THE PLANAR CANTOR SETS OF ZERO ANALYTIC CAPACITY AND THE LOCAL $T(b)$-THEOREM

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ABSTRACT. In this paper we obtain rather precise estimates for the analytic capacity of a big class of planar Cantors sets. In fact, we show that analytic capacity and positive analytic capacity are comparable for these sets. The main tool for the proof is an appropriate version of the $T(b)$-Theorem.

1. INTRODUCTION

In this paper we characterize the planar Cantor sets of zero analytic capacity. Our main result answers a question of P. Mattila [Ma] and completes the solution of a long standing open problem with a curious history, which goes back to 1972. We refer the reader to [I2, p. 153] and [Ma] for more details. Moreover, we confirm a conjecture of Eiderman [E] concerning the analytic capacity of the N-th approximation of a Cantor set.

Before formulating our main results we recall the definition of the basic objects involved.

The analytic capacity of a compact subset E of the complex plane $\mathbb{C}$ is

$$(1) \qquad \gamma(E) = \sup |f'(\infty)|,$$

where the supremum is taken over all analytic functions f on $\mathbb{C} \backslash E$ such that $|f| \leqslant 1$ on $\mathbb{C} \backslash E$. Although there has recently been important progress on our understanding of analytic capacity (see the survey papers [D], [V3] and the references given there), many basic questions about γ remain unanswered. One of the oldest is the semi-additivity problem, that is, the problem of showing the existence of an absolute constant C such that

$$(2) \qquad \gamma(E \cup F) \leqslant C \left\{ \gamma(E) + \gamma(F) \right\},$$

for all compact sets E and F. If (2) were true, then one would have powerful geometric criteria for rational approximation, which are otherwise missing (see [V2] and [Vi2]).

On the other hand, it has recently been established that a close variant of γ, called positive analytic capacity, is indeed semi-additive. The positive analytic capacity of a compact set E is

$$\gamma^+(E) = \sup \mu(E),$$

where the supremum is taken over the positive Borel measures μ supported on E such that the Cauchy potential $f = \dfrac{1}{z} * \mu$ is a function in $L^\infty(\mathbb{C})$ with $\|f\|_\infty \leqslant 1$. Since $(\dfrac{1}{z} * \mu)'(\infty) = \mu(E)$, we clearly have $\gamma^+(E) \leqslant \gamma(E)$.

Since γ^+ is semi-additive, as shown in [T1], it is clear that (2) follows from the inequality

(3) $$\gamma(E) \leqslant C\gamma^+(E), \quad E \text{ compact } \subset \mathbb{C},$$

where the positive constant C does not depend on E. We will see below that our main result provides a proof of (3) for a particular (but significant) class of sets E. To the best of our knowledge, the first mention of (3) that can be found in the literature is in [DO]. An equivalent form of (3), which involves Menger curvature, has recently been conjectured by Melnikov (see [D]).

Now we turn our attention to Cantor sets. Given a sequence $\lambda = (\lambda_n)_{n=1}^\infty$, $0 \leqslant \lambda_n \leqslant 1/3$, we construct a Cantor set by the following algorithm. Consider the unit square $Q^0 = [0,1] \times [0,1]$. At the first step we take 4 closed squares inside Q^0, of side-length λ_1, with sides parallel to the coordinate axes, such that each square contains a vertex of Q^0. At step 2 we apply the preceding procedure to each of the 4 squares produced at step 1, but now using the proportion factor λ_2. Then we obtain 16 squares of side-length $\sigma_2 = \lambda_1\lambda_2$. Proceeding inductively, we have at the n-th step 4^n squares Q_j^n, $1 \leqslant j \leqslant 4^n$, of side-length $\sigma_n = \prod_{j=1}^n \lambda_j$. Write

$$E_n = E(\lambda_1, \ldots, \lambda_n) = \bigcup_{j=1}^{4^n} Q_j^n,$$

and define the Cantor set associated to the sequence $\lambda = (\lambda_n)_{n=1}^\infty$ by the identity

$$E = E(\lambda) = \bigcap_{n=1}^\infty E_n.$$

Our main result reads as follows.

Theorem 1. *The Cantor set $E(\lambda)$ has zero analytic capacity if and only if*

$$\sum_{n=1}^\infty \frac{1}{(4^n \sigma_n)^2} = \infty.$$

The assumption $\lambda_n \leqslant 1/3$ for the Cantor sets $E(\lambda)$ is purely technical. Actually Theorem 1 (as well as Theorem 2 below) holds for any sequence

$(\lambda_n)_n$ with $0 < \lambda_n < 1/2$. See Remark 2 at the end of the paper for more details.

Mattila showed in [Ma] that the above condition is necessary and our contribution in this paper is proving the sufficiency. The special case $\lambda_n = 1/4$, $n \geqslant 1$, was obtained independently by Garnett [G1] and Ivanov [I1] in the 1970's and, since then, the "corner quarters" Cantor set has become the favorite example of a set of zero analytic capacity and positive length. P. Jones gave in [J] an alternative proof of Garnett's result, based on harmonic measure. Recently Jones' approach has been used to establish the vanishing of the analytic capacity of $E(\lambda)$ for some special classes of sequences $\lambda = (\lambda_n)_{n=1}^{\infty}$ with $4^n\sigma_n$ tending to infinity [GY].

Theorem 1 follows from a more precise result on the analytic capacity of the set $E_N = E(\lambda_1, \ldots, \lambda_N)$. The asymptotic behaviour of $\gamma^+(E_N)$ is completely understood: for some constant $C > 1$ and all $N = 1, 2, \ldots$ one has

$$(4) \qquad C^{-1}\Big(\sum_{n=1}^{N} \frac{1}{(4^n\sigma_n)^2}\Big)^{-1/2} \leqslant \gamma^+(E_N) \leqslant C\Big(\sum_{n=1}^{N} \frac{1}{(4^n\sigma_n)^2}\Big)^{-1/2}.$$

The upper estimate is due to Eiderman [E] and a different proof has been given in [T2]. The lower estimate was proved by Mattila in [Ma]. However, the result was not explicitly stated in [Ma], presumably because at that time the main object of interest was γ rather than γ^+. An indication of how one proves the lower estimate in (4) will be provided at the end of Section 2. See also [E, p. 821].

Theorem 1 follows from the upper estimate in (4) and the next result.

Theorem 2. *There exists a positive constant C_0 such that*

$$(5) \qquad\qquad\qquad \gamma(E_N) \leqslant C_0\gamma^+(E_N),\ N = 1, 2, \ldots$$

If $\lambda_n = 1/4$, $n \geqslant 1$, then combining Theorem 2 with (4) we get

$$\gamma(E_N) \leqslant \frac{C}{\sqrt{N}},\ N = 1, 2, \ldots,$$

which improves considerably Murai's inequality [Mu]

$$\gamma(E_N) \leqslant \frac{C}{\log N},\ N = 2, 3, \ldots,$$

the best estimate known up to now.

The main tool used in our proof of Theorem 2 is the local $T(b)$-Theorem of M. Christ [CH2], a particular version of which will be discussed and stated in Section 2. Section 2 contains also some basic facts on the Cauchy transform and the Plemelj formulae. The proof of Theorem 2 is presented in Section 3.

Our notation and terminology are standard. For example $D(z, r)$ is the open disk centered at z and of radius r, ds is the arclength measure on a

rectifiable arc and $P \simeq Q$ means that $C^{-1}Q \leqslant P \leqslant CQ$ for some absolute constant $C > 1$.

The symbols $C, C', C'', C_0, C_1, \dots$ stand for absolute constants with a definite value. We will also use the symbol A to denote an absolute constant that may vary at different occurrences.

Remark 1. The second author [T3] has recently proved that Theorem 2 also holds for a general compact set. In particular, this implies the semiadditivity of analytic capacity. The proof in [T3] also involves and induction argument and an appropriate $T(b)$-Theorem as in the present paper.

2. Background results

2.1. **Cauchy Integrals.** Fix an integer $M > 0$ and let $E_M = E(\lambda_1, \dots, \lambda_M)$ be the M-th approximation of the Cantor set associated to the sequence $(\lambda_n)_{n=1}^{\infty}$. Then E_M is the union of 4^M closed squares Q_j^M, $1 \leqslant j \leqslant 4^M$, and ∂E_M is the union of the 4^M closed piecewise linear curves ∂Q_j^M. For a Borel measure μ supported on ∂E_M set

$$C(\mu)(z) = \int \frac{d\mu(\zeta)}{\zeta - z}, \quad z \notin \partial E_M,$$

and

$$C(\mu)(z) = \lim_{\varepsilon \to 0} \int_{|\zeta - z| > \varepsilon} \frac{d\mu(\zeta)}{\zeta - z}, \quad z \in \partial E_M,$$

whenever the principal value integral exists. Let $C^+(\mu)(z)$ (respectively $C^-(\mu)(z)$) stand for the non-tangential limit of $C(\mu)(w)$ as w tends to z from the interior of E_M (respectively, from the complement of E_M). It follows from standard classical results that $C\mu(z)$, $C^+\mu(z)$ and $C^-\mu(z)$ exist for almost all z with respect to arclength measure ds on ∂E_M. Moreover one has the Plemelj formulae (see [V3])

$$(6) \qquad \begin{cases} C^+\mu(z) = C\mu(z) + \pi i f(z) \\ C^-\mu(z) = C\mu(z) - \pi i f(z), \end{cases}$$

where the identities hold for ds-almost all $z \in \partial E_M$ and $\mu = f(z)dz + \mu_s$, f being integrable and μ_s being singular with respect to ds.

Assume that one has $\mu_s = 0$ and

$$|f(z)| \leqslant A, \text{ for } ds\text{-almost all } z \in \partial E_M,$$

and that one wants to show

$$(7) \qquad |C(\mu)(z)| \leqslant A, \text{ for } ds\text{-almost all } z \in \partial E_M.$$

Then one only has to check that

$$|C(\mu)(z)| \leqslant A, \text{ for } z \notin E_M,$$

because then

$$|C^-(\mu)(z)| \leqslant A, \text{ for } ds\text{-almost all } z \in \partial E_M,$$

and thus the second identity in (6) gives (7).

2.2. **The local $T(b)$-Theorem.** The local $T(b)$-Theorem is a criterion for the L^2 boundedness of a singular integral that was proved originally by M. Christ in the setting of homogeneous spaces [CH2]. We state below a very particular version of Christ's result, which is adapted to the principal value Cauchy Integral and to a measure μ supported on ∂E_M. The reader may think that μ is of the form $\mu = c \, ds_{|\partial E_M}$ for some (small) positive constant c. However, one should keep in mind that, for $4^n \sigma_n \nearrow \infty$, $ds_{|\partial E_M}$ does not satisfy condition (i) in the statement below with a constant independent of M. As it will become clear later, an appropiate choice of c is required to get (i) and (iii) with absolute constants.

Theorem. *(Christ). Let μ be a positive Borel measure supported on ∂E_M satisfying, for some absolute constant C, the following conditions:*

  (i) $\mu(D(z,r)) \leqslant Cr$, $z \in \partial E_M$, $r > 0$,
  (ii) $\mu(D(z,2r)) \leqslant C\mu(D(z,r))$, $z \in \partial E_M$, $r > 0$,
  (iii) *For each disc D centered at a point in ∂E_M there exists a function b_D in $L^\infty(\mu)$, b_D supported on D, satisfying $|b_D| \leqslant 1$ and $|C(b_D\mu)| \leqslant 1$ μ-almost everywhere on ∂E_M, and $\mu(D) \leqslant C \left| \int b_D d\mu \right|$.*
    Then

$$(8) \qquad \int |C(f\mu)|^2 \, d\mu \leqslant C' \int |f|^2 d\mu, \quad f \in L^2(\mu),$$

  for some absolute constant C' (depending only on C).

The relevance of inequality (8) for our problem lies in the fact that it implies

$$(9) \qquad \mu(K) \leqslant C''\gamma^+(K), \quad K \text{ compact } \subset \partial E_M,$$

for some absolute constant C'' (depending only on C').

The derivation of (9) from (8) goes through a well-known path: first, by classical Calderón-Zygmund Theory one gets a weak $(1,1)$ inequality from (8); then, a surprisingly simple method to dualize a weak $(1,1)$ inequality leads immediately to (9). The original argument is in [DO]. Some years before [DO] Uy found a slightly different way of dualizing a weak $(1,1)$ inequality, which, however, does not yield (9) (see [Uy]). The interested reader will find additional information in [CH1], [T1] and [V1].

Inequality (9) explains also why the lower estimate in (4) follows from Mattila's arguments in [Ma]: see Theorem 3.7 in p. 202 and the first paragraph after it.

3. Proof of Theorem 2

We first give a sketch of the argument. Assume that one can find a positive Borel measure μ supported on ∂E_N, $E_N = E(\lambda_1, \ldots, \lambda_N)$, which satisfies (i) and (ii) with M replaced by N, such that $\|\mu\| = \gamma(E_N)$ and the

Cauchy Integral is bounded on $L^2(\mu)$. Then we get (9) with M replaced by N, as we explained in section 2. For $K = \partial E_N$ (9) yields

$$\gamma(E_N) = \|\mu\| \leqslant A\gamma^+(E_N),$$

as desired. In the actual argument we do not construct μ on E_N. For reasons that will become clear later we are forced to work in E_M with M smaller than N. On the other hand, M cannot be much smaller than N, because in the course of the subsequent reasoning one needs to have $\gamma^+(E_M) \leqslant A\gamma^+(E_N)$. Hence M has to be chosen carefully, in such a way that the local $T(b)$-Theorem can be applied to get the boundedness of the Cauchy Integral on $L^2(\mu)$, with an absolute constant.

Now we start the proof of Theorem 2.

Set $a_n = 4^n \sigma_n$ and

$$S_n = \frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2}.$$

We can assume, without loss of generality, that for each $N > 1$, there exists M, $1 \leqslant M < N$, such that

$$(10) \qquad\qquad S_M \leqslant \frac{S_N}{2} < S_{M+1}.$$

Otherwise $\dfrac{S_N}{2} < S_1$ and thus, by (4), $\gamma^+(E_N) \geqslant A^{-1}\lambda_1$. On the other hand, taking into account the obvious estimate of analytic capacity by length, we clearly have

$$\gamma(E_N) \leqslant \gamma(E_1) \leqslant \frac{1}{2\pi}\,\text{length}(\partial E_1) = \frac{8}{\pi}\lambda_1.$$

Therefore (5) is trivial in the present case, provided C_0 is chosen to satisfy $C_0 \geqslant \dfrac{8}{\pi}A$.

Assume, then, that (10) holds and let's proceed to prove (5) by induction on N. The case $N = 1$ is obviously true. The induction hypothesis is

$$\gamma(E_n) \leqslant C_0\gamma(E_n^+), \quad 0 < n < N,$$

where the precise value of the absolute constant C_0 will be determined later.

We distinguish two cases.

Case 1: For some absolute constant C_1, to be determined later,

$$(11) \qquad\qquad a_M\gamma(E(\lambda_{M+1}, \ldots, \lambda_N)) \leqslant C_1\gamma(E_N).$$

Case 2: (11) does not hold.

We deal first with Case 2. By the induction hypothesis applied to the sequence $\lambda_{M+1}, \ldots, \lambda_N$ and by (4) we have

$$\gamma(E_N) \leqslant C_1^{-1} a_M \gamma(E(\lambda_{M+1}, \ldots, \lambda_N)) \leqslant$$

$$\leqslant C_1^{-1} C_0 a_M \gamma^+(E(\lambda_{M+1}, \ldots, \lambda_N)) \leqslant$$

$$\leqslant C_1^{-1} C_0 A \frac{a_M}{\left(\sum_{n=M+1}^{N} \frac{1}{(4\lambda_{M+1} \ldots 4\lambda_n)^2} \right)^{1/2}} =$$

$$= C_1^{-1} C_0 A \frac{1}{\left(\sum_{n=M+1}^{N} \frac{1}{a_n^2} \right)^{1/2}}.$$

Clearly, the inequality $S_M \leqslant \dfrac{S_N}{2}$ is equivalent to

$$\frac{1}{\left(\sum_{n=M+1}^{N} \frac{1}{a_n^2} \right)^{1/2}} \leqslant \frac{\sqrt{2}}{\left(\sum_{n=1}^{N} \frac{1}{a_n^2} \right)^{1/2}},$$

and so, again by (4),

$$\gamma(E_N) \leqslant C_1^{-1} C_0 A \gamma^+(E_N).$$

If $C_1 = A$, where A is the constant in the preceding inequality, we get (5), as desired.

Let's now consider Case 1. Set

$$\mu = \frac{\gamma(E_N)}{\text{length}(\partial E_M)} ds_{|\partial E_M},$$

so that $\|\mu\| = \gamma(E_N)$. To check condition (i) in the local $T(b)$-Theorem of Section 2 we consider two cases. If $r \leqslant \sigma_M$, then

$$\mu(D) \leqslant \frac{\gamma(E_N)}{\text{length}(\partial E_M)} 2r \leqslant Cr,$$

because $\gamma(E_N) \leqslant \gamma(E_M) \leqslant \text{length}(\partial E_M)$.

For $r > \sigma_M$ we can replace arbitrary discs centered at points in ∂E_M by the squares Q_j^n, $0 \leqslant n \leqslant M$, $0 \leqslant j \leqslant 4^n$. In other words, it suffices to prove

(12) $$\mu(Q_j^n) \leqslant C\ell(Q_j^n), \ 0 \leqslant n \leqslant M, \ 1 \leqslant j \leqslant 4^n.$$

To show this, given a disc D of radius r centered at $z \in \partial E_M$, one considers a square $Q_j^n \supset D$, where n is chosen so that $\ell(Q_j^n)$ is comparable to r, Then,

$$\mu(Q_j^n) = \gamma(E_N) \frac{1}{4^n} = \frac{4\gamma(E_N)}{\text{length}(\partial E_n)} \ell(Q_j^n) \leqslant \frac{4\gamma(E_n)}{\text{length}(\partial E_n)} \ell(Q_j^n) \leqslant \frac{2}{\pi} \ell(Q_j^n).$$

and so (12) is proved.

It is also a simple matter to ascertain that (ii) holds.

Now our goal is to prove that the hypothesis (iii) of the local $T(b)$-Theorem is satisfied. Once this has been verified, (9) applied to $K = \partial E_M$ yields

$$(13) \qquad \gamma(E_N) = \|\mu\| \leqslant A\gamma^+(E_M) \leqslant \frac{A}{S_M^{1/2}}.$$

Assuming that (13) holds, to complete the proof we distinguish again two cases, according to whether $1/a_{M+1}^2$ is greater than S_M or not.

If $1/a_{M+1}^2 > S_M$, then

$$S_{M+1} = S_M + \frac{1}{a_{M+1}^2} \simeq \frac{1}{a_{M+1}^2},$$

and so

$$\gamma^+(E_{M+1}) \simeq \frac{1}{S_{M+1}^{1/2}} \simeq a_{M+1} = \frac{1}{4}\operatorname{length}(\partial E_{M+1}).$$

Hence

$$\gamma(E_N) \leqslant \gamma(E_{M+1}) \leqslant \frac{1}{2\pi}\operatorname{length}(\partial E_{M+1}) \simeq \gamma^+(E_{M+1}) \simeq \gamma^+(E_N),$$

and thus (5) holds for a sufficiently big constant C_0.

If $1/a_{M+1}^2 \leqslant S_M$, then $S_{M+1} \simeq S_M$ and so

$$\gamma^+(E_N) \simeq \frac{1}{S_N^{1/2}} \simeq \frac{1}{S_M^{1/2}},$$

which gives (5), with a big enough C_0, by (13).

Summing up, we have reduced the proof of Theorem 2 to checking that the hypothesis (iii) of the local $T(b)$-Theorem is satisfied. As we already remarked when dealing with hypothesis (i), in proving (iii) we can replace discs centered at points in ∂E_M by squares Q_j^n, $1 \leqslant j \leqslant 4^n$, $0 \leqslant n \leqslant M$. In other words, it is enough to show that, given a square Q_j^n, $0 \leqslant n \leqslant M$, $1 \leqslant j \leqslant 4^n$, there exists a function b_j^n in $L^\infty(\mu)$, supported on Q_j^n, satisfying $|b_j^n| \leqslant 1$ and $|C(b_j^n \mu)| \leqslant 1$ $d\mu$-almost everywhere on ∂E_M and such that

$$\mu(Q_j^n) \leqslant C \left| \int b_j^n d\mu \right|.$$

Let f be the Ahlfors function of E_N. Then f is analytic on $\mathbb{C}\backslash E_N$, $|f(z)| \leqslant 1$, $z \notin E_N$, $f(\infty) = 0$ and $f'(\infty) = \gamma(E_N)$. The non-tangential boundary value of f at $\zeta \in \partial E_N$, which exists for ds-almost all $\zeta \in \partial E_N$, is denoted by $f(\zeta)$. Set $\nu = \frac{1}{2\pi i}f(\zeta)d\zeta_{|\partial E_N}$, so that

$$f(z) = \int \frac{1}{z - \zeta}d\nu(\zeta), \quad z \notin E_N.$$

Fix a generation n, $0 \leqslant n \leqslant M$. Then, for some index k, $1 \leqslant k \leqslant 4^n$,

$$\gamma(E_N) = \sum_{j=1}^{4^n} \nu(Q_j^n) \leqslant 4^n |\nu(Q_k^n)|,$$

or, equivalently, $\mu(Q_k^n) \leqslant |\nu(Q_k^n)|$.

To define the function b_k^n associated to Q_k^n we need to describe a simple preliminary construction.

Take a compactly supported C^∞ function φ on $\mathbb{C}$, $0 \leqslant \varphi \leqslant 1$, $\int_{\partial Q^0} \varphi ds \geqslant 1$, such that φ vanishes on $\cup_{j=1}^4 D(z_j, 1/4)$, where the z_j are the vertices of Q^0. Then $|C(\varphi ds_{|\partial Q_0})| \leqslant A$, as one checks easily. Set

$$\varphi_j^M(z) = \varphi\left(\frac{z - v_j^M}{\sigma_M}\right)\chi_{Q_j^M},$$

where v_j^M is the left lower vertex of Q_j^M. Hence $|C(\varphi_j^M d\mu)| \leqslant A$ and

$$\int \varphi_j^M d\mu = \frac{1}{\text{length}(\partial Q_j^M)} \int \varphi_j^M ds \quad \mu(Q_j^M) \geqslant \frac{1}{4}\mu(Q_j^M).$$

Define

$$b = b_k^n = \sum_{Q_j^M \subset Q_k^n} \nu(Q_j^M)\frac{\varphi_j^M}{\int \varphi_j^M d\mu}.$$

For $j \neq k$ we construct b_j^n by simply translating b_k^n. We have $Q_j^n = w_j^n + Q_k^n$, for some complex number w_j^n. Set

$$b_j^n(z) = b_k^n(z - w_j^n), \ z \in \mathbb{C}.$$

Now we will prove that b_k^n satisfies condition (iii). Clearly,

$$\left|\int b d\mu\right| = |\nu(Q_k^n)| \geqslant \mu(Q_k^n).$$

To show that b is bounded it suffices to prove

(14) $$|\nu(Q_j^M)| \leqslant A\mu(Q_j^M), \quad 1 \leqslant j \leqslant 4^M,$$

and for this we first remark that $|C(\chi_{Q_j^M}\nu)(z)| \leqslant A$, $z \notin E_N$. This is proved in [G2, Lemma 2.3 (a), p. 90]. Since $C(\chi_{Q_j^M}\nu)$ is analytic outside $Q_j^M \cap E_N$, we conclude that

$$|\nu(Q_j^M)| \leqslant A\gamma(Q_j^M \cap E_N).$$

Now notice that the set $Q_j^M \cap E_N$ can be obtained from $E(\lambda_{M+1}, \ldots, \lambda_N)$ by a dilation of factor σ_M and a translation. Hence, recalling (11),

$$\gamma(Q_j^M \cap E_N) = \sigma_M \gamma(E(\lambda_{M+1}, \ldots, \lambda_N)) \leqslant C_1 \frac{1}{4^M}\gamma(E_N) = C_1\mu(Q_j^M),$$

which gives (14). It is worth pointing out at this point that the above inequality explains why M cannot be taken to be N.

Thanks to the discussion in Section 2 on Cauchy Integrals and the Plemelj formulae, it becomes clear that we are only left with the task of proving

$$|C(bd\mu)(z)| \leqslant A, \quad z \notin E_M \cap Q_k^n.$$

Since $|C(\chi_{Q_k^n}\nu)(z)| \leqslant A$, $z \notin E_N \cap Q_k^n$, we only need to estimate, for $z \notin E_M \cap Q_k^n$, the difference

$$(15) \qquad C(bd\mu)(z) - C(\chi_{Q_k^n}\nu)(z) = \sum_{Q_j^M \subset Q_k^n} C(\alpha_j^M)(z),$$

where

$$\alpha_j^M = \nu(Q_j^M)\frac{\varphi_j^M d\mu}{\int \varphi_j^M d\mu} - \chi_{Q_j^M}\nu.$$

We have $\int d\alpha_j^M = 0$ and $|C(\alpha_j^M)(z)| \leqslant A$, $z \notin Q_j^M$, $1 \leqslant j \leqslant 4^M$, using again [G2, Lemma 2.3 (a), p. 90].

Thus, if z_j^M is the center of Q_j^M,

$$(16) \qquad |C(\alpha_j^M)(z)| \leqslant A\frac{\sigma_M^2}{\text{dist}(z, Q_j^M)^2}, \quad |z - z_j^M| > \sigma_M.$$

By the maximum principle, in estimating (15) we can assume that $|z - z_j^M| \leqslant \sigma_M$ for some j with $Q_j^M \subset Q_k^n$. Hence (15) is not greater than

$$(17) \qquad A + A\sum_{l \neq j} \frac{\sigma_M^2}{\text{dist}(z, Q_l^M)^2}.$$

For $0 \leqslant n \leqslant M$ let Q^n be the square in the n-th generation that contains Q_j^M. We can estimate (17) by

$$A + A\sum_{n=0}^{M-1} \sum_{Q_l^M \subset Q^n \backslash Q^{n+1}} \frac{\sigma_M^2}{\text{dist}(z, Q_l^M)^2} \leqslant$$

$$\leqslant A + A\sum_{n=0}^{M-1} \frac{\sigma_M^2}{\sigma_n^2}4^{M-n} \leqslant A + A\sum_{n=0}^{M-1}\left(\frac{4}{9}\right)^{M-n} \leqslant A,$$

because $\sigma_M = \sigma_n\lambda_{n+1}\ldots\lambda_M \leqslant \dfrac{\sigma_n}{3^{M-n}}$, $0 \leqslant n \leqslant M$.

This completes the construction of the function b_k^n associated to the square Q_k^n as required by hypothesis (iii) in the local $T(b)$-Theorem.

Now, by translation invariance it is clear that b_j^n for $j \leqslant k$ also satisfies (iii).

This shows that the local $T(b)$-Theorem can be applied to μ and thus completes the proof of Theorem 2. $\square$

Remark 2. For simplicity, we assumed above that $\lambda_n \leqslant 1/3$ for all n. However, both Theorem 1 and Theorem 2 hold for $0 < \lambda_n < 1/2$. Let us sketch the changes needed in the proof.

First it should be noticed that the estimate (4) for $\gamma^+(E_N)$ holds for $0 < \lambda_n < 1/2$. Indeed, the arguments for the left inequality in (4) in [Ma] are valid in this case. On the other hand, the right inequality in (4) is also true for $0 < \lambda_n < 1/2$. For example, arguing as in [T2] one can easily check that

$$\gamma^+(E_N) \leqslant A \left(1 + \sum_{\substack{1 \leqslant n \leqslant N, \\ \lambda_n \leqslant 1/3}} \frac{1}{(4^n \sigma_n)^2} \right)^{-1/2} \approx \left(\sum_{1 \leqslant n \leqslant N} \frac{1}{(4^n \sigma_n)^2} \right)^{-1/2}.$$

The other places where the assumption $\lambda_n \leqslant 1/3$ has been used are (14) and (15). The inequality (14) also holds for $0 < \lambda_n < 1/2$. It follows from Vitushkin's estimates for the integral $\int_\Gamma f(z)\,dz$ for piecewise Lyapunov curves Γ [Vi1] (in our case $\Gamma = \partial Q_j^M$). To prove (15) one can use the sharper estimate

$$|C(\alpha_j^M)(z)| \leqslant A \frac{\sigma_M \, \gamma(Q_j^M \cap E_N)}{\operatorname{dist}(z, Q_j^M)^2} \leqslant A C_1 \frac{\sigma_M \, \mu(Q_j^M)}{\operatorname{dist}(z, Q_j^M)^2}, \quad |z - z_j^M| > \sigma_M,$$

instead of (16) (see [G2, p.12–13], for example). We leave the details for the reader.

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