L^2 BOUNDEDNESS OF THE CAUCHY INTEGRAL AND MENGER CURVATURE

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ABSTRACT. In this paper we explain the relevance of Menger curvature in understanding the L^2 boundedness properties of the Cauchy Integral Operator. After introducing Menger curvature and describing its basic properties we proceed to prove the Coifman-McIntosh-Meyer Theorem on the Cauchy Integral on a Lipschitz graph. From this circle of ideas comes a new simple approach to the L^2 boundedness of the first Calderón commutator. We point out that the L^2 boundedness of the Cauchy Integral on a Lipschitz graph can be easily reduced to the boundedness of the first commutator. In the last section we describe the various steps in the solution of Vitushkin's conjecture on analytic capacity paying special attention to the role played by Menger curvature.

1. INTRODUCTION

The Menger curvature associated to a triple z_1, z_2 and z_3 of distinct points in the plane is the inverse of the radius of the circle passing through the given points. The Cauchy kernel $1/z$ and Menger curvature are related by a remarkable identity that has emerged recently and that has been shown to be extremely useful in dealing with L^2 boundedness of the Cauchy Integral Operator and, consequently, with analytic capacity. New striking results have been proved and new light has been shed on some classical theorems. The purpose of this article is to describe some of the main ideas behind these developments and try to make them accessible to analysts not necessarily familiar with the subject. On the other hand, we do not aim at any kind of completeness.

One intriguing fact about Menger curvature is that, up to now, no useful higher dimensional analogue that relates well to the Riesz kernels has been found. Therefore, for the Riesz transforms in higher dimensions no geometric approach is known that parallels the route available for the Cauchy kernel that will be described below. In fact, many higher dimensional versions of the results that will be presented in the next sections are still open.

2. MENGER CURVATURE

During the last few years the Menger's name has occurred in several parts of classical analysis. Some biographic notes will help place the person into a historical perspective.

Karl Menger was born in Vienna in 1902 and died in Chicago in 1985. He wrote a Thesis on the definition of dimension for separable metric spaces under the direction of H. Hahn and became soon a mathematician of broad scientific interests. He worked in topology, differential geometry, calculus of variations, logic, graph theory and economics. Before the age of 30 he had written 65 papers. In 1937 he emigrated to the United States escaping from the deteriorated political

and social atmosphere created in Austria by Hitler's regime. His name is attached, besides to a notion of curvature in general metric spaces, to the universal curve or Menger sponge [BM] and to some quantities, called Caley-Menger determinants [B], he used to characterize Euclidean spaces among metric spaces. The interested reader is referred to [K] where more detailed information on Menger's life and his mathematical work is provided and further references are given.

Given three distinct points in the plane, say z_1, z_2 and z_3 the Menger curvature associated to them is

$$c(z_1, z_2, z_3) = R^{-1},$$

where R is the radius of the circle passing through z_1, z_2 and z_3. If two of the three points coincide then we set $c(z_1, z_2, z_3) = 0$. In this way the condition $c(z_1, z_2, z_3) = 0$ is equivalent to the three points being collinear.

The picture below shows examples of triples having small curvature. In each case the reader is invited to imagine the circle passing through the three points and to realize that its radius is large. Indeed, the two pictures are essentially the same modulo a rotation and a dilation.
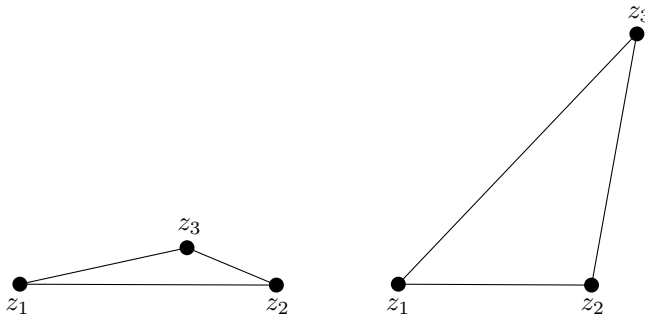

FIGURE 1

There are a couple of beautiful formulas for Menger curvature coming from elementary geometry that help in understanding the size of $c(z_1, z_2, z_3)$ in concrete examples. The first one is

$$(2.1) \qquad c(z_1, z_2, z_3) = 2\,\frac{\sin \alpha}{l}$$

where l is the length of a side of the triangle T determined by z_1, z_2 and z_3 and α is the angle opposite to that side. The second formula, which can be deduced readily from (2.1), is

$$(2.2) \qquad c(z_1, z_2, z_3) = \frac{4\,\mathrm{area}(T)}{|z_1 - z_2||z_1 - z_3||z_2 - z_3|}.$$

From (2.1) (or (2.2)) is completely obvious that the curvature of the two triples shown in Figure 1 becomes small provided z_1 and z_2 are kept fixed and z_3 tends, in the first picture, to a given point in between z_1 and z_2, and to infinity in the second picture. As a last example take $z_1 = -1$, $z_2 = 1$ and $z_3 = e^{i\epsilon}$, $\epsilon \in \mathbb{R}$. As ϵ tends to 0, $c(z_1, z_2, z_3)$ is always 1, in spite of the fact that z_3 gets very close to the straight line determined by z_1 and z_2.

From Heron's formula relating the area of a triangle to its perimeter, one can write down an expression for $c(z_1, z_2, z_3)$ involving only the mutual distances between z_1, z_2 and z_3. This remark is not going to be used in the sequel but it could be useful if one were interested in considering some of the problems we are dealing with in more general metric spaces. It also helps in getting an intuition of why Menger curvature and Caley-Menger determinants are so useful in characterizing Euclidean spaces among metric spaces.

The relationship between Menger curvature and the Cauchy kernel appears as follows. Take three points z_1, z_2 and z_3 in the complex plane and consider the Cauchy matrix associated to them

$$C = \begin{pmatrix} 0 & \frac{1}{z_1 - z_2} & \frac{1}{z_1 - z_3} \\ \frac{1}{z_2 - z_1} & 0 & \frac{1}{z_2 - z_3} \\ \frac{1}{z_3 - z_1} & \frac{1}{z_3 - z_2} & 0 \end{pmatrix}.$$

Let $\mathbf{1}$ denote the vector $(1, 1, 1) \in \mathbb{C}^3$, set

$$(2.3) \qquad \langle z, w \rangle = \sum z_i \overline{w_i}, \text{ for } z, w \in \mathbb{C}^3,$$

and let C^* stand for the adjoint of C with respect to the standard hermitian product (2.3) in $\mathbb{C}^3$.

With the notation just established one gets

$$\|C(\mathbf{1})\|^2 = \langle C^* C(\mathbf{1}), \mathbf{1} \rangle = \sum_{i \neq j} |z_i - z_j|^{-2} + \sum_\sigma \frac{1}{(z_{\sigma(2)} - z_{\sigma(1)})(\overline{z_{\sigma(3)} - z_{\sigma(1)}})},$$

where in the second term the sum is over all permutations of $\{1, 2, 3\}$. Consequently, the square of the norm of the vector $C(\mathbf{1})$ is given by the sum of two terms, the first of which is the trace of $C^* C$ and turns out to be a positive number, in accordance with the fact that $C^* C$ is a positive matrix. The second term is the sum of the six entries of $C^* C$ which are not on the main diagonal. Since $C^* C$ is positive, this quantity must be real. The absolutely astonishing fact, for which the author has no explanation besides the blind computation performed below, is that the second term is a non-negative number. More precisely, we have the following.

Lemma 2.1. *Given three points z_1, z_2 and z_3 in the plane one has*

$$(2.4) \qquad \sum_\sigma \frac{1}{(z_{\sigma(2)} - z_{\sigma(1)})(\overline{z_{\sigma(3)} - z_{\sigma(1)}})} = c(z_1, z_2, z_3)^2,$$

where the sum is over all permutations of $\{1, 2, 3\}$.

The above identity was discovered by Melnikov in the course of his research on discrete expressions for analytic capacity [Me]. That (2.4) could be used in the *Calderón-Zygmund* theory of the Cauchy Integral was realized later by the author ([V2] and [MV]).

Proof of the Lemma. Coupling each term in (2.4) with its complex conjugate we see that the left hand side of (2.4) is equal to

$$(2.5) \qquad 2 \sum_{i=1}^3 \left(\frac{z_j - z_i}{|z_j - z_i|}, \frac{z_k - z_i}{|z_k - z_i|} \right) \frac{1}{|z_j - z_i||z_k - z_i|}$$

where (,) stands for the standard scalar product in $\mathbb{R}^2 = \mathbb{C}$, and, for each given i, j and k are the other two indices in $\{1, 2, 3\}$. Let l_i, $i = 1, 2, 3$, denote the lengths

of the sides of the triangle determined by z_1, z_2 and z_3 and let θ_i be the angle opposite to the side of length l_i. Then (2.5) is equal to

$$2\left(\frac{\cos\theta_1}{l_2 l_3} + \frac{\cos\theta_2}{l_1 l_3} + \frac{\cos\theta_3}{l_1 l_2}\right),$$

which by (2.1) is

$$c(z_1, z_2, z_3)^2 \, \frac{\sin(2\theta_1) + \sin(2\theta_2) + \sin(2\theta_3)}{4\sin\theta_1 \sin\theta_2 \sin\theta_3} = c(z_1, z_2, z_3)^2$$

where the last identity follows by eliminating θ_3 and working out elementary trigonometric formulas. □

3. The Cauchy Integral on Lipschitz graphs: background

In this section we introduce some background facts on the L^2 boundedness of the Cauchy Integral on a Lipschitz graph: we describe two possible routes to get to the problem and we briefly sketch some relevant parts of the long history of the proof.

Let Γ be a simple closed oriented rectifiable curve in the plane or a simple oriented locally rectifiable curve through infinity. One should have in mind the model examples of the unit circle $\{z : |z| = 1\}$ and the real axis. Because of rectifiability there is an arc-length measure on Γ, which we denote by $|dz| = ds$. Take $f \in L^1(ds)$ and consider its Cauchy Integral

$$(3.1) \qquad Cf(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta, \quad z \notin \Gamma.$$

We would like to understand the behaviour of $Cf(w)$ as the point $w \notin \Gamma$ tends to some point z on Γ. Thus our first task is to discuss the possibility of giving a sense to the integral in the right hand side of (3.1) for $z \in \Gamma$. It becomes clear immediately that the integral in (3.1) is not necessarily absolutely convergent if $z \in \Gamma$, because the kernel is not absolutely integrable with respect to ds. Because of this, one looks at the truncated integrals

$$C_\epsilon f(z) = \frac{1}{2\pi i} \int_{|\zeta - z| > \epsilon} \frac{f(\zeta)}{\zeta - z}\, d\zeta, \quad z \in \Gamma, \quad \epsilon > 0,$$

which are always absolutely convergent, and sets

$$(3.2) \qquad Cf(z) = \lim_{\epsilon \to 0} C_\epsilon f(z), \quad z \in \Gamma,$$

whenever the limit exists. In other words, the integral in (3.1) should be understood in the principal value sense whenever $z \in \Gamma$ and so the problem of existence of principal values of Cauchy Integrals arises naturally. Now, if f has some smoothness, say f is the restriction to Γ of a function of class C^1 on the whole plane (compactly supported if Γ is unbounded), then (3.2) exists for all $z \in \Gamma$. To prove this simple fact, denote by Ω^+ and Ω^- the domains on the left and right hand sides of Γ respectively. The argument becomes easier if Γ is compact and so we concentrate on this case. Assume also that the orientation has been chosen so that Ω^+ is bounded. Setting $\gamma_\epsilon = \{\zeta : |\zeta - z| > \epsilon\} \cap \Gamma$ and $\sigma_\epsilon = \{\zeta : |\zeta - z| = \epsilon\} \cap \Omega^+$ we have

$$\int_{\gamma_\epsilon} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \int_{\gamma_\epsilon} \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta + f(z) \int_{\sigma_\epsilon} \frac{d\zeta}{\zeta - z}$$
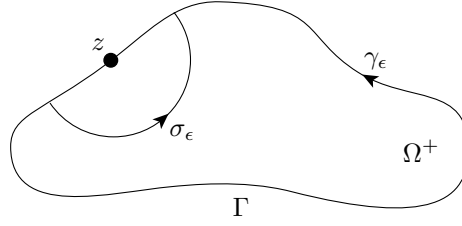
FIGURE 2

and thus, dividing by $2\pi i$ and letting $\epsilon \to 0$, we get

$$(3.3) \qquad Cf(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta) - f(z)}{\zeta - z} \, d\zeta + \frac{f(z)}{2},$$

provided Γ has a tangent at the point z.

Let us keep the smoothness assumption on f and consider what happens to $Cf(w)$ when $w \notin \Gamma$ tends to z. Notice that

$$Cf(w) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta) - f(w)}{\zeta - w} \, d\zeta + f(w), \quad w \in \Omega^+,$$

and

$$Cf(w) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta) - f(w)}{\zeta - w} \, d\zeta, \quad w \in \Omega^-.$$

If we let $C^\pm f(z)$ stand for the limits of $Cf(w)$ as w tends to z from $\Omega^\pm$, then by dominated convergence

$$C^+ f(z) = \frac{1}{2\pi i} \int_\Omega \frac{f(\zeta) - f(z)}{\zeta - z} \, d\zeta + f(z)$$

and

$$C^- f(z) = \frac{1}{2\pi i} \int_\Omega \frac{f(\zeta) - f(z)}{\zeta - z} \, d\zeta.$$

Comparing with (3.3) we obtain the elegant Plemelj's formulas (Sojotsky's in Russia)

$$(3.4) \qquad \begin{cases} C^+ f(z) = Cf(z) + \frac{1}{2} f(z) \\ C^- f(z) = Cf(z) - \frac{1}{2} f(z) \end{cases}.$$

Indeed, we have proved (3.4) under the hypothesis that there is a tangent to Γ at the point z, but this occurs almost everywhere on Γ because of rectifiability.

What happens if f is a general function on $L^1(ds)$ without any additional smoothness properties? Experience with the model cases shows that then one has to restrict the way in which w approaches Γ. It turns out that only a non-tangential approach can be allowed if limits are required to exist. Fortunately, there is no problem at all in formalizing this notion for a general rectifiable curve, because tangents, and thus normals, exist at almost all points of the curve. Privalov was able to show that for $f \in L^1(ds)$ and for almost all $z \in \Gamma$ one has

$$Cf(w) - C_\epsilon f(z) \to \pm \frac{f(z)}{2}$$

as w tends non-tangentially to z from $\Omega^\pm$, where $\epsilon = 2|w - z|$.

Denote by $C^{\pm}f(z)$ the limits, if they exist, of $Cf(w)$ as w tends non-tangentially to z from $\Omega^{\pm}$. We conclude that a.e. on Γ the existence of one of the limits $Cf(z)$, $C^+f(z)$, $C^-f(z)$ implies the existence of the other two and the validity of the formulas (3.4). The following basic question then arises:

(3.5)   Does the principal value integral $Cf(z)$ exist almost everywhere

$$\text{for any } f \text{ in } L^1(ds)?$$

There is an argument to reduce the above problem to the case of C^1 curves that will be outlined briefly below. But assume for the moment that the reduction has been performed. Since a simple C^1 curve is locally a rotation of a C^1 graph we can suppose, without loss of generality, that Γ is the graph of a C^1 function A. Localizing and rotating again if necessary, we can further assume that the slopes of our graph are as small as we wish or, in other words, that $\|A'\|_{\infty}$ is arbitrarily small. After a short moment of reflection one concludes that there is no essential difference, for the kind of problem we are facing, between the assumptions of continuity and of boundedness of A'. As is well known, for a locally integrable function A the boundedness of A' and the Lipschitz condition

$$|A(x) - A(y)| \le M|x - y|, \quad x, y \in \mathbb{R},$$

are equivalent. Therefore our problem (3.5) has been reduced to Lipschitz graphs with an arbitrarily small Lipschitz constant M.

We now give an indication of how one can reduce the problem from the general rectifiable case to the C^1 case. A rectifiable curve has, by definition, a parametrization given by a function of bounded variation. By means of the arclength parameter s one finds a parametrization $z(s)$ with bounded $z'(s)$, which is not good enough because we are aiming at continuity of $z'(s)$, not just boundedness. Nevertheless, using Egoroff's Theorem, is not difficult to see that one can get the following Lusin type result: given $\epsilon > 0$ there is a closed subset E in the arclength parameter interval $[0, L]$ such that the Lebesgue measure of $[0, L] \backslash E$ is less than ϵ and the restriction of $z'(s)$ to E is continuous there. Moreover, one also gets

$$z(s) - z(t) = z'(t)(s - t) + o(s - t), \quad s, t \in E.$$

Whitney's extension theorem now provides a C^1 curve Γ' whose intersection with Γ is so big that $\Gamma \backslash \Gamma'$ has arclength measure less than ϵ. Thus Γ can be expressed as $\bigcup\limits_{n=1}^{\infty} (\Gamma \cap \Gamma_n) \cup N$ where N has vanishing arclength and Γ_n is a C^1 curve. Using this one can complete the reduction without much difficulty.

We can now try to answer (3.5) when Γ is a Lipschitz graph (with small constant). To get more insight on the difficulties of the question we are considering, one can parametrize Γ by $\gamma(x) = x + iA(x)$, $x \in \mathbb{R}$, and write, for $z = \gamma(x)$,

$$\int_{|\zeta - z| > \epsilon} \frac{f(\zeta) - f(z)}{\zeta - z} \, d\zeta = \int_{|y - x| > \epsilon} \frac{f(\gamma(y))(1 + iA'(y))}{y - x + i(A(y) - A(x))} \, dy.$$

Absorbing the bounded factor $1 + iA'(y)$ in f, one is then led to inquire about the existence of the principal value integral

$$(3.6) \qquad Tf(x) = \lim_{\epsilon \to 0} \int_{|y - x| > \epsilon} \frac{f(y) \, dy}{y - x + iA(y) - A(x)},$$

for $f \in L^1(\mathbb{R})$. Since the kernel in (3.6) is odd, it is a simple matter to convince oneself, even without resorting to what we did before, that the limit in (3.6) exists for all x, provided f is a compactly supported function of class C^1 on $\mathbb{R}$. Now, there are well known classical real variable techniques that tell us that the a.e. existence of the principal value integral (3.6) follows from an estimate for the truncated operators

$$T_\epsilon f(x) = \int_{|y-x|>\epsilon} \frac{f(y)\,dy}{y - x + i(A(y) - A(x))}, \quad x \in \mathbb{R},$$

of the type

$$(3.7) \qquad |\{x : |T_\epsilon f(x)| > t\}| \leq Ct^{-1}\|f\|_1, \quad t > 0,$$

with C independent of ϵ. In turn (3.7) follows, via the famous Calderón-Zygmund decomposition, from the L^2 estimate

$$(3.8) \qquad \int_{-\infty}^{\infty} |T_\epsilon f(x)|^2\,dx \leq C \int |f(x)|^2\,dx,$$

with C again independent of ϵ. Hence Privalov's central question (3.5) can be answered in the positive sense, once one knows (3.8) to hold.

What we have done actually is reduce Privalov's problem to estimating in L^2 the singular integral operator of Calderón-Zygmund type T. Notice that T is not a convolution operator and so Fourier transform techniques seem to lose all their power.

On the other hand, since one is particularly interested in the case in which the graph of A is extremely close to the real axis, one can view the kernel of T as a very small perturbation of $\frac{1}{y-x}$, the kernel of the familiar Hilbert transform. This naive approach works when $A \in C^{1+\epsilon}$ but breaks down in the limiting case $\epsilon = 0$.

Another route that takes us directly to the problem of L^2 boundedness of the Cauchy Integral on Lipschitz graphs is the one discovered by Calderón. He was interested in constructing algebras of singular integral operators to develop a symbolic calculus that would then be applied to extend some fundamental results of classical PDE theory to equations with minimal smoothness requirements on their coefficients. This is very nicely explained in [C3] and [S].

If one looks at the composition of the simplest operators belonging to the class arising in Calderón's work in dimension one, then one is led to the commutator between the operator of multiplication by the Lipschitz function A and $H\frac{d}{dx}$, H being the Hilbert transform. It is a simple matter to write this operator as

$$(3.9) \qquad C_1 f(x) = \left(AH\frac{d}{dx} - H\frac{d}{dx}A \right)(f) = \text{P.V.} \int_{-\infty}^{\infty} \frac{A(y) - A(x)}{(y - x)^2} f(y)\,dy,$$

where P.V. stands for principal value and f is a test function. The problem here is the $L^2(\mathbb{R})$ boundedness of C_1. This was achieved in 1965 by Calderón [C1] using complex analytic techniques involving the Hardy space H^1 and the Lusin area function. The higher order commutators

$$(3.10) \qquad C_n f(x) = \text{P.V.} \int_{-\infty}^{\infty} \left(\frac{A(y) - A(x)}{y - x} \right)^n \frac{f(y)}{y - x}\,dy$$

were not amenable, however, to these methods. Notice that the Cauchy Integral (3.6) and the family of commutators (3.10) can be written as

$$Tf(x) = \text{P.V.} \int_{-\infty}^{\infty} F\left(\frac{A(y) - A(x)}{y - x}\right) \frac{f(y)}{y - x}\, dy$$

for appropriate choices of F, namely $F(t) = (1 + it)^{-1}$ or $F(t) = t^n$, and thus, in some sense, they are close relatives. In fact, if $\|A'\|_\infty < 1$ then, expanding the kernel of the Cauchy Integral, we have

$$(3.11) \qquad\qquad Tf(x) = \sum_{n=0}^{\infty} (-i)^n C_n f(x).$$

If you are able to prove that C_n is bounded on $L^2(\mathbb{R})$ with an estimate on its norm of the type $\|C_n\| \leq C^n \|A'\|_\infty^n$, then you can sum up the series (3.11) and show the boundedness of T, at least for graphs satisfying $C\|A'\|_\infty < 1$.

To get boundedness of the Cauchy Integral without any restriction on $\|A'\|_\infty$ you would need to show polynomial estimates of the kind

$$\|C_n\| \leq C_0(1 + n)^k \|A'\|_\infty^n,$$

where k is a positive integer independent of n. This is exactly what Coifman, McIntosh and Meyer did in 1982 (with $k = 9$) in their famous paper [CMM]. Five years before, in 1977, Calderón had been able to prove, again by complex analytic methods, that the Cauchy Integral is bounded for Lipschitz graphs with sufficiently small constant, thus solving Privalov's problem [C2] and [C3]. Around that time (and later) remarkable achievements were obtained by Coifman and Meyer on commutators related to pseudodifferential operators, paraproducts and multilinear singular integrals.

4. L^2 boundedness of the Cauchy Integral on a Lipschitz graph

In this section we discuss the author's proof of the L^2 boundedness of the Cauchy Integral T on the graph of the Lipschitz function $y = A(x)$ (see [V2] and [MV]).

Given an interval I and a function $f \in L^2(I)$ one starts by trying to find a good expression for the norm of $T_\epsilon f$ in $L^2(I)$. It is possible to obtain the formula (χ_I is the characteristic function of I)

$$(4.1) \qquad\qquad 2\int_I |T_\epsilon(f)|^2 + 4\,\text{Re} \int_I T_\epsilon(f)\overline{T_\epsilon(\chi_I)}f = G + M,$$

where G is a term of a geometric nature, containing basically Menger curvature, and M is a very good term under control. In fact M is closely related to the Hardy-Littlewood maximal operator (see [V3]). It is also important to notice that the second term on the left hand side of (4.1) is harmless, because of the Cauchy-Schwarz inequality, once one knows how to estimate the norm of $T_\epsilon(\chi_I)$ in $L^2(I)$; for more details on this see the argument that leads to (4.7) below. To feel the potential force of (4.1) let us look at the case $A \equiv 0$, so that the graph is just the real axis: then one can see that G vanishes, because a straight line has zero curvature, and that (4.1) yields

$$\int_I (H_\epsilon f)^2 \leq C \int_I (Mf)^2,$$

where $H_\epsilon = T_\epsilon$ is the truncated Hilbert transform and Mf the Hardy-Littlewood maximal operator applied to f. The idea is then that in the general case the term G

will have the appropriate control thanks to Menger curvature, which one hopes to be related to the geometry of the graph.

In the particularly important special case $f = \chi_I$ (4.1) reduces to

$$\text{(4.2)} \qquad 6 \int_I |T_\epsilon(\chi_I)|^2 = G + M$$

and M can be shown to be to less than $C|I|$.

Let us proceed to discuss in some detail the case $f = \chi_I$. Parametrizing the graph by $\gamma(x) = x + iA(x)$ we have

$$\int_I |T_\epsilon(\chi_I)|^2 = \int_I T_\epsilon(\chi_I)(x)\overline{T_\epsilon(\chi_I)(x)}\, dx$$

$$= \int_I \int_{I_\epsilon(x)} \int_{I_\epsilon(x)} \frac{dx\, dy\, dz}{(\gamma(y) - \gamma(x))\overline{(\gamma(z) - \gamma(x))}},$$

where $I_\epsilon(x) = \{t \in I : |t - x| > \epsilon\}$. In order to symmetrize the domain we set

$$S_\epsilon = \{(x, y, z) \in I^3 : |y - x| > \epsilon, |z - x| > \epsilon, |z - y| > \epsilon\}.$$

Standard estimates show that

$$\int_I |T_\epsilon(\chi_I)|^2 = \iiint_{S_\epsilon} \frac{dx\, dy\, dz}{(\gamma(y) - \gamma(x))\overline{(\gamma(z) - \gamma(x))}} + E$$

where E is an error term satisfying $|E| \leq C|I|$. By permutating the positions of the three variables in the integrand of the right hand side of the preceding formula we get 6 different expressions for the left hand side. Using Lemma 2.1 we then see that

$$\text{(4.3)} \qquad \int_I |T_\epsilon(\chi_I)|^2 = \frac{1}{6} \iiint_{S_\epsilon} c^2(\gamma(x), \gamma(y), \gamma(z))\, dx\, dy\, dz + O(|I|),$$

which is (4.2) if we let G be the triple integral above.

Hence it is enough to prove the inequality

$$\text{(4.4)} \qquad \iiint_{I^3} c^2(\gamma(x), \gamma(y), \gamma(z))\, dx\, dy\, dz \leq C\|A'\|_\infty^2 |I|.$$

The area of the triangle with vertices $\gamma(x)$, $\gamma(y)$ and $\gamma(z)$ is

$$|(A(y) - A(x))(z - x) - (A(z) - A(x))(y - x)|$$

and thus, because of (2.2),

$$\text{(4.5)} \qquad c(\gamma(x), \gamma(y), \gamma(z)) \leq 4 \left| \frac{\dfrac{A(y) - A(x)}{y - x} - \dfrac{A(z) - A(x)}{z - x}}{z - y} \right|.$$

We need now the following lemma whose proof is not much more than a beautiful exercise with Plancherel's Theorem.

Lemma 4.1. *If a is a locally integrable function with derivative $a' \in L^2(\mathbb{R})$, then*

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \frac{\dfrac{a(y) - a(x)}{y - x} - \dfrac{a(z) - a(y)}{z - y}}{z - x} \right|^2 dx\, dy\, dz = c_0 \int_{-\infty}^{\infty} |a'(x)|^2\, dx,$$

c_0 *being a numerical constant.*

Our next step will be to localize the L^2 identity provided by the lemma to the given interval I. This is a kind of technique one uses often when working with BMO. Indeed, in the next inequality the BMO norm of A' will show up naturally.

Apply the lemma to the function $a = \chi_I(A - P_I)$, where $P_I(x) = A'_I(x-\alpha) + A(\alpha)$, $A'_I = \frac{1}{|I|} \int_I A'$, $I = (\alpha, \beta)$. The result is

$$\iiint_{I^3} c^2(\gamma(x), \gamma(y), \gamma(z)) \, dx \, dy \, dz \leq C \int_I |A' - A'_I|^2 \leq C\|A'\|_\infty^2 |I|,$$

which completes the proof of the estimate

$$(4.6) \qquad \int_I |T_\epsilon(\chi_I)|^2 \leq C(1 + \|A'\|_\infty^2)|I|, \text{ for all intervals } I.$$

If one is willing to apply the $T(1)$-Theorem of David and Journé, the proof is complete. Otherwise, there is a simple way of avoiding $T(1)$ which we now describe. Take a (real) function $b \in L^\infty(I)$ and try to express the $L^2(I)$ norm of $T_\epsilon(b)$ in terms of $c^2(\gamma(x), \gamma(y), \gamma(z))$ using the permutation trick described above. One can do that easily and one obtains the identity

$$2 \int_I |T_\epsilon(b)|^2 + 4 \operatorname{Re} \int_I T_\epsilon(b)\overline{T_\epsilon(\chi_I)}b$$
$$= \iiint_{S_\epsilon} c^2(\gamma(x), \gamma(y), \gamma(z))b(x)b(y) \, dx \, dy \, dz + O\left(\int_I |b|^2\right),$$

which is a concrete version of (4.1) with f replaced by b.

Hence, using (4.4) and (4.6),

$$\int_I |T_\epsilon(b)|^2 \leq C(1 + \|A'\|_\infty^2)^{1/2} \left(\int_I |T_\epsilon(b)|^2\right)^{1/2} \|b\|_\infty |I|^{1/2}$$
$$+ C(1 + \|A'\|_\infty^2)\|b\|_\infty^2|I|,$$

which clearly gives

$$(4.7) \qquad \int_I |T_\epsilon(b)|^2 \leq C(1 + \|A'\|_\infty^2)\|b\|_\infty^2|I|.$$

Now (4.7) implies that T_ϵ sends boundedly $H^1(\mathbb{R})$ into $L^1(\mathbb{R})$ (use atoms) and $L^\infty(\mathbb{R})$ into $\operatorname{BMO}(\mathbb{R})$. By interpolation we finally obtain that T_ϵ maps boundedly $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$ (in fact $L^p(\mathbb{R})$ into $L^p(\mathbb{R})$, $1 < p < \infty$).

In the next section we will show that the quantity in the right hand side of (4.5) is intimately connected to the first Calderón commutator C_1. In fact it turns out that (4.3) and (4.5) say that the Cauchy Integral is dominated by the first commutator. Thus in some ironic sense, the 1965 paper [C1] is the beginning and almost the end of the story. Of course, this was far from being even conceivable in the sixties and, indeed, much clever and hard work had to be done for many years to fully understand the Cauchy Integral. The beautiful quotation from Montaigne that opens the book by Meyer and Coifman [MC] suits perfectly, I believe, to the long and complicated process of understanding the Cauchy Integral.

5. L^2 boundedness of the first Calderón commutator

In this section we present a simple proof of the L^2 boundedness of the first Calderón commutator (3.9) and we show that the Cauchy Integral is controlled by the first commutator. As the reader will see, we basically reinterpret the ideas and

results introduced in the preceding section. Other proofs of the L^2 boundedness of the first commutator can be found in [Mu].

Call

$$K(x, y) = \frac{A(y) - A(x)}{(y - x)^2}$$

the kernel of the first commutator. We want to exploit further the symmetrization idea described before.

Write

$$C_{1,\epsilon}(f)(x) = \int_{|y-x|>\epsilon} K(x, y) f(y) \, dy, \quad x \in \mathbb{R}.$$

Given an interval I, we have

$$\int_I C_{1,\epsilon}(\chi_I)^2 = \int_I \int\int_{I_\epsilon(x)} \int_{I_\epsilon(x)} K(x, y) K(x, z) \, dx \, dy \, dz$$

$$= \iiint_{S_\epsilon} K(x, y) K(x, z) \, dx \, dy \, dz + O(|I|),$$

where $I_\epsilon(x)$ and S_ϵ are defined in the preceding section. Notice that the integrand in the triple integral above is already symmetric in y and z. If we interchange x by y, and x by z, we get two new different expressions for $\int_I C_{1,\epsilon}(\chi_I)^2$. Therefore

$$\int_I C_{1,\epsilon}(\chi_I)^2 = \frac{1}{3} \iiint_{S_\epsilon} S(x, y, z) \, dx \, dy \, dz + O(|I|)$$

where

$$S(x, y, z) = K(x, y)K(x, z) + K(y, x)K(y, z) + K(z, y)K(z, x).$$

It turns out that, as a straightforward computation shows,

$$S(x, y, z) = \left(\frac{\dfrac{A(y) - A(x)}{y - x} - \dfrac{A(z) - A(y)}{z - y}}{z - x} \right)^2,$$

and so Lemma 4.1 gives, after localization,

$$\int_I C_{1,\epsilon}(\chi_I)^2 \le C(1 + \|A'\|_\infty^2)|I|.$$

The L^2 boundedness is again a consequence of the $T(1)$-Theorem or of the (simpler) fact that one can interpolate between H^1 and BMO. On the other hand (4.3) and (4.5) yield

$$\int_I |T_\epsilon(\chi_I)|^2 \le C \int_I C_{1,\epsilon}(\chi_I)^2 + O(|I|),$$

thus showing that the Cauchy Integral can be controlled by the first commutator.

Joan Mateu and the author have recently applied the symmetrization technique to the n-th Calderón commutator obtaining what seems to be the best possible asymptotic estimate

$$\|C_n\| \le C_0(1 + n)\|A'\|_\infty^n.$$

6. The solution of Vitushkin's conjecture

Menger curvature has been one of the key ingredients in the proof of Vitushkin's conjecture, which has been recently completed by G. David [D2]. When one considers the proof in its totality one realizes that it is an impressive achievement, because it involves several deep ideas of different nature that various people have been introducing at successive stages, sometimes for seemingly unrelated purposes. Some parts of the proof are still too involved technically and some further work seems necessary to perceive more clearly the real importance of each of the ideas involved and purify the technical development of the proof. A good step in this direction has been taken by Nazarov, Treil and Volberg in the nice article [NTV2]. An excellent exposition by G. David, including a historical perspective, is available at present [D3]. After reading that paper you get the feeling that no human being could have proved the result even two decades after it was conjectured because too many essential steps were still missing. Although this is a well known and common phenomenon in mathematics, is still impressive to perceive it directly in a concrete instance that you can seize, at least approximately.

In this section we will try to give an idea, even if vague, of some of the main steps in the proof of the conjecture.

Vitushkin's conjecture is the solution, in a particular case, of Painlevé's problem. Painlevé proved that planar compact sets of zero length (one dimensional Hausdorff measure) are removable for bounded analytic functions. Removability of a compact set E means that if an open set Ω contains E and $f \in H^\infty(\Omega \backslash E)$, then f can be continued analytically to Ω. In fact, it is not difficult to see that one only needs to check the above condition for $\Omega = \mathbb{C}$ and so (by Liouville) that E is removable if and only if $H^\infty(\mathbb{C} \backslash E)$ is reduced to constants. The first trace of the notion of removability goes back to Riemann who proved, as we all know, that a point is removable. Painlevé asked, more than one hundred years ago, if removable sets can be described in geometric or metric terms, and nobody has been able yet to provide an answer to a question of such an innocous appearance.

That Painlevé's result is, in some sense, sharp becomes apparent by considering an interval I on the real axis: I has positive length and is not removable, because the conformal mapping from the complement of I in the extended complex plane onto the unit disc cannot be continued analytically through I.

Vitushkin discovered that the converse of Painlevé's result is not true, that is, that you can build a removable compact set of positive length. The simplest possible example of such a phenomenon was found independently by Garnett and Ivanov. It turns out to be the planar Cantor set one generates by the procedure of taking "corner quarters". Since it is an important object for us, we briefly review its definition.

We start with the unit square $Q = [0,1] \times [0,1]$. We divide Q into sixteen squares of side length $1/4$ and we take those four that contain the vertices of Q. We repeat the procedure in each of these four squares and we get sixteen squares of side length $1/16$. Proceeding inductively we have at the n-th generation 4^n squares of side length 4^{-n}. Let E_n denote their union and set $E = \bigcap_{n=1}^{\infty} E_n$, the Cantor set we are interested in.

The orthogonal projection of each E_n, and consequently of E, on the line L shown in Figure 3 is some interval (the same for each n). Therefore the length of
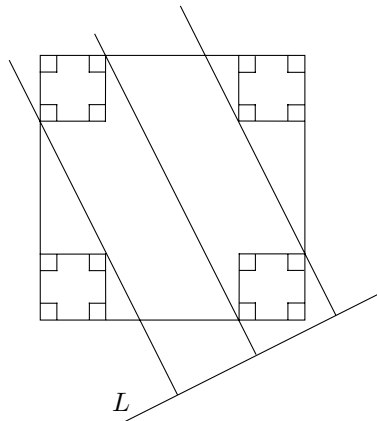
E is positive because a projection does not increase length. Moreover, it is easily seen that E has finite length. It can also be shown, and this is not obvious, that E is removable [G1].

The right question is now the following: what smallness property does the set E enjoy that implies removability and that is not shared by an interval on the real line? An indication of what the answer might be is given by Besicovitch theory of (measurable) sets of finite length [F] and [Ma]. Besicovitch showed in the thirties that any set of finite length E can be expressed as $E = N \cup R \cup I$ where N has vanishing length, and R and I are (either the empty set or) sets of positive length of opposite extremal nature. The set R is rectifiable, that is, is a subset of a rectifiable curve. The set I is irregular, in the sense that intersects all rectifiable curves in zero length. Alternatively, in spite of having positive length, I projects into sets of zero length in almost all directions. It turns out that the Cantor set we considered before is irregular and this fact explains why its behaviour, as far as removability is concerned, is so different from that of an interval.

Vitushkin's conjecture now becomes natural. It reads as follows [Vi].

If E is a compact set of positive finite length, then E is removable if and only if E is irregular in Besicovitch's sense.

Notice that Besicovitch irregularity is invariant by bilipschitz mappings and thus is a metric condition.

Calderón's contribution is the proof of the necessity of Besicovitch irregularity. Indeed, assume that E is not irregular. Then E intersects some rectifiable curve Γ in positive length. In fact Γ can be taken to be a C^1 graph with small constant because of the reduction argument described in section 3. Set $K = E \cap \Gamma$ and consider $f(z) = C(\chi_K)(z)$, $z \notin K$. Unfortunately f is not bounded, but, appealing to the L^2 boundedness of the Cauchy Integral on Γ, it can be seen to be in $H^2(\mathbb{C}\backslash K)$. An analytic function on $\mathbb{C}\backslash K$ is said to be in $H^2(\mathbb{C}\backslash K)$ if its L^2 norms on a certain sequence of curves surrounding and approximating K are uniformly bounded. Thus f is in $H^2(\mathbb{C}\backslash K)$ and cannot be continued analytically through K,

because otherwise it would vanish identically by Liouville. That f is not identically zero becomes clear from the fact that $\lim_{z \to \infty} z f(z) = \text{length}(K) \neq 0$.

This is enough to show the non-removability of K, and thus that of E, thanks to a deep theorem of Garabedian (see [G2]) stating that the removable sets for H^∞ and H^2 are the same. It is worth mentioning that this is basically the argument to prove the Denjoy conjecture from Calderón's Theorem on the Cauchy Integral (see [M] and [V1] for more details).

Hence proving the sufficiency in the statement of Vitushkin's conjecture is the real difficult problem. One has to show that if E is non-removable, then there exists a rectifiable curve that intersects E in positive length. If one thinks for a while at how such a curve could be constructed, one realizes that in fact it is not obvious at all even how to begin.

The beta numbers of Peter Jones provide a hint of what could be tried. Introduced to deal with the Cauchy Integral on a Lipschitz graph [J1], the beta numbers play an essential role in deciding whether or not a given set is contained in a rectifiable curve, a question related to the famous traveling salesman problem of computational geometry [J2].

For a compact set E and a dyadic square Q, the beta number associated to E and Q is

$$(6.1) \qquad \beta(Q) = \inf_L \sup_{z \in E \cap Q} \text{dist}(z, L)\ell(Q)^{-1},$$

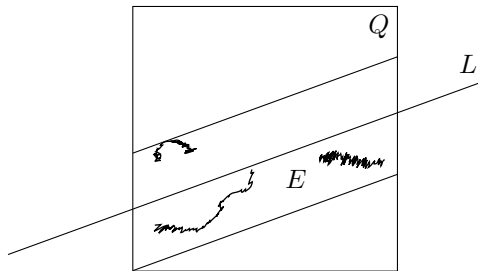where the infimum is taken over all straight lines L intersecting Q.

FIGURE 4

Therefore $\beta(Q)$ measures, in a scale invariant way, how much the set E deviates from a line at the scale and location determined by Q. It turns out that there exists a rectifiable curve Γ containing E if and only if $\sum_Q \beta^2(Q)\ell(Q)$ is finite, where the sum is taken over all dyadic squares [J2].

We are now much better equipped to confront Vitushkin's conjecture. Indeed, given a non-removable set E we have at our disposal the beta numbers to identify a rectifiable piece inside E. It is still unclear how one can get any information from non-removability. **M. Christ** made an important contribution in this direction [Ch]. In his original result he had to impose an additional requirement on E, namely that E has positive finite length in a uniform fashion:

$$C^{-1}r \leq \text{length}(E \cap D(z,r)) \leq Cr, \quad z \in E, \quad 0 < r \leq \text{diam}(E),$$

where $D(z,r)$ is the open disc of radius r centered at z. The above condition, called Ahlfors regularity, had been shown by G. David to characterize those rectifiable

curves in the plane for which the Cauchy Integral is bounded on $L^2(ds)$ [D1]. Before stating Christ's result we need to establish some terminology. Take a positive Radon measure μ in the plane, and a positive number ϵ. For as compactly supported function f on $L^2(\mu)$ define

$$C_\epsilon(f\mu)(z) = \int_{|\zeta-z|>\epsilon} \frac{f(\zeta)}{\zeta-z} \, d\mu(\zeta), \quad z \in \mathbb{C}.$$

Then one says that the Cauchy Integral is bounded on $L^2(\mu)$ provided

(6.2) $$\int |C_\epsilon(f\mu)|^2 \, d\mu \le C \int |f|^2 \, d\mu,$$

with C independent of ϵ and f. If μ has no atoms, then a necessary condition for (6.2) is

(6.3) $$\mu(D(z,r)) \le Cr, \quad z \in \mathbb{C}, \quad r > 0.$$

For example, if μ is the length measure restricted to the Cantor set of Figure 3 then (6.3) holds but it can be proved that (6.2) does not.

Theorem 6.1 (Christ). *If E is a non-removable Ahlfors regular compact set then there exists another Ahlfors regular compact set E' such that* length$(E \cap E') > 0$ *and the Cauchy Integral is bounded on $L^2(\mu)$, where μ is length (one dimensional Hausdorff measure) restricted to $E \cap E'$.*

Hence non-removability is equivalent, at least for the class of Ahlfors regular sets, to L^2 boundedness of the Cauchy Integral with respect to the length measure on a non-trivial piece of the set (non-removability follows from L^2 boundedness by a variation of the argument we used in the proof of the necessary condition in Vitushkin's conjecture).

The reason for requiring Ahlfors regularity is that the proof uses an extremely strong criterion for L^2 boundedness of singular integrals of Calderón-Zygmund type known as the $T(b)$-Theorem, which is due to David, Journé and Semmes. In the original formulation the $T(b)$-Theorem required the underlying space to be of homogeneous type, that is, a metric space endowed with a Borel positive measure μ satisfying the doubling condition

(6.4) $$\mu(2B) \le C\mu(B), \text{ for all balls } B.$$

The length measure on an Ahlfors regular set is clearly doubling, but one finds without difficulty examples of sets of finite length for which the length measure restricted to the set does not satisfy (6.4).

If the kernel of the operator T is odd the $T(b)$-Theorem basically asserts that T is bounded on L^2 provided $T(b)$ is in BMO (in particular in L^∞) for some bounded function b satisfying a non-triviality condition called para-accretivity:

(6.5) $$\frac{1}{\mu(B)} \left| \int_B b \, d\mu \right| \ge \delta > 0, \text{ for all balls } B.$$

We are now ready to get an idea of what non-removability has to do with the $T(b)$-Theorem. Assume that E is non-removable (and Ahlfors regular). Then there is a non-constant bounded holomorphic function B on $\mathbb{C}\backslash E$, which can be seen to be of the form

$$B(z) = \int_E \frac{b(\zeta)}{\zeta-z} \, d\mu(\zeta), \quad z \notin E,$$

μ being the length measure on E and b some function in $L^\infty(\mu)$.

When z tends to some point in E we are in trouble, because in this general context there is nothing like the Plemelj's formula. However, one still has

$$|C_\epsilon(b\mu)(z)| \leq C\|B\|_\infty, \quad z \in E, \quad \epsilon > 0,$$

and this can be understood, for each ϵ and in a uniform way, as "$T(b)$ is bounded" for the operator defined by $T(f) = C_\epsilon(f\mu)$.

Although the non-constancy of B translates into $\int_E b\,d\mu \neq 0$, b does not need to have locally a non zero integral in a uniform way, as in (6.5). In other words, b is not necessarily para-accretive and this justifies the need for a modification process, based on a stopping time argument, that in the end will transform E into E'. How Ahlfors regularity can be dispensed with will be mentioned later on.

Therefore, to complete the proof of Vitushkin's conjecture we need to establish a link between L^2 boundedness of the Cauchy Integral and Jones' beta numbers. It turns out that Menger curvature is the perfect technical device to do this. In the Ahlfors regular case the argument is rather transparent [MMV] if one is willing to use the David-Semmes theory of uniform rectifiability [DS]. Although there is now a direct way of relating Menger curvature to the beta numbers, we will sketch the original one. See [Fa] for the best result known before the Menger era.

Let E be an Ahlfors regular set. A first fact, of a technical nature but extremely useful, is that one has grids of "dyadic squares" in E, as in the Euclidean setting. A dyadic square in our context is a subset of E with no particular geometric shape. To understand what we mean it is convenient to consider the example of the Cantor set described at the beginning of this section. The dyadic squares of size 4^{-n} are the intersections of E with each of the 4^n squares appearing at the n-th generation. Then they have diameter and length comparable to 4^{-n}.

The dyadic squares of a fixed size form a partition of E and one still has the basic fact that if two given dyadic squares are not disjoint then one is contained in the other. The reader is referred to [DS] for complete details.

Let μ be the length measure on E and assume that the Cauchy Integral is bounded on $L^2(\mu)$, that is, that (6.2) holds. **David and Semmes** proved that E is (locally) contained in a rectifiable curve provided an additional condition, called the **weak geometric lemma**, holds [DS]. The result is not at all easy to prove, on the contrary, it is the product of a rather sophisticated strategy involving what has been called "Corona decompositions". To formulate the weak geometric lemma, set, for any given dyadic square Q_0 in E and any $\epsilon > 0$,

$$D(Q_0, \epsilon) = \{Q : Q \text{ dyadic square} \subset Q_0 \text{ such that } \beta(Q) > \epsilon\},$$

where $\beta(Q)$ is defined (essentially) as in (6.1) replacing $\ell(Q)$ by $\mathrm{diam}(Q)$. The weak geometric lemma is the following assertion, which can be true or not depending on the nature of the given Ahlfors regular set E.

For each dyadic square Q_0 and each $\epsilon > 0$ one has

$$(6.6) \qquad \sum_{Q \in D(Q_0, \epsilon)} \mu(Q) \leq C\mu(Q_0).$$

The Carleson type condition (6.6) is clearly a weaker form of the "geometric lemma"

$$\sum_{Q \subset Q_0} \beta^2(Q) \operatorname{diam}(Q) \leq C \operatorname{diam}(Q_0),$$

that looks like a variation of the Jones condition.

It turns out that with the help of Menger curvature one can prove [MMV] that the weak geometric lemma is a consequence of the boundedness of the Cauchy Integral on $L^2(\mu)$. To explain this we start by noticing that playing the permutations game that we described in section 4, one can get without any additional effort

$$\int_D |C_\epsilon(\chi_D \mu)|^2 \, d\mu = \iiint_{S_\epsilon} c^2(z, w, \zeta) \, d\mu(z) \, d\mu(w) \, d\mu(\zeta) + O(\mu(D)),$$

where D is a disc and

$$S_\epsilon = \{(z, w, \zeta) \in D^3 : |z - w| > \epsilon, \ |z - \zeta| > \epsilon \text{ and } |w - \zeta| > \epsilon\}.$$

If the Cauchy Integral is bounded on $L^2(\mu)$, then

$$(6.7) \qquad \iiint_{D^3} c^2(z, w, \zeta) \, d\mu(z) \, d\mu(w) \, d\mu(\zeta) \leq C\mu(D), \text{ for each disc } D.$$

To check (6.6), let Q be a dyadic square in E, and take two points $a_1, a_2 \in Q$ such that

$$d \equiv |a_1 - a_2| \simeq \operatorname{diam}(Q) \simeq \mu(Q).$$

Let L be the straight line passing through a_1 and a_2, and let a_3 be a point in Q such that $h \equiv \operatorname{dist}(a_3, L) = \sup_{z \in Q} \operatorname{dist}(z, L)$. Recall that $Q \subset E$ and so h may be much smaller than d. Set $D_i = D(a_i, \frac{h}{3})$, $1 \leq i \leq 3$. Then we have

$$c(z_1, z_2, z_3) \geq Chd^{-2},$$

for some positive constant C and all $(z_1, z_2, z_3) \in D_1 \times D_2 \times D_3 \equiv T(Q)$.
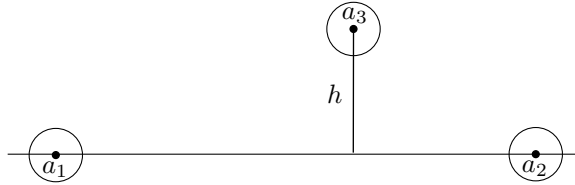

FIGURE 5

Assume now that $Q \subset Q_0$ and $\beta(Q) > \epsilon$. Since clearly $Chd^{-1} \geq \beta(Q) > \epsilon$,

$$\iiint_{T(Q)} c^2(z_1, z_2, z_3) \, d\mu(z_1) \, d\mu(z_2) \, d\mu(z_3) \geq C(hd^{-2})^2 h^3$$

$$= Ch^5 d^{-4}$$

$$\geq C\epsilon^5 d$$

$$\geq C\epsilon^5 \mu(Q).$$

It is not difficult to convince oneself that the $T(Q)$ have finite overlapping as Q varies.

Hence

$$\sum_{Q \in D(Q_0, \epsilon)} \mu(Q) \le C\epsilon^{-5} \sum_{Q \subset Q_0} \iiint_{T(Q)} c^2(z_1, z_2, z_3) \, d\mu(z_1) \, d\mu(z_2) \, d\mu(z_3)$$

$$\le C\epsilon^{-5} \iiint_{(4Q_0)^3} c^2(z_1, z_2, z_3) \, d\mu(z_1) \, d\mu(z_2) \, d\mu(z_3)$$

$$\le C\epsilon^{-5} \mu(Q_0),$$

where the last inequality follows from (6.7). This completes the proof of (6.6).

The main steps in the proof of Vitushkin's conjecture in the Ahlfors regular case can be summarized in the following scheme.
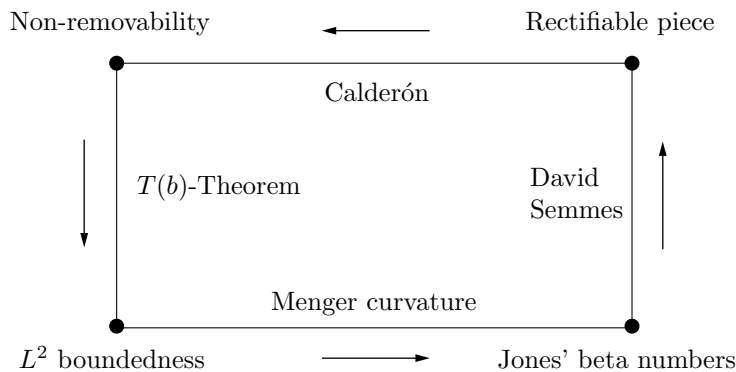

FIGURE 6

Getting rid of the Ahlfors regularity assumption requires new ideas in order to overcome the considerable difficulties caused by the lack of homogeneity. In particular one needs a $T(b)$-Theorem without the assumption that the underlying measure satisfies the doubling condition. That this could be proved looked rather unlikely at first glance, because one was used to believe that spaces of homogeneous type were the right environment in which Calderón-Zygmund Theory could be developed. Several people (David, Nazarov, Mattila, Tolsa, Treil and Volberg among others) have contributed during the last few years to show that, surprisingly enough, substantial parts of classical Calderón-Zygmund Theory hold without the doubling condition (see, for example, [T1], [T2], [NTV1] and [D3]). In particular the $T(b)$-Theorem holds in the non-homogeneous setting, as David proved in [D2]. For more details on the difficulties one has to face in the non-doubling context the reader is invited to consult the nice survey paper [D3] and the preprints [NTV2], [NTV3].

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