# ANALYTIC CAPACITY AND THE CAUCHY SINGULAR INTEGRAL

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The pictures were drawn on the actual transparencies so they are missing here.

## Removable sets for bounded analytic functions

A compact set $K$ in the plane is said to be **removable** when

$$f \in H^{\infty}(\Omega \setminus K) \Longrightarrow f \text{ is analytic on } \Omega$$

Examples: a point is removable (Riemann). A disc is not removable:

**Painlevé (1888)** : a set of zero length is removable.
Length = one dimensional Hausdorff measure

**Painleve's problem (Ahlfors (1947))**:describe in *geometric* terms the removable sets.

To find more natural the definition of analytic capacity:
Remark: $K$ is removable if and only if

$$H^\infty(\mathbb{C} \setminus K) = \{\text{constants}\}.$$

**Analytic capacity**: given a compact subset $K$ of the plane

$$\gamma(K) = \sup |f'(\infty)|$$

where the sup is on those $f$ in $H^\infty(\mathbb{C} \setminus K)$, $\|f\|_\infty \leq 1$ ($f(\infty) = 0$).
The extremal is called the Ahlfors function.

Easy : $\gamma(K) = 0$ iff $K$ is removable.

**Examples**

$$\gamma(\bar{D}(a,r)) = r \text{ and } f(z) = \frac{r}{z-a}.$$

For a continuum $K$, $\gamma(K) \simeq \text{diam}(K)$ and the Ahlfors function is the conformal mapping $f$ of $\mathbb{C}_\infty \setminus K$ into the unit disc with $f(\infty) = 0$.

For $K \subset \mathbb{R}$, $\gamma(K) = \text{length}(K)/4$. (Pommerenke)

For $[-1, 1]$ :
$$\frac{1 - \sqrt{\frac{z-1}{z+1}}}{1 + \sqrt{\frac{z-1}{z+1}}}$$

$$\frac{1 - \exp\left(\frac{1}{2}\int_K \frac{dt}{t-z}\right)}{1 + \exp\left(\frac{1}{2}\int_K \frac{dt}{t-z}\right)}$$

If $\dim_H(K) > 1$, then $\gamma(K) > 0$.

**Proof of Painlevé's theorem:**
length$(K) = 0 \Longrightarrow \gamma(K) = 0$
Cover the set $K$ by squares $Q_j$.

If $f \in H^\infty(\mathbb{C} \setminus K)$ and $\|f\|_\infty \leq 1$, then

$$\begin{aligned}
|f'(\infty)| &= \Big|\frac{1}{2\pi\imath}\int_{\partial(\cup_j Q_j)} f(z)dz\Big| \\
&\leq \sum_j \frac{1}{2\pi}\int_{\partial Q_j} |f(z)||dz| \\
&\leq \sum_j \frac{2}{\pi} l(Q_j).
\end{aligned}$$

$$\gamma(K) \leq \frac{2}{\pi}\ \inf \sum_j l(Q_j) = \frac{2}{\pi}\ H^1(K)$$

where $H^1(K)$ is the Hausdorff content of K.

**Proof of** $\dim_H(K) > 1 \implies \gamma(K) > 0$**.**

Take a positive measure $\mu$ supported on $K$ satisfying for some $\alpha > 1$

$$\mu\, D(z,r) \le r^{\alpha},\ r > 0,\ z \in \mathbb{C}.$$

$$\text{Set} \quad f(z) = \int \frac{1}{\zeta - z} d\mu(\zeta),\ z \notin K.$$

$$\begin{aligned} |f(z)| &\le \int \frac{1}{|\zeta - z|} d\mu(\zeta) \\ &\preceq \int_0^d \frac{1}{t}\, dt^{\alpha} = \int_0^d t^{\alpha-2}\, dt = C(\alpha) \end{aligned}$$

In fact you get the inequality

$$\left(H^{\alpha}(K)\right)^{\frac{1}{\alpha}} \le C(\alpha)\,\gamma(K), \text{for every } \alpha > 1.$$

Only sets of Hausdorff dimension 1 and positive length are intersting

- Rectifiable curves

- Invisible sets (Besicovitch irregular): sets of finite positive length which project into sets of zero length in almost all directions.

**Denjoy proved:** $\frac{1}{4}\,\text{length}(K) \leq\ \gamma(K),\ \ K \subset \mathbb{R}.$

Start with an interval $K = [a, b]$.

$$\int_a^b \frac{1}{t-z}\,dt = \log\frac{b-z}{a-z} = \log\left|\frac{b-z}{a-z}\right| + \imath\,\arg\left(\frac{b-z}{a-z}\right)$$

If $K = \cup_j [a_j, b_j]$ then

$$\left|\text{Im}\,\frac{1}{2}\int_K \frac{1}{t-z}\,dt\right| = \frac{1}{2}\left|\sum_j arg(\frac{b_j - z}{a_j - z})\right| \leq\ \frac{\pi}{2}.$$

$$\left|\frac{1-\exp(\frac{1}{2}\int_K \frac{1}{t-z}\,dt)}{1+\exp(\frac{1}{2}\int_K \frac{1}{t-z}\,dt)}\right| \leq 1$$

**The Denjoy conjecture:** If $\Gamma$ is a rectifiable curve then $\gamma(K)=0$ iff length $(K)=0$, $K\subset\Gamma$.

It can be easily reduced to $\Gamma$ a Lipschitz graph

The Cauchy Integral on the curve $\Gamma$ is

$$C_\varepsilon f(z)=\int_{|\zeta-z|>\varepsilon}\frac{f(\zeta)}{\zeta-z}\,|d\zeta|\,,\qquad z\in\mathbb{C},\qquad \varepsilon>0$$

If $\int_\Gamma |C_\varepsilon f(z)|^2 \, |dz| \;\leq\; C \int_\Gamma |f(z)|^2 \, |dz|$ then the Denjoy conjecture is solved.

Calderón (1977), Coifman-McIntosh-Meyer (1982).

Garabedian ($\simeq 50$): $H^\infty$ removability $= H^2$ removability. (Hahn-Banach)

$$f(z) = C(\chi_K)(z) = \int_K \frac{|d\zeta|}{\zeta - z}, \quad z \notin K \subset \Gamma,$$

is analytic on $\mathbb{C}_\infty \backslash K$, $f'(\infty) = -\text{length}\,(K)$ and is in $H^2(\mathbb{C} \setminus K)$:

You get $\quad \text{length}\,(K) \leq C(\Gamma)\gamma(K)$.

## Another way (Uy, Davie-Oksendal).

$C\colon L^2(\Gamma) \longrightarrow L^2(\Gamma) \Longrightarrow C\colon L^1(\Gamma) \longrightarrow L^{1,\infty}(\Gamma)$

by standard Calderón-Zygmnund Theory

$$\left|\left\{z \in \Gamma :\ |C_\varepsilon f(z)| > t\right\}\right| \leq \frac{C}{t}\, ||f||_1$$

dualizes to (Hahn-Banach)

$$|K| \leq C \ \sup\left\{\left|\int_K b(z)|dz|\right| :\ \operatorname{spt} b \subset K\,,\right.$$
$$\left. 0 \leq b \leq 1\,,\ ||C_\varepsilon(b)||_\infty \leq 1\,,\, \varepsilon > 0\right\}$$

Then $\quad C(b)(z) = \int_K \frac{b(\zeta)}{\zeta - z}\,|d\zeta|\,, z \notin K \subset \Gamma,$

is in $H^\infty(\mathbb{C} \setminus K)$ and $C(b)'(\infty) = -\int_K b(z)\,|dz|$.

Thus lenght $(K) \leq C(\Gamma)\gamma(K)$.

## Uy's Theorem (1979).

$K$ compact set of zero area.

$$|f(z) - f(w)| \leq C|z - w|\,, \quad z, w \in \mathbb{C}\,,$$

$f$ analytic on $\mathbb{C} \setminus K$.

Then $\bar{\partial} f \in L^\infty(\mathbb{C})$ and supported on $K$. Thus $f$ is analytic on $\mathbb{C}$ (Weyl's lemma). Conversely, if $K$ has positive area can we find $f$ lipschitz on $\mathbb{C}$, analytic on $\mathbb{C} \setminus K$, not of the form $a + bz$?

The obvious try $f(z) = \frac{1}{\pi} \int_K \frac{dA(\zeta)}{z - \zeta}$ does not work because $\bar{\partial}\, f = \chi_K$ but

$$\partial\, f(z) = P.V. - \frac{1}{\pi} \int_K \frac{dA(\zeta)}{(z - \zeta)^2} = B(\chi_K)(z)\,,$$

which is not necessarily bounded.

The Beurling transform satisfies

$$\Big|\Big\{z \in \mathbb{C} : \ |Bg(z)| > t\Big\}\Big| \le \frac{C}{t}\,||g||_1$$

and Uy showed that this dualizes to

$$|K| \le C\,\sup\Big\{\Big|\int_K g\,dA\Big| : \ \mathrm{spt}\,g \subset K\,, \\ ||g||_\infty \le 1\,,\ ||Bg||_\infty \le 1\Big\}$$

Take $g$ attaining the supremun above, set

$$f(z) = \frac{1}{\pi}\int \frac{g(\zeta)}{z-\zeta}\,dA(\zeta)\,.$$

Then $\quad ||\bar\partial f||_\infty = ||g||_\infty \le 1$

and $\quad ||\partial f||_\infty = ||Bf||_\infty \le 1.$

Thus $f$ is lipschitz on $\mathbb{C}$, and analytic on $\mathbb{C}\setminus K$ and not of the form $a + bz$.

Open problem
extremely interesting
because it must be difficult.

**Garnett-Jones.**
Given a compact $K$ in a Lipschitz graph **construct** a non-constant bounded analytic function on $\mathbb{C} \setminus K$.

Recent paper by Jones-Mueller.

# Menger curvature.

Given three distinct points in the plane consider

$$C = \begin{pmatrix} 0 & \frac{1}{z_1-z_2} & \frac{1}{z_1-z_3} \\ \frac{1}{z_2-z_1} & 0 & \frac{1}{z_2-z_3} \\ \frac{1}{z_3-z_1} & \frac{1}{z_3-z_2} & 0 \end{pmatrix}$$

$$C^*C = \begin{pmatrix} \frac{1}{|z_2-z_1|^2} + \frac{1}{|z_3-z_1|^2} & \frac{1}{(z_3-z_2)\overline{(z_3-z_1)}} & \frac{1}{(z_3-z_2)} \\ & & \frac{1}{\overline{(z_1-z_2)}} \\ & & \end{pmatrix}$$

Set $\mathbf{1} = (1,1,1) \in \mathbb{C}^3$. Then

$$||C\mathbf{1}||^2 = \langle C\mathbf{1}, C\mathbf{1} \rangle = \langle C^*C\mathbf{1}, \mathbf{1} \rangle$$

$$= \sum_{i+j} \frac{1}{|z_i - z_j|^2} + \sum_{\sigma} \frac{1}{(z_{\sigma(2)} - z_{\sigma(1)})\overline{(z_{\sigma(3)} - z_{\sigma(1)})}}$$

**Melnikov**

$$\sum_{\sigma} \frac{1}{(z_{\sigma(2)} - z_{\sigma(1)})(\overline{z_{\sigma(3)} - z_{\sigma(1)}})}$$

$$= \left( \frac{4A(z_1, z_2, z_3)}{|z_2 - z_1|\,|z_3 - z_1|\,|z_3 - z_2|} \right)^2 = \frac{1}{R^2}$$

$$c\,(z_1, z_2, z_3) = \frac{1}{R} \qquad \text{Menger curvature}$$

**Proof:** Take $z_1 = 0$, $z_2 = 1$, $z_3 = z$

$$\frac{1}{(z_2 - z_1)(\overline{z_3 - z_1})} + \frac{1}{(z_1 - z_2)(\overline{z_3 - z_2})}$$

$$+ \frac{1}{(z_1 - z_3)(\overline{z_2 - z_3})} = \frac{1}{\bar{z}} - \frac{1}{\overline{z - 1}} + \frac{1}{\overline{z(z-1)}}$$

$$= \frac{z|z-1|^2 - |z|^2(z-1) + \bar{z}(z-1)}{|z|^2|z-1|^2}$$

Twice the real part is $\frac{4(\operatorname{Im} z)^2}{|z|^2|z-1|^2}$

$\mu$ positive Radon measure satisfying

$$\mu\, D(z,r) \le C\, r\,, \quad z \in \mathbb{C},\, r > 0\,.$$

$$C_\varepsilon(f\mu)(z) = \int_{|\zeta - z|>\varepsilon} \frac{f(\zeta)}{\zeta - z} d\,\mu(\zeta)\,, z \in \mathbb{C},\, \varepsilon > 0\,.$$

$$\int_D |C_\varepsilon(\chi_D \cdot \mu)|^2 d\,\mu = \int_D C_\varepsilon(\chi_D)(z)\overline{C_\varepsilon(\chi_D)(z)} d\mu(z)$$

$$= \int_D \int_{D_\varepsilon} \int_{D_\varepsilon} \frac{1}{(\zeta - z)} \frac{1}{(\overline{w - z})} d\mu(\zeta) d\mu(w) d\mu(z)$$

$$\int_D |C_\varepsilon(\chi_D\, \mu)|^2 d\mu$$

$$= \int \underset{T_\varepsilon}{\int} \int c^2(z,\zeta,w) d\mu(\zeta) d\mu(w) d\mu(z) + O(\mu D)$$

$$T_\varepsilon = \{(z,\zeta,w) \in D^3 \colon |z - \zeta| > \varepsilon\,, |z - w| > \varepsilon \\ |\zeta - w| > \varepsilon\}$$

What happens on the graph of $y = A(x)$?

$$C\left(\gamma(x),\gamma(y),\gamma(z)\right)^2 \leq 16\left(\frac{\frac{A(y)-A(x)}{y-x} - \frac{A(z)-A(x)}{z-x}}{z-y}\right)^2$$

$$\int\int_{I^3}\int c^2(\gamma(x),\gamma(y),\gamma(z))dxdydz \leq C\,|I|$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\frac{a(y)-a(x)}{y-x} - \frac{a(z)-a(x)}{z-x}}{z-y}\right)^2 dxdydz$$
$$= c\,||a'||_2^2$$

Given $I$

$$\int\int_{I^3}\int\left(\frac{\frac{A(y)-A(x)}{y-x}-\frac{A(z)-A(x)}{z-x}}{z-y}\right)^2 dxdydz \leq$$

$$C\,||(A-P_I)'||^2_{L^2(I)} \leq C\,||A'||^2_\infty\,|I|$$

Remark: A Lipschitz ($A' \in L^\infty(\mathbb{R})$) can be replaced by $A' \in$ BMO ($\mathbb{R}$).

## Total Menger curvature

Pointwise Menger curvature

$$c_\mu^2(z) = \int\int c^2(z,\zeta,w)d\mu(\zeta)d\mu(w)\,, \quad z \in \mathbb{C}$$

Total Menger curvature

$$\int\int\int c^2(z,\zeta,w)\,d\mu(z)d\mu(\zeta)d\mu(w) = \int c_\mu^2(z)d\mu(z)$$

We have proved that

$$\int\int_{J^3}\int c^2(z,\zeta,w)|dz||d\zeta||dw| \le C \text{ length}(J)$$

$J =$ arc of Lipschitz graph.

The operator norm of the Cauchy Integral with respect to $\mu$ is

$$||C||^2_{L^2(\mu),L^2(\mu)} \simeq$$

$$||M\mu||_\infty + \sup_D \frac{1}{\mu D} \iiint_{D^3} c^2(z,\zeta,w)\,d\mu(z)\,d\mu(\zeta)\,d\mu(w)$$

and $M\mu(z) = \sup_{r>0} \frac{\mu D(z,r)}{r}, \quad z \in \mathbb{C}.$

## First Calderón commutator

$$C_1 f(x) = \text{ P.V.} \int_{-\infty}^{\infty} \frac{A(y) - A(x)}{(y - x)^2} f(y) dy$$

Calderón proved (1965) $C_1 \colon L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$

$$C_n f(x) = \text{ P.V.} \int_{-\infty}^{\infty} \frac{(A(y) - A(x))^n}{(y - x)^{n+1}} f(y) dy$$

Set $K(y,x) = \frac{A(y)-A(x)}{(y-x)^2}$ and symmetrize:

$$K(y,x)K(z,x) + K(x,y)K(z,y) + K(x,z)K(y,z) =$$

$$\left( \frac{\frac{A(y)-A(x)}{(y-x)} - \frac{A(z)-A(x)}{(z-x)}}{z - y} \right)^2$$

and so the Cauchy integral is controlled by the first commutator.

Quotation from Montaigne taken from Y. Meyer's book on Wavelets.

Ce a quoi l'un s'était failli, l'autre y est arrivé et ce qui était inconnu à un siècle, le siècle suivant l'à eclairci, et les sciences et les arts ne se jettent pas en moule mais se forment et figurent en les maniant et polissant a plusieurs fois ...

## Other kernels

$$\gamma(K) = \sup |f'(\infty)| =$$

$$\sup\Big\{|\langle T, 1\rangle| \ : \text{spt}\, T \subset K\,, \Big\|\frac{1}{z} * T\Big\|_\infty \leq 1\Big\}$$

$$T = \frac{1}{\pi}\bar{\partial} f \qquad f = \frac{1}{z} * T$$

$$\gamma_+(K) = \sup\Big\{||\mu||: \text{spt}\, \mu \subset K\,, \|\tfrac{1}{z} * \mu\|_\infty \leq 1\Big\}$$

$$C(K) = \sup\Big\{||\mu||: \text{spt}\, \mu \subset K\,, \|\tfrac{1}{|z|} * \mu\|_\infty \leq 1\Big\}$$

Take $0 < \alpha < n$ and $K_\alpha(x) = \frac{x}{|x|^{1+\alpha}}\,, x \in \mathbb{R}^n$.

$$\gamma_\alpha(K) = \sup\Big\{|\langle T, 1\rangle|: \text{spt}\, T \subset K, \Big\|\frac{x}{|x|^{1+\alpha}} * T\Big\|_\infty \leq 1\Big\}$$

**Laura Prat (2003)**

$$K_\alpha(y-x)K_\alpha(z-x)+K_\alpha(x-y)K_\alpha(z-y)+ \\ K_\alpha(x-z)K_\alpha(y-z)\equiv P_\alpha(x,y,z)$$

$$\frac{2-2^\alpha}{L(x,y,z)^{2\alpha}}\le P_\alpha(x,y,z)\le\frac{2^{1+\alpha}}{L(x,y,z)^{2\alpha}},\ 0<\alpha<1.$$

$P_\alpha(x,y,z)$ changes sign if $\alpha>1$.

$$\gamma_\alpha(K)\simeq C_{\frac{2}{3}(n-\alpha),\frac{3}{2}}(K)\,,\quad 0<\alpha<1.$$

$$C_{s,p}(K)=\inf\left\{||f||_p^p:\frac{1}{|x|^{n-s}}*f\ge 1\text{ on }K\right\}$$

Then $\Lambda^\alpha(K)<\infty\Longrightarrow\gamma_\alpha(K)=0,\ 0<\alpha<1.$

$\gamma_\alpha$ is semiadditive amb bilipschitz invariant.

**Case** $\alpha = n - 1$

Volberg: $\gamma_{n-1}$ is semiadditive (2003).

The kernel is $\frac{x}{|x|^n} = \bigtriangledown\left(\frac{1}{|x|^{n-2}}\right)$

$\gamma_{n-1}$ is the capacity associated to lipschitz harmonic functions.

$$T * \frac{x}{|x|^n} \in L^\infty \Longleftrightarrow T * \frac{1}{|x|^{n-2}} \text{ Lipschitz}$$

Very likely $\gamma_\alpha$ is semiadditive for all $\alpha$.

## The Garnett set

$$E_N = \bigcup_{j=1}^{4^N} Q_j^N \qquad\qquad E = \bigcap_{N=1}^{\infty} E_N$$

Take circles instead of squares (sorry!)

$\mu = \mu_N = |dz_{\partial E_N}|$

$$||C||_{L^2(\mu),L^2(\mu)} \simeq \sqrt{N}$$

$$c_\mu^2(z) = \iint_{E_N^2} c^2(z,\zeta,w)\,d\mu(\zeta)\,d\mu(w)$$

pointwise curvature.

Then $c_\mu^2 \simeq N$ and so

$$\iiint_{E_N^3} c^2(z,\zeta,w)\,d\mu(z)\,d\mu(\zeta)\,d\mu(w) \simeq N.$$

Take squares (discs) $Q_j$ in the $j$-th generation

$$Q_0 \supset Q_1 \supset \ldots \supset Q_N \ni z.$$

$$c_\mu^2(z) = 2 \iint_{|w-z|\geq|\zeta-z|} c^2(z,w,\zeta)\,d\mu(w)\,d\mu(\zeta)$$

$$= 2\sum_{n=0}^{N-1} \iint_{\substack{w\in Q_n\setminus Q_{n+1}\\ |w-z|\geq|\zeta-z|}} c^2(z,w,\zeta)\,d\mu(w)\,d\mu(\zeta)$$

$$+ 2\iint_{Q_N^2} c^2(z,w,\zeta)\,d\mu(\zeta)\,d\mu(w)$$

$$\leq 2\sum_{n=0}^{N-1} \frac{1}{\left(\frac{1}{4^n}\right)^2}\,\mu(Q_n)^2 + 2 \simeq N$$

Choose two squares in the $n+1$ generation not containing $z$ and contained in $Q_n$.

$$c_\mu^2(z) \geq \sum_{n=0}^{N-1} \int\limits_{Q_1^{n+1}} \int\limits_{Q_2^{n+1}} c^2(z,w,\zeta)\, d\mu(w)\, d\mu(\zeta)$$

$$\geq C \sum_{n=0}^{N-1} \frac{1}{\left(\frac{1}{4^n}\right)^2}\, \mu(Q_1^{n+1})\mu(Q_2^{n+1}) \simeq N$$

Thus Cauchy is unbounded on $L^2(E, d\bigwedge^1)$ in spite of

$$\bigwedge^1 (E \cap D(z,r)) \simeq r\,, \quad z \in E\,, \quad 0 < r \leq 1.$$

**Vitushkins's conjecture**: If $0 < \bigwedge^1(E) < \infty$, then $\gamma(E) = 0 \Longleftrightarrow E$ is invisible.

**Proof of** $\gamma(E) = 0 \Longrightarrow E$ invisible.

By Besicovitch theory

$$E \text{ invisible} \iff \bigwedge^1 (E \cap \Gamma) = 0$$

for every rectifiable $\Gamma$.

If $E$ is not invisible, then the Denjoy conjecture says that $\gamma(E) > 0$.

## The $T(b)$-Theorem

The operator $T$ will be the Cauchy Integral. The underlying measure $\mu$ satisfies

$$\mu D(z,r) \le Cr\,, \quad r>0\,, \quad z\in\mathbb{C}$$

and the doubling condition (if we are lucky)

$$\mu(D(z,2r)) \le C\mu D(z,r)\,, \quad r>0\,, \quad z\in\mathbb{C}$$

Assume that $b\in L^\infty(\mu)$, $||b||_\infty \le 1$ such that $||C_\varepsilon(b)||_\infty \le 1$, $\varepsilon>0$.
Assume also that $b$ is para-accretive, that is,

$$\frac{1}{|D|}\Big|\int_D b\,d\mu\Big| \ge \delta > 0\,,$$

for each disc $D$ centered at a point of the support of $\mu$.

Then $C\colon L^2(\mu) \longrightarrow L^2(\mu)$.

$b = 1$ $\quad$ $T(1)$-**Theorem** David-Journé

$T(b)$-**Theorem**: David-Journé-Semmes.

Non-doubling context: Tolsa, Nazarov, Treil and Volberg.

**Local** $T(b)$-**Theorem** (Christ)
Assume that for each disc $D$ there is

$$b_D \in L^\infty(\mu)\,, \quad \operatorname{spt} b_D \subset D\,, \quad ||b_D||_\infty \le 1\,,$$

$$||C_\varepsilon(b_D)||_\infty \le 1\,, \quad \varepsilon > 0\,,$$

and

$$\frac{1}{\mu D}\Big|\int_D b_D\Big| \ge \delta > 0\,.$$

Then $C\colon L^2(\mu) \longrightarrow L^2(\mu)$ and

$$||C||_{L^2(\mu),L^2(\mu)} \le C\Big(\frac{1}{\delta}\Big)^{10}\,.$$

Let's show $\quad \gamma(E_N) \leq C \frac{1}{N^{\frac{1}{20}}}$.

(Indeed, we know $\gamma(E_N) \simeq \frac{1}{\sqrt{N}}$), but this is much more difficult)

Take $f \in H^\infty(\mathbb{C} \setminus E_N)$,

$$||f||_\infty \leq 1\,, \quad f(\infty) = 0\,, \quad f'(\infty) = \gamma(E_N).$$

Express

$$f(z) = \frac{-1}{2\pi i} \int_{\partial E_N} \frac{f(\zeta)}{\zeta - z}\, d\zeta\,, \quad z \notin E_N.$$

Set $b(z) = -\frac{f(\zeta)\dot{z}(\zeta)}{2\pi i}$

Then $\quad ||b||_\infty \leq 1,\ ||C(b)||_\infty \leq 1$ and

$$\Big| \int_{\partial E_N} b ds \Big| = \gamma(E_N)\,.$$

Take now $Q_j^n \cap E_N$, $0 \leq n \leq N$:
it is a dilated $(\frac{1}{4^n})$ translation of $E_{N-n}$. Therefore:

There is $b_j^n \in L^\infty(ds)$, $\operatorname{spt} b_j^n \subset Q_j^n \cap E_N$,

$$||b_j^n||_\infty \leq 1 \qquad ||C(b_j^n)||_\infty \leq 1$$

and

$$\int b_j^n ds = \gamma(E_{N-n}) \frac{1}{4^n} \geq C\gamma(E_N) l(Q_j^n) \,.$$

By the local $T(b)$-Theorem:

$$\sqrt{N} \simeq ||C||_{L^2(ds),L^2(ds)} \leq C \frac{1}{\gamma(E_N)^{10}}$$

which is $\gamma(E_N) \leq C \frac{1}{N^{\frac{1}{20}}}$.