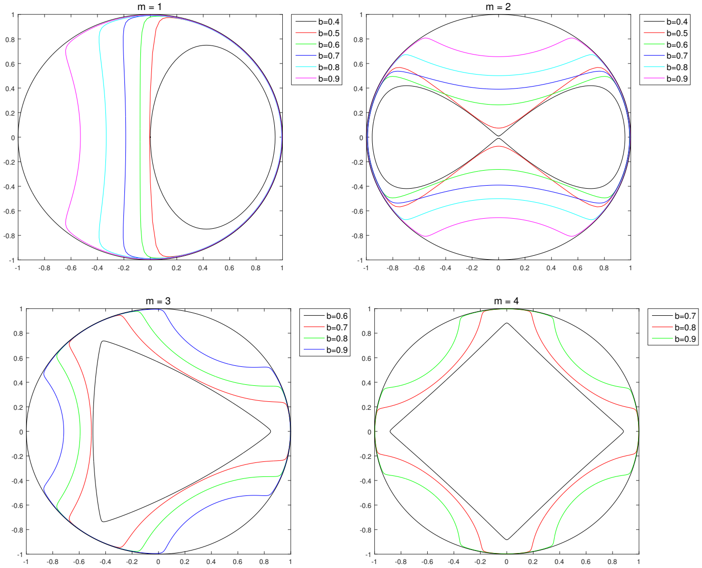
- de la Hoz-Hassainia-H-Mateu (2015). Let $m \geq 1$, then there exists m-fold V-states bifurcating from the trivial solution $\omega_0 = \mathbf{1}_{\mathbb{D}_b}$ at the angular velocity

$$\Omega_m \triangleq \frac{m - 1 + b^{2m}}{2m}.$$

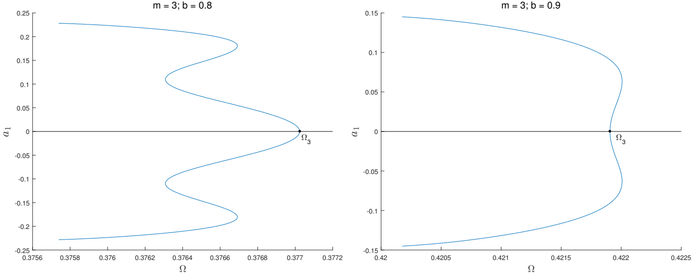
- Remarks :

  1. As $b \to 0$ we get Burbea eigenvalues.
  2. In the plane $\mathbb{R}^2$, $m \geq 2$ and there is no V-states with only one axis of symmetry.

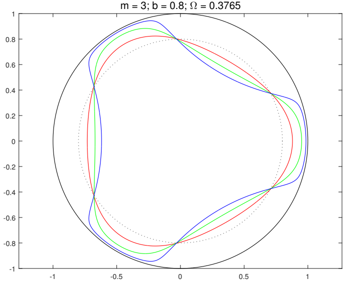
I) Limiting V-states



II) Bifurcation diagram



III) V-states with the same Ω



Doubly-connected V-state

Goal : find in the plane rotating patches in the form

$$\omega_0 = \mathbf{1}_{D_1 \setminus D_2}, \quad D_2 \Subset D_1,$$

with D_1, D_2 two bounded simply-connected domains.

- The annuli are explicit rotating patches (stationary).
- To date, no other explicit solutions are known !

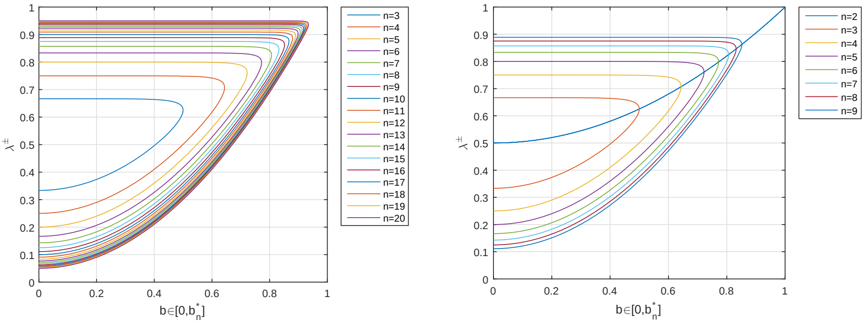
- de la Hoz-H.-Mateu-Verdera 2014 :
  Let $\mathcal{C}(b, 1)$ be the annulus of small radius b. Define

$$\Delta_m = \left[\frac{m}{2}(1 - b^2) - 1\right]^2 - b^{2m}$$

  and take $m \geq 3$ such that $\Delta_m > 0$. Then there are two branches of non trivial m-fold doubly connected V-states bifurcating from the annulus at the angular velocities $\Omega_m^{\pm}$

$$\Omega_m^{\pm} = \frac{1 - b^2}{4} \pm \frac{1}{2m}\sqrt{\Delta_m}.$$

Structure of the eigenvalues



- For given b, $\exists m_b$ such that the bifurcation holds for any $m \geq m_b$.
- Monotonicity : $m \mapsto \Omega_m^- \searrow$; $\quad m \mapsto \Omega_m^+ \nearrow$

- The bifurcation to 4−fold holds if

$$0 < b < \sqrt{\sqrt{2} - 1} \triangleq b_4^\star \approx 0.6435$$

Numerical experiments :

- For $b << b_4^\star$, corners appear in the limiting V-states. For $b = 0.4$.

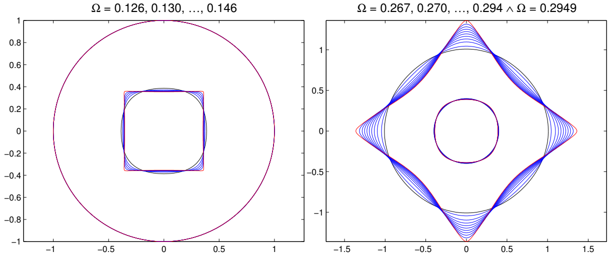


FIGURE – Left : V-states bifurcating from Ω_4^-. Right : V-states bifurcating from Ω_4^+

- If $b \approx b_4^\star$,($\Omega_m^+ \approx \Omega_m^-$) then the two branches merge forming small loop (proved with Renault 2016).

  $b = 0.63$

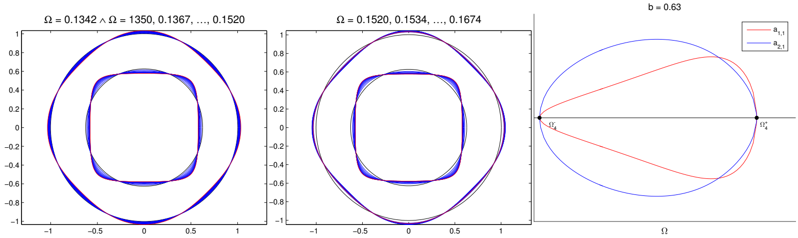


FIGURE – Left : V-states bifurcating from Ω_4^-. Right : V-states bifurcating from Ω_4^+

- For the degenerate case $b = b_4^\star$, there is no bifurcation, (proved with Mateu 2015)