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A Geometric Proof of the L^2 Boundedness of the Cauchy Integral on Lipschitz Graphs

Mark S. Melnikov and Joan Verdera

1 Introduction

In this paper we give a new proof of the L^2 boundedness of the Cauchy integral on Lipschitz graphs (and chord-arc curves). Our method consists in controlling the Cauchy integral by an appropiate square function measuring the curvature of the graph. The square function is then estimated via a Fourier transform computation.

Let $\Gamma = \{(x, y) \in \mathbb{R}^2 : y = A(x)\}$ be the graph of a Lipschitz function A defined on the real line. Then A is locally absolutely continuous and A' is bounded. The Cauchy integral of $f \in L^2(\Gamma)$ is

$$Cf(z) = \lim_{\varepsilon\to 0} \int_{|\zeta - z| > \varepsilon} \frac{f(\zeta)}{\zeta - z}\, d\zeta, \qquad z \in \Gamma, \tag{1}$$

where $d\zeta = d\zeta_{|\Gamma}$. The almost everywhere existence (with respect to arclength) of the limit in (1) is a deep result, which in fact is a consequence of L^2 estimates, via standard real-variable methods. Thus, instead of considering the principal value integral (1), one looks at the truncated Cauchy integral

$$\int_{|\zeta - z| > \varepsilon} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \int_{|z(y) - z(x)| > \varepsilon} \frac{f(z(y))(1 + iA'(y))}{z(y) - z(x)}\, dy,$$

where Γ has been parametrized by $z(x) = x + iA(x)$. Neglecting the bounded factor $1 + iA'(y)$ and slightly modifying the domain of integration, one is led to consider the truncated operators

$$C_\varepsilon f(x) = \int_{|y - x| > \varepsilon} \frac{f(y)}{z(y) - z(x)}\, dy, \qquad f \in L^2(\mathbb{R}), \quad \varepsilon > 0.$$

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Theorem. For some constant C depending only on $\|A'\|_\infty$, one has

$$\int_{-\infty}^{\infty} |C_\varepsilon f(x)|^2 \, dx \leq C \int_{-\infty}^{\infty} |f(x)|^2 \, dx. \tag{2}$$

□

Calderôn proved inequality (2) when $\|A'\|_\infty$ is sufficiently small [C], and Coifman, McIntosh, and Meyer settled the general case some years later [CMM]. Since then, many other proofs of (2) have been found (see [CJS], [D], [J], [Mu], and [S]). In this paper we add a new proof to the list. In our opinion, the geometric idea behind it is interesting in its own right, and we believe that it should have other applications.

2 The proof

Our first task is to find a good expression for $\int_I |C_\varepsilon(\chi_I)|^2$, χ_I being the characteristic function of the interval I. Clearly,

$$\begin{aligned}\int_I |C_\varepsilon(\chi_I)|^2 &= \int_I C_\varepsilon(\chi_I)(x)\overline{C_\varepsilon(\chi_I)}(x)\, dx \\ &= \iiint_{T_\varepsilon} \frac{1}{z(y)-z(x)} \frac{1}{\overline{z(t)-z(x)}} \, dx\, dy\, dt,\end{aligned} \tag{3}$$

where $T_\varepsilon = \{(x,y,t) \in I^3 : |y-x| > \varepsilon \text{ and } |t-x| > \varepsilon\}$. The triple integral in (3) is not symmetric, either in the domain or in the kernel. To symmetrize the domain, set

$$S_\varepsilon = \{(x,y,t) \in I^3 : |y-x| > \varepsilon,\ |t-x| > \varepsilon \text{ and } |t-y| > \varepsilon\}.$$

We claim that

$$\int_I |C_\varepsilon(\chi_I)|^2 = \iiint_{S_\varepsilon} \frac{1}{z(y)-z(x)} \frac{1}{\overline{z(t)-z(x)}} \, dx\, dy\, dt + O(|I|). \tag{4}$$

The claim follows from the inequalities

$$\iiint_{U_{\varepsilon,j}} \frac{1}{|z(y)-z(x)|\,|z(t)-z(x)|} \, dx\, dy\, dt \leq C|I|, \qquad j = 1, 2, \tag{5}$$

where

$$U_{\varepsilon,1} = \{(x,y,t) \in I^3 : |y-x| > \varepsilon,\ |t-x| > 2\varepsilon \text{ and } |t-y| < \varepsilon\}$$

and

$$U_{\varepsilon,2} = \{(x,y,t) \in I^3 : |y-x| > \varepsilon,\ \varepsilon < |t-x| < 2\varepsilon \text{ and } |t-y| < \varepsilon\}.$$

On $U_{\varepsilon,2}$ the integrand in (5) is not greater than ε^{-2}. Hence (5) is obvious for $j = 2$. For $j = 1$, the triple integral in (5) can be estimated by the iterated integral

$$\int_I \int_{|t-x|>2\varepsilon} \int_{|y-t|<\varepsilon} \frac{2}{|t-x|^2}\, dy\, dt\, dx.$$

Therefore, (5) holds also for $j = 1$.

To symmetrize the kernel in (4), we permute the position of the three variables in all possible ways, and we get

$$6\int_I |C_\varepsilon(\chi_I)|^2 = \iiint_{S_\varepsilon} \left(\sum_\sigma \frac{1}{z(x_{\sigma(2)}) - z(x_{\sigma(1)})} \frac{1}{\overline{z(x_{\sigma(3)}) - z(x_{\sigma(1)})}} \right) dx_1\, dx_2\, dx_3 + O(|I|), \tag{6}$$

where the sum is taken over the six permutations of the set $\{1, 2, 3\}$.

We need now two lemmas. The first computes the kernel of the triple integral in (6), and the second provides an estimate for the integral.

Lemma 1 [M]. Given three pairwise distinct points z_1, z_2, and z_3 in the plane, we have

$$\sum_\sigma \frac{1}{z_{\sigma(2)} - z_{\sigma(1)}} \frac{1}{\overline{z_{\sigma(3)} - z_{\sigma(1)}}} = \left(\frac{4S(z_1, z_2, z_3)}{|z_2 - z_1|\,|z_3 - z_1|\,|z_2 - z_3|} \right)^2,$$

where $S(z_1, z_2, z_3)$ is the area of the triangle with vertices at z_1, z_2, and z_3. □

By elementary geometry, the quantity

$$c(z_1, z_2, z_3) \equiv \frac{4S(z_1, z_2, z_3)}{|z_2 - z_1\, |z_3 - z_1|\,|z_2 - z_3|}$$

turns out to be equal to R^{-1}, where R is the radius of the circle passing through z_1, z_2, and z_3, and also equal to

$$\frac{2\sin\alpha_{ij}}{|z_i - z_j|}, \qquad i \neq j,$$

where α_{ij} is the angle, in the triangle determined by z_1, z_2, and z_3, opposite to the side $z_i z_j$. In some geometry textbooks, $c(z_1, z_2, z_3)$ is called the Menger curvature associated to the points z_1, z_2, z_3. (See [BM, p. 361] for a reference to the original paper by Menger in which $c(z_1, z_2, z_3)$ was first introduced.) A notion of "curvature of a measure" involving $c(z_1, z_2, z_3)$ was considered in [M] in connection with analytic capacity.

Proof of Lemma 1. Set $a = z_2 - z_1$ and $b = z_3 - z_1$. A simple computation gives

$$\mathrm{Re}\left(\frac{1}{ab} + \frac{1}{b(\overline{b} - \overline{a})} - \frac{1}{(b-a)\overline{a}} \right) = 2\left(\frac{|a|^2|b|^2 - \mathrm{Re}(a\overline{b})^2}{|a|^2|b|^2|b-a|^2} \right),$$

from which Lemma 1 follows readily. ■

Using the formula

$$S(z_1, z_2, z_3) = |(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)|, \tag{7}$$

where $x_j = \mathrm{Re}(z_j)$, $y_j = \mathrm{Im}(z_j)$, $j = 1, 2, 3$, one gets

$$c(z(x), z(y), z(t)) \leq 2 \left| \frac{\dfrac{A(y) - A(x)}{y - x} - \dfrac{A(t) - A(x)}{t - x}}{t - y} \right|.$$

Lemma 2. Let a be a locally absolutely continuous function on the real line such that $a' \in L^2(\mathbb{R})$. Then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{\dfrac{a(y) - a(x)}{y - x} - \dfrac{a(t) - a(x)}{t - x}}{t - y} \right)^2 dx\, dy\, dt = \alpha \|a'\|_2^2, \tag{8}$$

for some numerical constant α. □

Proof. Introducing new variables $h = y - x$ and $k = t - x$ and applying Plancherel in x, one shows that the triple integral in (8) is equal to

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left| \frac{\dfrac{e^{i\xi h} - 1}{\xi h} - \dfrac{e^{i\xi k} - 1}{\xi k}}{h - k} \right|^2 |a'(\xi)|^2\, d\xi\, dh\, dk$$

$$= \|a'\|_2^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left| \frac{\dfrac{e^{iu} - 1}{u} - \dfrac{e^{iv} - 1}{v}}{u - v} \right|^2 du\, dv,$$

where $u = \xi h$ and $v = \xi k$.

If $E(u) = (e^{iu} - 1)/iu$, then the above double integral is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left| \frac{E(u + t) - E(u)}{t} \right|^2 du\, dt,$$

which, by Plancherel and the identity $\hat{E} = 2\pi\chi_{(0,1)}$, turns out to be

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\hat{E}(\xi)|^2 \left| \frac{e^{i\xi t} - 1}{t} \right|^2 d\xi\, dt = 2\pi \int_{-\infty}^{\infty}\int_{0}^{1} \left| \frac{e^{i\xi t} - 1}{t} \right|^2 d\xi\, dt.$$

That the last integral is finite follows from the fact that the integrand can be estimated by $4t^{-2}$ if $|t| > 1$ and by a constant times ξ^2 if $|t| \leq 1$. ■

Corollary. For some universal constant C, one has, for each interval I,

$$\int_I \int_I \int_I c^2(z(x), z(y), z(t))\, dx\, dy\, dt \le C\|A'\|_\infty^2 |I|. \tag{9}$$

$\square$

Proof. Given $I = [\alpha, \beta]$, consider the first-degree polynomial $P_I(x) = A(\alpha) + A'_I(x - \alpha)$, where $A'_I = (1/|I|)\int_I A' = (A(\beta) - A(\alpha))/(\beta - \alpha)$, and set $a = (A - P_I)\chi_I$. Then the left-hand side of (9) is not greater than

$$4 \int_I \int_I \int_I \left(\frac{\frac{A(y) - A(x)}{y - x} - \frac{A(t) - A(x)}{t - x}}{t - y} \right)^2 dx\, dy\, dt$$

$$\le 4 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{\frac{a(y) - a(x)}{y - x} - \frac{a(t) - a(x)}{t - x}}{t - y} \right)^2 dx\, dy\, dt$$

$$= 4\alpha \|(A' - A'_I)\chi_I\|_2^2 \le 16\alpha \|A'\|_\infty^2 |I|.$$

$\blacksquare$

Remark. The reader should compare the last inequality with [Ch, Proposition 16, p. 32], in which an essentially equivalent estimate for the β's of P. Jones is discussed.

Combining (6), Lemma 1, and (9), we obtain

$$\int_I |C_\varepsilon(\chi_I)|^2 \le C|I|, \tag{10}$$

for each interval I, with $C = C(\|A'\|_\infty)$.

Notice that (10) is equivalent to saying that $C_\varepsilon(1) \in BMO(\mathbb{R})$, with BMO-norm bounded independently of ε. The T1-Theorem [DJ] now concludes the proof of (2). However, with a little more effort we can avoid appealing to the T1-Theorem. Let b be a real function in $L^\infty(I)$. It is easily seen that

$$2 \int_I |C_\varepsilon(b)|^2 + 4 \operatorname{Re} \int_I C_\varepsilon(b) \overline{C_\varepsilon(\chi_I)} b$$

$$= \iiint_{S_\varepsilon} c^2(z(x), z(y), z(t)) b(y) b(t)\, dx\, dy\, dt + O(\|b\|_\infty^2 |I|).$$

Then, by (9) and (10),

$$\int_I |C_\varepsilon(b)|^2 \le C\|b\|_\infty |I|^{1/2} \left(\int_I |C_\varepsilon(b)|^2 \right)^{1/2} + C\|b\|_\infty^2 |I|,$$

and thus $\int_I |C_\varepsilon(b)|^2 \le C\|b\|_\infty^2 |I|$. Hence

$$\int_I |C_\varepsilon(b)| \le \left(\int_I |C_\varepsilon(b)|^2\right)^{1/2} |I|^{1/2} \le C\|b\|_\infty |I|,$$

and this easily implies that C_ε boundedly sends $L^\infty(\mathbb{R})$ into $BMO(\mathbb{R})$ and $H^1(\mathbb{R})$ into $L^1(\mathbb{R})$, with bounds independent of ε. By interpolation we finally get (2). ■

3 Chord-arc curves

A (locally) rectifiable curve Γ passing through ∞ is said to be a chord-arc curve provided the length $(\widehat{ab}) \le (1+c)|a-b|$, for some positive constant c and all $a, b \in \Gamma$, where $\widehat{ab}$ denotes the arc contained in Γ joining a and b. In terms of the arclength parametrization $z(t)$ of Γ, the chord-arc condition is

$$|t-s| \le (1+c)|z(t)-z(s)|, \qquad t, s \in \mathbb{R}. \tag{11}$$

The L^2 boundedness of the Cauchy integral on a chord-arc curve is equivalent to the following.

Theorem. If

$$C_\varepsilon f(t) = \int_{|s-t|>\varepsilon} \frac{f(s)\,ds}{z(s)-z(t)}, \qquad f \in L^2(\mathbb{R}), \quad \varepsilon > 0,$$

then

$$\int_{-\infty}^{\infty} |C_\varepsilon f(t)|^2\,dt \le C \int_{-\infty}^{\infty} |f(t)|^2\,dt,$$

for some $C = C(\Gamma)$ independent of ε. □

Using (7) and (11), one gets

$$c(z(t), z(s), z(u)) \le C \left| \frac{\dfrac{z(t)-z(s)}{t-s} - \dfrac{z(u)-z(s)}{u-s}}{t-u} \right|,$$

and thus, by Lemma 2 and localization,

$$\int_I \int_I \int_I c^2(z(t), z(s), z(u))\,dt\,ds\,du \le C|I|, \quad \text{for any interval } I.$$

The proof of the theorem is now completed as in Section 1.

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Department of Mathematics, Universitat Autonoma de Barcelona, 08193 Bellaterra, Barcelona, Spain