On the $T(1)$-Theorem for the Cauchy Integral

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Abstract

The main goal of this paper is to present an alternative, real variable proof of the $T(1)$-Theorem for the Cauchy Integral. We then prove that the estimate from below of analytic capacity in terms of total Menger curvature is a direct consequence of the $T(1)$-Theorem. An example shows that the L^∞-BMO estimate for the Cauchy Integral does not follow from L^2 boundedness when the underlying measure is not doubling.

Introduction

In this paper we present an alternative proof of the $T(1)$-Theorem for the Cauchy Integral Operator with respect to an underlying measure which is not assumed to satisfy the standard doubling condition. This result has been proved recently in [T1] and, independently, in [NTV1] where fairly general Calderón-Zygmund operators are considered. The proof in [T1] exploits a tool specific to the Cauchy kernel, called Menger curvature (see section 1 for the definition) and is based on two main ingredients: a good λ inequality and a special argument, which is designed to make the transition from an L^2 estimate to a weak $(1,1)$ inequality. This argument involves analytic capacity (concretely, the inequality (18) below) and consequently is of a complex analytic nature. Our approach avoids use of complex analysis. In fact, our strategy consists in finding in any given disc a "big piece", in the sense of Guy David [D1, D2], on which the operator is bounded on L^2. We then plug in the standard good λ inequality to control the maximal Cauchy Integral by the centered maximal operator, as in [D1, D2]. In this second step one only needs to check that the doubling condition is not really used in the classical arguments. Thus our proof is actually reduced to the construction

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of a "big piece", which turns out to be fairly simple because of the good positivity properties of Menger curvature. We proceed now to state precisely the main result.

Let μ be a positive Radon measure in the plane. Our goal is to estimate the Cauchy integral operator on $L^2(\mu)$. In view of the singularity of the Cauchy kernel $\frac{1}{z-\zeta}$ we assume that μ satisfies the growth condition

$$\mu(D) \leq Cr(D), \text{ for each disc } D, \tag{1}$$

where $r(D)$ stands for the radius of D and C is some positive constant independent of D. Indeed, if μ has no atoms then (1) is necessary for the $L^2(\mu)$ boundedness of the Cauchy Integral [D2, p. 56]. We say that the Cauchy integral operator is bounded on $L^2(\mu)$ whenever for some positive constant C one has

$$\int |C_\varepsilon(f\mu)|^2\, d\mu \leq C \int |f|^2\, d\mu, \quad f \in L^2(\mu),\, \varepsilon > 0, \tag{2}$$

where

$$C_\varepsilon(f\mu)(z) = \int_{|\zeta-z|>\varepsilon} \frac{f(\zeta)}{\zeta - z}\, d\mu(\zeta), \quad z \in \mathbb{C}. \tag{3}$$

Notice that the integral in (3) is absolutely convergent for each z, as can readily be seen applying the Schwarz inequality and then using (1).

A necessary condition for (2) is obtained by taking as f the characteristic function χ_D of a disc D and restricting the domain of integration in the left hand side of (2) to D:

$$\int_D |C_\varepsilon(\chi_D\mu)|^2\, d\mu \leq C\mu(D), \text{ for each disc } D,\, \varepsilon > 0. \tag{4}$$

The $T(1)$-Theorem for the Cauchy integral can now be stated as follows.

Theorem. *Let μ be a positive Radon measure satisfying (1). Then (2) follows from (4).*

We remark that if μ satisfies the doubling condition

$$\mu(2D) \leq C\mu(D), \text{ for each disc } D, \tag{5}$$

where $2D$ stands for the disc concentric with D of twice the radius, then (4) is easily seen to be equivalent to requiring that $C_\varepsilon(\mu)$ belongs to $\mathrm{BMO}(\mu)$, uniformly in ε. Hence we recover the familiar condition in the standard formulation of the $T(1)$-Theorem for the operator $T = C$ [D2, p. 30]:

$$C(1) \equiv C(\mu) \text{ belongs to } \mathrm{BMO} \equiv \mathrm{BMO}(\mu).$$

In the doubling context the Theorem can readily be proved using Menger curvature and interpolation between H^1 and BMO (see section 4).

In section 1 we gather some preliminaries including notation, terminology and background. Section 2 contains the proof of the Theorem. In section 3 we remark that the estimate from below for analytic capacity in terms of Merger curvature (inequality (18) below) follows readily from the theorem by purely real variable arguments. Section 4 shows that if the doubling condition (5) fails then L^2 boundedness of C does not imply the L^∞-BMO estimate.

1 Preliminaries

Given three distinct points z_1, z_2, $z_3 \in \mathbb{C}$ one has the identity [Me]

$$(6) \qquad \sum_\sigma \frac{1}{(z_{\sigma(2)} - z_{\sigma(1)})\overline{(z_{\sigma(3)} - z_{\sigma(1)})}} = c(z_1, z_2, z_3)^2$$

where the sum is taken over the six permutations of $\{1, 2, 3\}$ and $c(z_1, z_2, z_3)$ is the Menger curvatuve of the given triple, that is, the inverse of the radius of the circumference passing through z_1, z_2 and z_3. For a positive Radon measure ν the quantity

$$c(\nu)^2 = \iiint c(z_1, z_2, z_3)^2 \, d\nu(z_1) \, d\nu(z_2) \, d\nu(z_3)$$

is called the total Menger curvature of ν or simply the curvature of ν. Note that we have not defined $c(z_1, z_2, z_3)$ for triples where at least two of the points are the same; for such triples we may set $c(z_1, z_2, z_3) = 0$.

The first application of (6) to the L^2 theory of the Cauchy Integral Operator was a new proof of the L^2 boundedness of the Cauchy Integral on Lipschitz graphs, (see [V2] and [MV]). There we showed that the arc length measure on an arc of a Lipschitz graph has finite Menger curvature.

Later on the identity (6) was used to obtain estimates from below for analytic capacity [Me] and to describe uniform rectifiability via the mapping properties of the Cauchy integral operator [MMV]. The results in [V2] or [MV] were explicitly mentioned in [Me], pp. 828-829, but unfortunately no reference was made to [V2] or [MV], which already existed in preprint form. This has caused some misunderstanding of the real sequence of events and some inaccuracies in attributing the results. Impressive progress has been made, using (6), in recent work by several authors [DM, JM, L, Ma, T1, T2, T3], culminating in David's solution of Vitushkin's conjecture [D3].

In our estimates we will use two variants of the Hardy-Littlewood maximal operator acting on a complex Radon measure ν, namely,

$$M\nu(z) = \sup_{r>0} \frac{|\nu|(D(z,r))}{r}, \quad z \in \mathbb{C},$$

and

$$M_\mu\nu(z) = \sup_{r>0} \frac{|\nu|(D(z,r))}{\mu(D(z,r))}, \quad z \in \operatorname{spt}\mu,$$

where $D(z,r)$ is the open disc centered at z of radius r and $\operatorname{spt}\mu$ is the closed support of μ.

It follows from the Besicovitch covering Lemma that M_μ satisfies the weak type estimate [J, p. 8]

$$\mu\{z : M_\mu\nu(z) > t\} \le Ct^{-1}\|\nu\|, \tag{7}$$

and since

$$M\nu(z) \le CM_\mu\nu(z), \quad z \in \operatorname{spt}\mu,$$

because of (5), (7) also holds when M_μ is replaced by M.

Actually the weak type $(1,1)$ estimate for M is a consequence of the simplest standard covering lemma [S, Lemma 1, p. 12] and so there is nothing deep in it. Although we could work only with M_μ we prefer to keep the distinction between M and M_μ to emphasize those steps where the Besicovitch covering lemma necessarily comes into play. Notice that (7) coupled with the obvious L^∞ estimate gives by interpolation the inequality

$$\int M_\mu(f\mu)^p\, d\mu \le C \int |f|^p\, d\mu, \quad 1 < p < \infty.$$

The letter C will denote either the Cauchy Integral Operator or a constant which may be different at each occurrence and that is independent of the relevant variables under consideration. The precise meaning of C will always be clear from the context.

2 The proof

Let ν be a complex Radon measure. Set

$$C_\varepsilon\nu(z) = \int_{|\zeta - z|>\varepsilon} \frac{d\nu(\zeta)}{\zeta - z}, \quad z \in \mathbb{C}. \tag{8}$$

The integral in (8) is absolutely convergent for all z provided ν is a finite measure or, more generally, provided

$$\int \frac{d|\nu|(\zeta)}{1+|\zeta|} < \infty. \tag{9}$$

Lemma 1. *Let ν_j, $j = 1, 2, 3$ be three real Radon measures satisfying (9) with ν replaced by ν_j, $j = 1, 2, 3$. Then*

$$\sum_\sigma \int C_\varepsilon(\nu_{\sigma(1)})\overline{C_\varepsilon(\nu_{\sigma(2)})}\, d\nu_{\sigma(3)} = \iiint_{S_\varepsilon} c^2(z_1, z_2, z_3)\, d\nu_1(z_1)\, d\nu_2(z_2)\, d\nu_3(z_3) + R,$$

where the sum is taken over the permutations of $\{1, 2, 3\}$,

$$S_\varepsilon = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1 - z_2| > \varepsilon, |z_1 - z_3| > \varepsilon \text{ and } |z_2 - z_3| > \varepsilon\}$$

and

$$|R| \le C \sum_\sigma \int M\nu_{\sigma(2)}(z_{\sigma(1)}) M\nu_{\sigma(3)}(z_{\sigma(1)})\, d\nu_{\sigma(1)}(z_{\sigma(1)}),$$

C being an absolute constant.

Proof. Set

$$T_\varepsilon = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1 - z_3| > \varepsilon \text{ and } |z_2 - z_3| > \varepsilon\},$$
$$U_\varepsilon = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1 - z_2| \le \varepsilon, |z_1 - z_3| > 2\varepsilon \text{ and } |z_2 - z_3| > \varepsilon\}$$

and

$$V_\varepsilon = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1 - z_2| \le \varepsilon, |z_1 - z_3| \le 2\varepsilon \text{ and } |z_2 - z_3| > \varepsilon\}.$$

Then

$$\int C_\varepsilon(\nu_1)\overline{C_\varepsilon(\nu_2)}\, d\nu_3 = \iiint_{T_\varepsilon} \frac{d\nu_1(z_1)\, d\nu_2(z_2)\, d\nu_3(\nu_3)}{(z_1 - z_3)\overline{(z_2 - z_3)}}$$
$$= \iiint_{S_\varepsilon} \cdots + \iiint_{U_\varepsilon} \cdots + \iiint_{V_\varepsilon} \cdots = \iiint_{S_\varepsilon} \cdots + I_\varepsilon + II_\varepsilon$$

where the last identity is a definition of I_ε and II_ε. To estimate I_ε and II_ε we assume, without loss of generality, that the ν_j are positive measures. Then

$$\begin{aligned}|I_\varepsilon| &\le C \iiint_{U_\varepsilon} \frac{d\nu_1(z_1)\, d\nu_2(z_2)\, d\nu_3(z_3)}{|z_1-z_3|^2} \\ &\le C \iint_{|z_1-z_2|\le\varepsilon} \varepsilon^{-1} M\nu_3(z_1)\, d\nu_1(z_1)\, d\nu_2(z_2) \\ &\le C \int M\nu_2(z_1) M\nu_3(z_1)\, d\nu_1(z_1).\end{aligned}$$

For II_ε we write

$$\begin{aligned}|II_\varepsilon| &\le \varepsilon^{-2} \iiint_{V_\varepsilon} d\nu_1\, d\nu_2\, d\nu_3 \\ &\le C\varepsilon^{-1} \iint_{|z_1-z_2|\le\varepsilon} M\nu_3(z_1)\, d\nu_1(z_1)\, d\nu_2(z_2) \\ &\le C \int M\nu_2(z_1) M\nu_3(z_1)\, d\nu_1(z_1).\end{aligned}$$

Operating in a similar way for any σ and then summing over σ we get the conclusion of the Lemma. □

We apply Lemma 1 to $\nu_1 = \nu_2 = f\mu$ with f a (real function) in $L^2(\mu)$ and $\nu_3 = \chi_D \mu$ with D a fixed disc. We then have

$$\begin{aligned}(10)\quad & 2\int_D |C_\varepsilon(f\mu)|^2\, d\mu + 2\,\mathrm{Re} \int C_\varepsilon(f\mu)\overline{C_\varepsilon(\chi_D\mu)} f\, d\mu \\ &= \iiint_{S_\varepsilon} c^2(z,w,\zeta) f(z) f(w) \chi_D(\zeta)\, d\mu(z)\, d\mu(w)\, d\mu(\zeta) + O\left(\int f^2\, d\mu\right).\end{aligned}$$

In particular taking $f = \chi_D$ one gets

$$6\int_D |C_\varepsilon(\chi_D\mu)|^2\, d\mu = \iiint_{S_\varepsilon\cap D^3} c^2(z,w,\zeta)\, d\mu(z)\, d\mu(w)\, d\mu(\zeta) + O(\mu(D)),$$

and thus

$$(11)\qquad \iiint_{D^3} c^2(z,w,\zeta)\, d\mu(z)\, d\mu(w)\, d\mu(\zeta) \le C\mu(D),$$

provided (4) holds.

It is worth pointing out that (11) was inexactly attributed in [NTV1, p. 705]. Indeed, a first version of (11) appears in [V2] and [MV] and later on in [MMV] in the form at hand.

We come now to the core of the argument that produces a "big piece" inside a given disc D.

Set

$$c_D^2(z) = \iint_{D^2} c^2(z,w,\zeta)\, d\mu(w)\, d\mu(\zeta), \quad z \in \mathbb{C}.$$

By Chebischev

$$\mu\{z \in D : c_D(z) > t \text{ or } |C_\varepsilon(\chi_D \mu)(z)| > t\}$$
$$\leq t^{-2} \left(\int_D c_D^2(z)\, d\mu(z) + \int_D |C_\varepsilon(\chi_D\mu)|^2\, d\mu \right) \leq C t^{-2} \mu(D).$$

Hence, given $0 < \theta < 1$ (θ will be chosen later), there exists a compact $E \subset D$ such that

$$c_D(z) \leq \sqrt{C/\theta} \text{ and } |C_\varepsilon(\chi_D\mu)(z)| \leq \sqrt{C/\theta}, \quad z \in E,$$

and

$$\mu(D \backslash E) \leq \theta \mu(D).$$

Set, as in [T1],

$$k(z,w) = \int_D c^2(z,w,\zeta)\, d\mu(\zeta),$$

so that

$$\int_E k(z,w)\, d\mu(w) \leq c_D^2(z) \leq C/\theta, \quad z \in E.$$

Since $k(z,w) = k(w,z)$, Schur's Lemma now gives that if $f \in L^2(E)$ $(= L^2(E, d\mu))$ then

$$\left| \iiint_{S_\varepsilon} c^2(z,w,\zeta) f(z) f(w) \chi_D(\zeta)\, d\mu(z)\, d\mu(w)\, d\mu(\zeta) \right|$$
$$\leq \int |f(z)| \int |f(w)| k(z,w)\, d\mu(w)\, d\mu(z) \leq C \int f^2\, d\mu,$$

where $C = C(\theta)$ does not depend on ε.

Therefore from (10)

$$\int_D |C_\varepsilon(f\mu)|^2\, d\mu \leq C \left(\int_D |C_\varepsilon(f\mu)|^2 \right)^{1/2} \left(\int f^2\, d\mu \right)^{1/2} + C \int f^2\, d\mu,$$

and consequently

$$\int_D |C_\varepsilon(f\mu)|^2\,d\mu \le C\int f^2\,d\mu, \quad f\in L^2(E).$$

By duality this implies

$$\int_E |C_\varepsilon(f\mu)|^2\,d\mu \le C\int f^2\,d\mu, \quad f\in L^2(D). \tag{12}$$

We now need an appropriate Cotlar type inequality. For a complex Radon measure ν satisfying (9) set, for $z\in\mathbb{C}$,

$$C^*_\varepsilon(z) = \sup_{\delta\ge\varepsilon}|C_\delta\nu(z)|$$

and

$$C^*\nu(z) = \sup_{\varepsilon>0} C^*_\varepsilon\nu(z).$$

Lemma 2. *Let μ and ν be positive Radon measures satisfying the growth condition*

$$\mu(D)+\nu(D)\le Cr(D), \text{ for each disc } D,$$

and such that for some $\varepsilon>0$

$$\int |C_\varepsilon(f\,d\mu)|^2\,d\nu \le C\int |f|^2\,d\mu, \quad f\in L^2(\mu).$$

Then

$$C^*_\varepsilon(f\mu)(z)\le C\{M_\nu(|C_\varepsilon(f\mu)|^2\,d\nu)^{1/2}+M_\nu(|f|^2\,d\mu)^{1/2}\}.$$

For a proof for the case $\mu=\nu$, which can be seen to work under our hypothesis, we refer the reader to [T2, Lemma 3 and Theorem 4].

Combining (12) with Lemma 2 applied to $\nu=\chi_E\mu$ and $\mu=\chi_D\mu$ we get, for each $f\in L^2(D)$,

$$\begin{aligned}\mu\{z\in E: C^*_\varepsilon(f\mu)(z)>t\} &\le Ct^{-2}\left(\int_E |C_\varepsilon(f\mu)|^2\,d\mu+\int f^2\,d\mu\right)\\ &\le Ct^{-2}\int f^2\,d\mu.\end{aligned} \tag{13}$$

We now want to have the above inequality at our disposal for a general $f\in L^2(\mu)$. This essentially means that, for each open disc D, C_ε maps boundedly $L^2(D^c)$ into $L^2(D)$ with constant independent of ε. This is clear if C_ε is replaced by

$$C(f\mu)(z) = \lim_{\varepsilon \to 0} C_\varepsilon(f\mu)(z), \quad z \in D,\ f \in L^2(D^c).$$

The reason is that, $C(f\mu)$ being holomorphic on D, we only need to apply Carleson's Theorem twice:

$$\int_D |C(f\mu)|^2 \, d\mu \leq C \int_{\partial D} |C(f\mu)|^2 |dz| \leq C \int_{D^c} |f|^2 \, d\mu.$$

However we wish to have a real variable proof, which could be extended to $\mathbb{R}^n$ and $n-1$ dimensional kernels. This can be done painlessly and in fact is implicit in David's paper [D1].

Lemma 3. *Let Δ be an open disc and let μ and ν be positive Radon measures satisfying*

$$\mu(D) + \nu(D) \leq Cr(D), \text{ for each disc } D,$$

and $\mu(\Delta^C) = \nu(\Delta) = 0$.

Then

$$\int C^*(f\nu)^2 \, d\mu \leq C \int |f|^2 \, d\nu, \quad f \in L^2(\nu).$$

Proof. Assume, without loss of generality, that Δ is centered at the origin and let r be its radius. Given $z \in \Delta$ let d be the distance from z to $\partial\Delta$. We claim that

$$C^*(f\nu)(z) \leq C^*(f\nu)(w) + CM(f\nu)(w), \quad |w - z| \leq 2d. \tag{14}$$

Fix $\varepsilon > 0$. Assume first that $\varepsilon < d$. Then for $|w - z| \leq 2d$,

$$|C_\varepsilon(f\nu)(z)| = |C_d(f\nu)(z)| \leq |C_{4d}(f\nu)(z)| + M(f\nu)(w).$$

The same inequality holds for $d \leq \varepsilon < 4d$, so that we are left with the case $4d \leq \varepsilon$. Set $f_1 = \chi_{D(z,\varepsilon)} f$, $f_2 = f - f_1$. Thus, for $|w - z| \leq 2d$,

$$|C_\varepsilon(f\nu)(z) - C_\varepsilon(f_2\nu)(w)| \leq CM(f\nu)(w)$$

and

$$|C_\varepsilon(f_2\nu)(w) - C_\varepsilon(f\nu)(w)| \leq CM(f\nu)(w)$$

because of standard simple estimates. Therefore the claim follows.

Set

$$F(w) = C^*(f\nu)(w) + CM(f\nu)(w).$$

Using (14), the simplest covering lemma [S, Lemma 1, p. 12] and the growth condition on μ, one proves that (see for example [S, p. 59–60])

$$\mu\{z \in \Delta : C^*(f\nu)(z) > t\} \leq C\,|\{w \in \partial\Delta : F(w) > t\}|\,,$$

where $|\cdot|$ denotes one dimensional Lebesgue measure. Then

$$\begin{aligned}\int_\Delta C^*(f\nu)^2(z)\,d\mu(z) &\leq C\left(\int_{\partial\Delta} C^*(f\nu)^2(w)|dw| + \int_{\partial\Delta} M(f\nu)^2(w)|dw|\right)\\ &\leq C\int |f|^2\,d\nu\end{aligned}$$

by [D1, Proposition 5, p. 164] and [D1, Proposition 3, p. 161]. □

Lemma 3 and (13) now give

$$\mu\{z \in E : C_\varepsilon^*(f\mu)(z) > t\} \leq Ct^{-2}\int f^2\,d\mu, \quad f \in L^2(\mu), \tag{15}$$

which shows that E is indeed a "big piece".

The proof of the Theorem is practically complete. One last step is left: we have to check that (15) allows us to prove an appropriate good λ inequality without resorting to a doubling condition on μ. For the reader's convenience we present the well known argument, which can be found in [D2, p. 61–62]. The good λ inequality we need is the following.

For each $\eta > 0$ there exists $\gamma = \gamma(\eta) > 0$ small enough so that

$$\begin{aligned}\mu\{z : C_\varepsilon^*(f\mu)(z) > (1+\eta)t \text{ and } M_\mu(f^2\mu)^{1/2}(z) \leq \gamma t\}&\\ \leq \frac{1}{2}\mu\{z : C_\varepsilon^*(f\mu)(z) > t\}.&\end{aligned} \tag{16}$$

Once (16) is established we deduce that C_ε^* satisfies the same L^p inequalities as $M_\mu(f^2\mu)^{1/2}$ [D2, p. 60]. Then

$$\int C_\varepsilon^*(f\mu)^p\,d\mu \leq C_p\int |f|^p\,d\mu, \quad 2 < p < \infty.$$

In particular

$$\int |C_\varepsilon(f\mu)|^p\,d\mu \leq C_p\int |f|^p\,d\mu, \quad 2 < p < \infty,$$

and by duality we get the same estimate for $1 < p < 2$ and so for $p = 2$ by interpolation.

Let's prove (16). The set $\Omega = \{z : C_\varepsilon^*(f\mu)(t) > t\}$ is open. Given $a \in S \cap \Omega$ let $D(a)$ be the disc with center a and radius $5^{-1}\operatorname{dist}(a, \Omega^c)$. By the Besicovitch covering lemma $\Omega \cap S$ can be covered by a family of discs $D_j = D(a_j)$ which is almost disjoint, that is, such that each point in the plane belongs to at most N discs D_j, N being an absolute constant. Notice that then the family $\{4D_j\}$ is almost disjoint too. This is one of the key facts in order to allow us to dispense with the doubling condition.

We are going to show that, given $\eta > 0$ and $0 < \alpha < 1$, there exists $\gamma = \gamma(\eta, \alpha) > 0$ such that, for all j,

(17)

$$\mu\{z \in S \cap D_j : C_\varepsilon^*(f\mu)(z) > (1+\eta)t \text{ and } M_\mu(f^2\mu)^{1/2}(z) \le \gamma t\} \le \alpha\mu(4D_j).$$

Then summing over j,

$$\mu\{z \in S : C_\varepsilon^*(f\mu)(z) > (1+\eta)t \text{ and } M_\mu(f^2\mu)^{1/2}(z) \le \gamma t\} \le \alpha N \mu(\Omega),$$

where N stands now for the constant of almost disjointness of $\{4D_j\}$. Choosing α so that $\alpha N = \frac{1}{2}$ we get (16).

Let's turn our attention to (17). Fix j and set $D = D_j$, $a = a_j$. Assume, without loss of generality, that there exists $b \in S \cap D$ such that $M_\mu(f^2\mu)^{1/2}(b) \le \gamma t$. Let w be a point in Ω^c such that $|w - a| = \operatorname{dist}(a, \Omega^c)$ and set $B = D(w, 9r)$ where $r = |w - a|/5$ is the radius of D. Hence $D \subset \Delta \equiv D(b, 3r) \subset 4D \subset B$. Set $f_1 = f\chi_B$ and $f_2 = f - f_1$. Then, for $z \in D$ and $\delta \ge \varepsilon$,

$$\begin{aligned} |C_\delta(f_1\mu)(z)| &= |C_\delta(\chi_\Delta f)(z)| + \frac{1}{r}\int_B |f(\zeta)|\, d\mu(\zeta) \\ &\le C_\varepsilon^*(\chi_\Delta f)(z) + CM(f\mu)(b) \\ &\le C_\varepsilon^*(\chi_\Delta f)(z) + C\gamma t, \end{aligned}$$

and so

$$|C_\delta(f\mu)(z)| \le |C_\delta(f_2\mu)(z)| + C_\varepsilon^*(\chi_\Delta f\mu)(z) + C\gamma t.$$

To compare $C_\delta(f_2\mu)(z)$ with $C_\delta(f\mu)(w)$ we use the standard arguments (see [D1] or [D2]). We obtain

$$|C_\delta(f_2\mu)(z) - C_\delta(f_2\mu)(w)| \le CM(f\mu)(b)$$

and

$$|C_\delta(f_2\mu)(w)| \le C_\varepsilon^*(f\mu)(w) \le t.$$

Therefore

$$C_\varepsilon^*(f\mu)(z) \le C_\varepsilon^*(\chi_\Delta f\mu)(z) + (1 + C\gamma)t, \quad z \in D.$$

Now choose γ so that $2C\gamma \le \eta$ and let E be a "big piece" associated to the disc D and the number θ. Then

$$\begin{aligned}\mu\{z \in D : C_\varepsilon^*(f\mu)(z) > (1+\eta)t\} &\le \mu(D \backslash E) \\ &\quad + \mu\left\{z \in E : C_\varepsilon^*(\chi_\Delta f\mu)(z) > \frac{\eta}{2}t\right\} \\ &\le \theta\mu(D) + C(\eta t)^{-2}\int_\Delta f^2\,d\mu \\ &\le \theta\mu(D) + C(\eta t)^{-2}\mu(\Delta)M_\mu(f^2\mu)(b) \\ &\le (\theta + C(\gamma/\eta)^2)\mu(4D) \le \alpha\mu(4D)\end{aligned}$$

provided θ and γ are chosen small snough so that $\theta + C(\gamma/\eta)^2 \le \alpha$.

3 Estimating analytic capacity from below

Let K be a compact subset of $\mathbb{C}$, $\gamma(K)$ its analytic capacity, and let μ be a positive measure supported in K satisfying $\mu(D) \le r(D)$ for each disc D and $c(\mu) < \infty$. Then [Me]

$$\gamma(K) \ge c\frac{\|\mu\|^{3/2}}{(\|\mu\| + c^2(\mu))^{1/2}}. \tag{18}$$

The original proof of (18) is rather simple but relies on the Garabedian's L^2 description of analytic capacity [G] and thus depends on complex analysis techniques. We give here a quick derivation of an inequality slightly better than (18) from the $T(1)$-Theorem described in the preceding sections, using purely real variable methods. Therefore the $T(1)$-Theorem for the Cauchy Integral and (18) are equivalent statements. Similar arguments have been used independently by Tolsa in [T3] for other purposes.

Given a compactly supported positive measure μ, set

$$E(\mu) = \int M\mu(z)\,d\mu(z) + \int c_\mu(z)\,d\mu(z),$$

where

$$c_\mu^2(z) = \iint c^2(z,w,\zeta)\,d\mu(w)\,d\mu(\zeta), \quad z \in \mathbb{C}.$$

The quantity $E(\mu)$ and the function $M_\mu(z) + C_\mu(z)$ seem to be appropriate candidates to play the roles of "energy" and "potential" associated to the kernel $1/z$.

Theorem. *For each compact subset K of the plane,*

$$\gamma(K) \geq C \sup\{E(\mu)^{-1} : \operatorname{spt}\mu \subset K \text{ and } \|\mu\| = 1\}. \tag{19}$$

If μ is a positive measure supported on K such $\mu(D) \leq r(D)$ for all discs D, then

$$E\left(\frac{\mu}{\|\mu\|}\right) = \|\mu\|^{-2}E(\mu) \leq \|\mu\|^{-2}(\|\mu\|+c(\mu)\|\mu\|^{1/2}) = \|\mu\|^{-3/2}(\|\mu\|^{1/2}+c(\mu))$$

and so (18) follows from (19).

Proof of the Theorem. Take a probability measure μ supported on K with $E(\mu) < \infty$. By Chebischev there exists a compact subset J of K such that $\mu(J) \geq 2^{-1}$, and $M\mu(z) \leq A$ and $c_\mu(z) \leq A$, for all $z \in J$, where $A = 2E(\mu)$.

Set $\nu = \mu|_J$. Then $\|\nu\| \geq 2^{-1}$, $\nu(D) \leq Ar(D)$ for each disc D and

$$c_\nu(z) \leq A, \quad z \in \operatorname{spt}\nu. \tag{20}$$

Clearly (20) gives (11) with μ replaced by ν and therefore the Cauchy Integral is bounded on $L^2(\nu)$ by the $T(1)$-Theorem discussed in the previous sections. We wish now to have the weak L^1 inequality

$$\nu\{z : |C_\varepsilon(\lambda)| > t\} \leq \frac{CA}{t}\|\lambda\|, \tag{21}$$

where λ is any finite measure in the plane and C some absolute constant.

This follows by standard Calderón-Zygmund theory if μ is "doubling" and by a simple argument found recently in [NTV2] in the general case. Dualizing the weak type inequality (21), as in [T1] or [V1], we obtain that there exists a ν-measurable function h, $0 \leq h < 1$, with $\nu(J) \leq 2\int h\,d\nu$ and $|C(h\,d\nu)(z)| \leq CA$, for each $z \in \mathbb{C}\setminus J$. Here $C(h\,d\nu)$ is just the locally integrable function $\frac{1}{z} * h\,d\nu$. Therefore, for some absolute constant C

$$\gamma(K) \geq CA^{-1} = CE(\mu)^{-1},$$

as desired. □

4 Failure of the L^∞-BMO estimate

When μ is a doubling measure the proof of the $T(1)$ Theorem for the Cauchy Integral is very simple, as showed in [V2] and [MV]. The reasoning goes as follows. Fix a disc D and take a bounded, μ-measurable function f supported on D. Because of (10) and (11) we get

$$\begin{aligned}\int_D |C_\varepsilon(f\mu)|^2\,d\mu &\le 2\left(\int_D |C_\varepsilon(f\mu)|^2\,d\mu\right)^{1/2}\left(\int |C_\varepsilon(\chi_D\mu)|^2\,d\mu\right)^{1/2}\\ &\quad + \|f\|_\infty^2 \iiint_{D^3} c^2(z,w,\zeta)\,d\mu(z)\,d\mu(w)\,d\mu(\zeta) + C\|f\|_\infty^2\mu(D)\\ &\le C\mu(D)^{1/2}\left(\int_D |C_\varepsilon(f\mu)|^2\,d\mu\right)^{1/2} + C\mu(D)\|f\|_\infty^2\end{aligned}$$

and so

$$\int_D |C_\varepsilon(f\mu)|^2\,d\mu \le C\|f\|_\infty^2\mu(D), \tag{22}$$

with C independent of ε.

The above inequality and standard arguments show that C_ε maps $L^\infty(\mu)$ boundedly into $\mathrm{BMO}(\mu)$ and maps the atomic version of $H^1(\mu)$ boundedly into $L^1(\mu)$ (see [J, p. 49]). Interpolation between BMO and L^1 now gives that C_ε maps L^2 into L^2.

By $\mathrm{BMO}(\mu)$ we understand the space of locally integrable functions with respect to μ, such that for each disc D centered at a point in $\operatorname{spt}\mu$ one has

$$\int_D |f(z) - f_D|\,d\mu(z) \le C\mu(D),$$

C being a positive constant independent of D and

$$f_D = \frac{1}{\mu(D)}\int_D f\,d\mu.$$

An atom is a μ-measurable function a, supported on some disc D centered at a point in $\operatorname{spt}\mu$, such that $|a| \le \mu(D)^{-1}$ and $\int a\,d\mu = 0$. The atomic version $H^1_{at}(\mu)$ of H^1 is then the set of functions of the form

$$\sum_{j=1}^\infty \lambda_j a_j,$$

where a_j is an atom for all j and $\sum_{j=1}^\infty |\lambda_j| < \infty$.

When the measure μ is non-doubling one can still obtain (22) from the hypothesis of the $T(1)$-Theorem, but we shall see that (22) implies neither the

L^∞-BMO nor the H^1_{at}-L^1 estimate. The example we shall describe is rather simple. In fact, the measure μ will be the one-dimensional Lebesgue measure restricted to a certain subset of the real line.

Set $\lambda_n = 4^{-2^n}$, $n = 0, 1, 2, \ldots$, and

$$I_n = [\lambda_{n-1}^2, 2\lambda_{n-1}^2],$$
$$J_n = \left[\frac{\lambda_{n-1}}{2}, \frac{\lambda_{n-1}}{2} + \lambda_{n-1}^2\right], \quad n = 1, 2, , 3 \ldots .$$

Define μ as the one-dimensional Lebesgue measure restricted to

$$(-1, 0) \cup \left(\bigcup_{n=1}^{\infty}(I_n \cup J_n)\right).$$

Let D_n denote the disc of radius $\frac{\lambda_{n-1}}{2}$ centered at the point $\frac{\lambda_{n-1}}{2}$. Then the function

$$h = \sum_{n=1}^{\infty}(\chi_{I_n} - \chi_{J_n})\mu(D_n)^{-1}2^{-n}$$

lies in $H^1_{at}(\mu)$. We claim that

$$\int_{-1}^{0} |C(h)(x)|\, dx = +\infty,$$

where for $f \in L^1(\mathbb{R})$ we write

$$C(f)(x) = \text{P.V.} \int_{-\infty}^{\infty} \frac{f(t)}{t - x}\, dt.$$

Fix a positive integer n, and abbreviate I_n as both I and (a, b) and J_n as J and (α, β). Then

$$C(\chi_I)(x) - C(\chi_J)(x) = \log\frac{b - x}{a - x} - \log\frac{\beta - x}{\alpha - x} \geq 0, \text{ for } x \leq 0.$$

A simple computation gives

$$\int_{-1}^{0} C(\chi_I)(x)\, dx = \int_{a}^{b} \log\left(1 + \frac{1}{t}\right)\, dt \geq l \log\left(1 + \frac{1}{2l}\right)$$

and

$$\int_{-1}^{0} C(\chi_J)(x)\, dx = \int_{\alpha}^{\beta} \log\left(1 + \frac{1}{t}\right)\, dt \leq l \log\left(1 + \frac{1}{\sqrt{l}}\right),$$

where $l = b - a$.

Since $\mu(D_n) \sim \lambda_{n-1}^2$, we conclude that

$$\int_{-1}^{0} |C(h)(x)|\, dx \geq C \sum_{n=1}^{\infty} 2^{-n} \log \lambda_{n-1}^{-1} = +\infty,$$

as claimed.

Thus C does not map $H_{at}^1(\mu)$ into $L^1(\mu)$, although it maps $L^2(\mu)$ into $L^2(\mu)$. To show that $L^\infty(\mu)$ is not mapped boundedly into $\mathrm{BMO}(\mu)$ we resort to the most elementary fact concerning the duality between $H_{at}^1(\mu)$ and $\mathrm{BMO}(\mu)$. Namely, given an atom a and a disc D as in the definition of atom, there exists a function b in $L^\infty(\mu)$, $\|b\|_\infty = 1$, for which one has

$$\begin{aligned}\int |C(a)|\, d\mu &= \int C(a) b\, d\mu \\ &= -\int a C(b)\, d\mu \\ &= -\int a(C(b) - C(b)_D)\, d\mu \\ &\leq \frac{1}{\mu(D)} \int_D |C(b) - C(b)_D|\, d\mu.\end{aligned}$$

Then the L^∞-BMO estimate would imply the H_{at}^1-L^1 estimate, which fails.

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