CLASSICAL POTENTIAL THEORY AND ANALYTIC CAPACITY

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ABSTRACT. Recently, analytic capacity has been proved to be semi-additive (that is, sub-additive modulo an absolute constant). This solves a long standing open problem in complex analysis using real variables methods. The proof of this remarkable result, which is the culmination of a long series of contributions by many authors, uses some ideas from Potential Theory and, in fact, the problem itself presents a definite formal analogy with some central aspects of Potential Theory. The purpose of this article is to review the basics of Classical Potential Theory in such a way that the formal analogy between analytic capacity and classical Wiener capacity becomes evident and the semi-additivity problem becomes natural. At the end one gives an idea of the main difficulty involved in its solution.

1. INTRODUCTION

This paper is an article version of a plenary lecture that I delivered at the "Seminar on Mathematical Analysis in Andalusia," held in Granada in the summer of 2004. In preparing the lecture I was first tempted to talk about the solution of the semi-additivity problem for analytic capacity, but I quickly realized that this would have necessarily brought the exposition into a jungle of technicalities with limited interest for the audience. Then I remembered that classical potential theory is a beautiful and powerful branch of analysis, which has been a permanent source of inspiration for many problems on analytic capacity. Therefore I planned the exposition so that it could serve as an introduction to classical potential theory for part of the audience and as an explanation of the difficulties connected with the study of analytic capacity for others. The present article follows this plan faithfully.

Section 2 contains the introduction to classical potential theory, which is inspired by [18]. Section 3 deals with analytic capacity. The interested reader is advised to read the surveys [3], [14] and [15], where further references will be found.

2. ELEMENTARY ELECTROSTATICS

2.1. Coulomb's Law. A charge q placed at the origin creates an electrostatic field. This means that if one places a test charge q' at the point $x \in R^3$, then a force acting on this charge appears. Coulomb's Law asserts that this force acts in the direction of the vector x (if q and q' have the same sign) and that its magnitude is proportional to the product $q q'$ and inversely proportional to the square of the distance between the charges q and q'. In other words, assuming for the sake of simplicity that q and q' are unit charges, the force acting on q' due to the presence of q is proportional to

$$\vec{E}(x) = \frac{x}{|x|} \frac{1}{|x|^2} = \frac{x}{|x|^3}.$$

One refers to the vector above as the electrostatic field created by a unit charge at the origin (we ignore proportionality factors). A completely analogous theory, based on Newton's Laws, can be developed when interactions due to charges are replaced by interactions due to masses, and then one talks about the gravitational field.

A simple computation shows that the electrostatic field above is the gradient of a function :

$$\frac{x}{|x|^3} = -\nabla\left(\frac{1}{|x|}\right).$$

There is an interesting interpretation of the function $\frac{1}{|x|}$ in terms of potential energy. Assume that a unit charge travels from a point x to a point y following a path $\gamma(t), a \leq t \leq b$. The work produced by the force $\vec{E}(x)$ is given by the line integral

$$W = \int_a^b \vec{E}(\gamma(t)) \cdot \gamma'(t)\, dt = \int_a^b -\frac{d}{dt}\left(\frac{1}{|x(t)|}\right) dt = \frac{1}{|x|} - \frac{1}{|y|}$$

Letting $y \to \infty$ one concludes that $\frac{1}{|x|}$ can be thought of as the work done by the electrostatic field when the unit charge travels from x to ∞. Alternatively, one may think that $\frac{1}{|x|}$ is the potential energy of a unit charge placed at the point x.

When the charge generating the field is placed at the point a the electrostatic field is

$$\vec{E}(x) = \frac{x-a}{|x-a|^3} = -\nabla\left(\frac{1}{|x-a|}\right)$$

and the potential energy, or simply "potential", is $\frac{1}{|x-a|}$.

Consider now a body in the space and a general charge distribution on it. In mathematical terms we are considering a compact subset K of R^3 and a signed (finite Borel regular) measure μ on K. Now the question is : what is the mathematical expression for the electrostatic field generated by the given charge distribution?. To get a quick answer we use an elegant heuristic argument very popular among physicists. Subdivide the body K into a large number of small pieces. Call q_j the total charge of the j-th piece and select a point a_j in it.
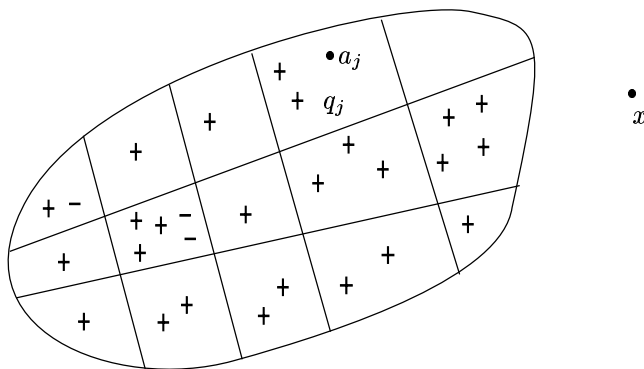

Figure 1

Then the distribution of charges μ is approximately $\sum_j q_j\, \delta_{a_j}$. Therefore the electrostatic field $\vec{E}(x)$ generated by μ is approximately the sum of the electrostatic

fields generated by point charges q_j at the points a_j, that is,

$$\vec{E}(x) \simeq \sum_j q_j \frac{x - a_j}{|x - a_j|^3}.$$

Observe that the sum in the equation above is a Riemann sum associated with a certain integral. Thus, when the pieces become arbitrarily small and one takes a limit the following identity arises :

$$\vec{E}(x) = \int \frac{x - y}{|x - y|^3} \, d\mu(y).$$

In other words, the electrostatic field generated by the charge distribution μ is the convolution of the measure μ with the vector valued kernel $\frac{x}{|x|^3}$. Since in the case at hand one can take derivatives inside the integral sign, we obtain

$$\vec{E}(x) = \int \frac{x - y}{|x - y|^3} \, d\mu(y) = -\nabla \left(\int \frac{d\mu(y)}{|x - y|} \right).$$

The argument presented before tells us that the value

$$U^\mu(x) = \int \frac{d\mu(y)}{|x - y|}$$

represents the potential energy of a unit charge placed at the point x or the work done by the unit charge when traveling from x to ∞ under the action of the field $\vec{E}(x)$. Because of that, $U^\mu(x)$ is called the (Newtonian) potential of the measure μ. From the mathematical perspective the Newtonian potential of μ is just the convolution of μ with the positive kernel $\frac{1}{|x|}$.

We now present some basic examples.

Example. *Take $\mu = \delta_0$, the Dirac delta at the origin. We are dealing with a unit charge placed at the origin and so the electrostatic field is $\vec{E}(x) = \int \frac{x}{|x|^3} \, d\delta_0(x) = \frac{x}{|x|^3}$ and the potential is $\frac{1}{|x|}$. Notice that $U^\mu(0) = \infty$, which shows that the potential energy at a point may be infinite.*

Example. *Let $\mu = d\sigma$ be the surface measure on the unit sphere $|x| = 1$. It is clear that the potential*

$$U^\mu(x) = \int_{|y|=1} \frac{1}{|x - y|} d\sigma(y)$$

depends only on $|x|$ for $|x| > 1$. In fact one can show that

$$U^\mu(x) = \frac{4\pi}{|x|}, \qquad |x| > 1,$$

$$U^\mu(x) = 4\pi, \qquad |x| \le 1.$$

Notice that in this case the potential is finite everywhere. This is due to the fact that the measure is much more dispersed than in the previous example, and hence is, in some sense, less singular. One obtains the electrostatic field by taking the gradient in the preceding identity:

$$\vec{E}(x) = \begin{cases} -4\pi \frac{x}{|x|^3}, & |x| > 1, \\ 0, & |x| \le 1. \end{cases}$$

Example. *Let $\mu = dt$ be the linear measure on the segment $K = \{(t, 0, 0): -1 \leq t \leq 1\}$. Notice that in this case the potential becomes infinite everywhere on the segment*

$$U^{\mu}(x) = \int_{-1}^{1} \frac{1}{|x_1 - t|}\, dt = \infty,\ x = (x_1, 0, 0) \in K.$$

This reflects the fact that the measure is much more concentrated than in the previous example.

2.2. The equilibrium potential.

Assume now that the body K is a conductor. What really happens when you distribute a charge, say the unit charge, on K?. The answer is that the charges move until they reach an **equilibrium distribution**. Call μ such an equilibrium charge distribution. If there were points in K with different potential energy, then this difference of potential would force the charges to move inside the conductor, which is impossible because we already reached equilibrium. Therefore $U^{\mu}(x) = V,\ x \in K$, where V is a constant. This is what Gauss proved assuming that the body and the potentials are as smooth as one may need. Now the following highly non-trivial mathematical problem arises.

The problem of the equilibrium distribution: *given a compact subset K of $\mathbb{R}^3$, show that there exists an "equilibrium" measure μ with total mass 1 such that the potential U^{μ} is* **constant** *on K.*

It turns out that an equilibrium measure exists, is unique and is a positive measure. Also, as we will see, the constant value of the "equilibrium" potential U^{μ} provides important information on K.

The problem of the equilibrium distribution was solved by Frostman, a student of Marcel Riesz, in his thesis (1935).

If the reader devotes some time to thinking about how this measure could be constructed, it becomes clear that the problem is far from easy. However, at least for the unit sphere $K = \{x : |x| = 1\}$ the equilibrium distribution can be found readily. It is the normalized surface measure. Its potential takes the constant value 1 on the sphere, as we said in Example 3 in subsection 2.1 .

The proof of the existence of an equilibrium distribution is based on the solution of an extremal problem that involves the notion of energy of a charge distribution, which we discuss in the next section.

2.3. Energy.

Consider a signed measure μ supported on a compact subset K of $\mathbb{R}^3$. Subdivide the body K into a large number of small pieces, take a point a_j in the j-th piece and assume that the charge of the j-th piece is q_j (see Figure 1 above). Thus μ is approximately the distribution $\sum_j q_j \delta_{a_j}$. Suppose that the charge q_j goes to ∞. Then the work done by the field generated by the charges $q_k, k \neq j$ is $\sum_{k \neq j} q_k \frac{1}{|a_k - a_j|}$. When we let all charges q_j go to infinity then the total work done is

$$\sum_j q_j \sum_{k \neq j} q_k \frac{1}{|a_k - a_j|}$$

In the computation of the work just outlined we do not consider the case $k = j$ because we would immediately get the value ∞, as remarked in Example 1. Now we recognize the sum above as a Riemann sum of a certain integral. Taking the limit as the pieces become arbitrarily small, we conclude that the work done by the

field generated by the distribution μ when all charges go to ∞ is exactly

$$\int U^{\mu}(x)\, d\mu(x).$$

Of course this integral should also be interpreted as the potential energy accumulated by the charge distribution μ. Hence we define the **energy** associated with the signed measure μ by the elegant expression

$$E(\mu) = \int U^{\mu}(x)\, d\mu\,(x) = \int\int \frac{d\mu(x)\, d\mu(y)}{|x-y|}.$$

Here are some examples.

If $\mu = \delta_0$, then

$$E(\delta_0) = \int \frac{1}{|x|}\, d\,\delta_0\,(x) = \infty\,.$$

If $\mu = dt$ on the segment $K = \{(t,0,0) : -1 \le t \le 1\}$, then

$$E(\mu) = \int_K U^{\mu}(x)\, d\mu\,(x) = \int_{-1}^{1} \infty\, dt = \infty\,.$$

The reason we get infinite energy in the two preceding examples is that the support of the charge distribution is too small. In the next two examples the charge distributions are sufficiently dispersed to yield finite energy.

Take μ to be the normalized surface measure on the unit sphere $K = \{x \in \mathbb{R}^3 : |x| = 1\}$. Since $U^{\mu}(x)$ is 1 on the sphere, we obtain

$$E(\mu) = \int_{|x|=1} U^{\mu}(x)\, d\mu(x) = 1$$

Let μ now be the volume measure (3-dimensional Lebesgue measure) restricted to the closed unit ball of $\mathbb{R}^3$. Then one can show that

$$U^{\mu}(x) \simeq 1\,, |x| \le 1 \qquad \text{and} \qquad U^{\mu}(x) \simeq \frac{1}{|x|}\,, |x| \ge 1,$$

and so

$$E(\mu) = \int_{|x| \le 1} U^{\mu}(x)\, d\mu\,(x) \simeq 1.$$

Here $A \simeq B$ means that there exists a constant $C \ge 1$, independent of the various parameters associated to the quantities A and B, such that $C^{-1} B \le A \le C\, B$.

We are now ready to use the notion of energy to present a sketch of the proof of the existence of an equilibrium measure. Given the conductor K, that is, given a compact subset K of $\mathbb{R}^3$, one tries to find an equilibrium distribution by considering a probability measure on K with minimal energy. In other words, one seeks a solution of the extremal problem

$$\inf\{\, E(\mu) : \|\mu\| = 1 \quad \text{and} \quad \operatorname{spt}\mu \subset K\},$$

where spt stands for "support," and tries to prove that it is an equilibrium distribution. A compactness argument in the unit ball of the space of finite measures in $\mathbb{R}^3$ easily shows that the infimum above is attained by a unique measure μ_e. A variational argument shows that the potential of μ_e is constant on K. In fact this is so only when the boundary of K has some mild regularity, but we postpone the

precise statement that holds in the general case until we have introduced the notion of capacity.

2.4. **Energy as a quadratic form in a Hilbert space.** One can easily see using Green's formula that, for some constant k depending only on dimension,

$$\Delta U^\mu = -k\,\mu\,,\tag{1}$$

in the sense of distributions, Δ being the Laplace operator

$$\Delta = \sum_i \frac{\partial^2}{\partial x_i^2}\,.$$

Since (1) allows us to replace $d\mu(x)$ by $-k^{-1}\Delta\,\mu(x)dx$, an integration by parts shows that

$$E(\mu) = \int U^\mu(x)\,d\mu(x) = -k^{-1}\int U^\mu(x)\,\Delta U^\mu(x)dx$$

$$= -k^{-1}\sum_j \int U^\mu(x)\frac{\partial^2}{\partial x_j^2} U^\mu(x)\,dx$$

$$= k^{-1}\sum_j \int \frac{\partial U^\mu}{\partial x_j}\,\frac{\partial U^\mu}{\partial x_j}\,dx$$

$$= k^{-1}\int |\nabla U^\mu(x)|^2\,dx.$$

This tells us that, modulo a fixed constant, the energy of a measure is the $L^2(\mathbb{R}^n)$ norm of the gradient of its Newtonian potential. Readers familiar with elementary properties of the Hilbert spaces may now understand more clearly the uniqueness property of measures minimizing the energy functional. The preceding identity also hints at the prominent role played by Sobolev spaces in potential theory. Recall that the L^2-Sobolev space of order 1 is defined as

$$W^{1,\,2}(\mathbb{R}^3) = \{f \in L^2(\mathbb{R}^3) : \frac{\partial f}{\partial x_j} \in L^2(\mathbb{R}^3)\,,\ 1 \le j \le 3\}.$$

There is a far-reaching version of the notion of energy associated to an index $p \neq 2$ and the L^p-Sobolev space of order 1

$$W^{1,\,p}(\mathbb{R}^3) = \{f \in L^p(\mathbb{R}^3) : \frac{\partial f}{\partial x_j} \in L^p(\mathbb{R}^3)\,,\ 1 \le j \le 3\},$$

which gives rise to a vast "non-linear" potential theory (see [AH]), with interesting connections to some aspects of PDE.

2.5. **Capacity.** A central role in Electrostatics is played by "condensers." The simplest condenser consists of two plates: one of them connected to earth, and the second charged by a battery.

If one charges the right plate by an amount Q, then a difference of potential V appears between the plates. It turns out that experimental measurements show that the quotient $\frac{Q}{V}$ is constant and this constant is called the *capacity* of the condenser,
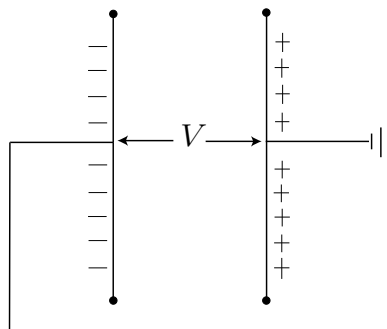
$$C = \frac{Q}{V}$$

Figure 2

Assume now that K is a conductor and that we take a charge distribution on K and let the charges move until they reach the equilibrium. Call μ the equilibrium distribution, so that $\mu(K)$ is the total charge. We know that the potential U^μ takes a constant value V on K. Define, following to Wiener, the **capacity** of K by

$$C(K) = \frac{\mu(K)}{V}.$$

One may imagine that the boundary of K and any sphere of large radius surrounding K are the plates of a condenser. Letting the radius of the sphere go to ∞, one gets an ideal condenser formed by the boundary of K and the point at ∞. Since the point at ∞ has zero potential ($U^\mu(\infty) = 0$), Wiener capacity may be understood as the capacity of this ideal condenser.

If you reach equilibrium from an initial unit charge distribution, then you get the equilibrium distribution μ_e of K. Its energy is precisely

$$E(\mu_e) = \int U^{\mu_e}\, d\mu_e = V.$$

Thus the constant value attained on K by the equilibrium distribution is exactly its energy. Since we learnt in the previous section that the equilibrium distribution minimizes energy, we obtain

$$C(K) = \frac{1}{V} = \frac{1}{\inf E(\mu)},$$

where the infimum is taken over the set of probability measures supported on K.

Sets of zero capacity play the role of negligible sets for potential theoretic questions, much in the same way sets of zero measure are negligible in measure theory. Clearly K has zero capacity if and only if any (positive) measure supported on K has infinite energy. This happens for a point and also for a segment, although the segment case needs a bit of argument (see page 13). A sphere has positive capacity and a ball too, according to the examples discussed in subsection 1.4. In fact the capacity of a sphere is exactly its radius and the same happens for a ball. That the capacity of a ball is the same as the capacity of the sphere can be easily understood by the following heuristic argument : if you distribute the unit charge on a conductor ball, then to reach equilibrium the charges (which we assume to be positive) will go first to the boundary, pushed by the mutual forces acting among them by Coulomb's Law, and then will move in the sphere to reach equilibrium.

The fact that a ball and its boundary have the same capacity immediately implies that **capacity is not an additive set function**, that is, capacity is not a measure.

The complete solution of the problem of existence of an equilibrium distribution was found by Frostman in his thesis (1935). Frostman proved the following.

Theorem. *Given a compact set $K \subset \mathbb{R}^3$ there exists a unique probability measure μ on K such that U^μ is constant C-almost everywhere (that is, except for a set of zero capacity) on K.*

2.6. Critical size of sets of zero capacity.

As we said before, a segment has zero capacity and a sphere has positive capacity. Now, a segment is a 1-dimensional object and a sphere is a 2-dimensional object. What can be said about sets with a fractional dimension between 1 and 2?.

The answer is as follows. If a set has (Hausdorff) dimension larger than 1, then it has positive capacity. If a set has finite 1-dimensional Hausdorff measure then it has zero capacity. Thus the critical dimension for capacity is exactly 1. Indeed, in dimension 1 one can find sets of vanishing capacity and also sets of positive capacity.

2.7. Sub-additivity of capacity.

It turns out that capacity is a sub-additive set function in the sense that

$$C(K_1 \cup K_2) \leq C(K_1) + C(K_2),$$

for all compact sets K_1 and K_2.

This follows readily once one has an alternative description of capacity due to La Vallée Poussin. Capacity is the maximal charge of a charge distribution with potential bounded by 1. In other words,

$$C(K) = \sup \{\mu(K) : \mu \geq 0, \ \mathrm{spt}\,\mu \subset K \text{ and } U^\mu(x) \leq 1, \ x \in \mathbb{C}\}.$$

Thus, in particular, a set K has positive capacity if and only if there is a positive measure supported on K with bounded potential.

To prove sub-additivity, take a positive measure μ supported on $K_1 \cup K_2$ such that $U^\mu(x) \leq 1, \ x \in K_1 \cup K_2$. Denote by μ_i the restriction of μ to K_i, $i = 1, 2$. Then obviously $U^{\mu_i}(x) \leq 1, \ x \in K_i$, $i = 1, 2$, and hence

$$\mu(K_1 \cup K_2) \leq \mu(K_1) + \mu(K_2) \leq C(K_1) + C(K_2).$$

Taking the supremum over such μ we get the sub-additivity inequality.

3. ANALYTIC CAPACITY

3.1. Removable sets for bounded analytic functions.

A compact subset K of the plane is said to be **removable** (for bounded analytic functions) provided that for each open set $\Omega \supset K$ and any bounded analytic function f on $\Omega \setminus K$ one may extend f to an analytic function on Ω.
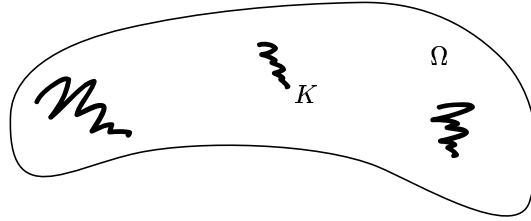
Figure 3

For example, a point is removable, according to a famous theorem of Riemann. A closed disc is not removable. Indeed, if the disc is centered at the origin it is clear that the function $f(z) = \frac{1}{z}$ is analytic and bounded on the complement of the disc, but it cannot be continued analytically to the whole plane.

It is not difficult to see that K is removable if and only if $H^\infty(\mathbb{C} \setminus K)$ (the set of bounded analytic analytic functions on $\mathbb{C} \setminus K$) is reduced to the constant functions. In other words, the test set Ω in the definition of removability can always be taken to be the whole plane.

The analytic capacity is a set function that quantifies the non-removability of a set. Since for a non-removable set there are non-constant bounded analytic functions on the complement of the set, it is natural to use derivatives to measure the non-removability of a set. The analytic capacity of a compact subset K of the plane is

$$\gamma(K) = \sup\{|f'(\infty)| : f \in H^\infty(\mathbb{C} \setminus K), \|f\|_\infty \le 1 \text{ and } f(\infty) = 0\},$$

where $f'(\infty)$ is the derivative of f at ∞. A compactness argument provides a function attaining the above supremum, and the extremal function can be shown to be unique provided $f'(\infty) > 0$. As in the Wiener capacity case, it turns out that the critical dimension for analytic capacity is 1. This is simply due to the fact that the homogeneity of the Cauchy kernel $\frac{1}{z}$ is -1, exactly as in the case of the Newtonian kernel. One also has

$$\gamma(\bar{D}(a,r)) = \gamma(\partial D(a,r)) = r, \text{ for each } a \in \mathbb{C} \text{ and } r > 0.$$

Indeed, there is a striking formal analogy between C and γ, as shown by the expression

$$(2) \qquad\qquad\qquad \gamma(K) = \sup |\langle T, 1 \rangle|,$$

where the supremum is taken over those distributions T supported on K such that the distribution $\frac{1}{z} * T$ is a measurable bounded function on the plane and $\|\frac{1}{z} * T\|_\infty \le 1$. This corresponds to the La Vallée Poussin expression for the Wiener capacity, with one important difference which will be discussed later on: positive measures are replaced by arbitrary distributions supported on K. The identity (2) is easy to prove. Given a bounded analytic function f on $\mathbb{C} \setminus K$ vanishing at ∞, one sets $T = \frac{1}{\pi} \bar{\partial} f$ so that $f = \frac{1}{z} * T$. In the opposite direction, given a distribution T with bounded Cauchy potential one simply sets $f = \frac{1}{z} * T$.

To stress the analogy between analytic and Wiener capacities, consider for compact subsets K of $\mathbb{R}^3$ the quantity

$$(3) \qquad\qquad\qquad C^*(K) = \sup |\langle T, 1 \rangle|,$$

where the supremum is taken over the class of distributions T supported on K, such that $\frac{1}{|x|} * T$ is a measurable bounded function on $\mathbb{R}^3$ and $||\frac{1}{|x|} * T||_\infty \leq 1$. Obviously $C(K) \leq C^*(K)$. We can easily show the following.

Proposition.
$$C^*(K) = C(K), \text{ for all compact } K \subset \mathbb{R}^3.$$

Proof. Consider a sequence of bounded open neighbourhoods Ω_n of K decreasing to K. We may assume, without loss of generality, that each Ω_n has a smooth boundary. Fix n and consider the equilibrium measure μ_e of the closure K_n of Ω_n. Then μ_e is a probability measure supported on K_n and its Newtonian potential takes a constant value V_n everywhere on K_n. Set $\mu = \frac{1}{V_n} \mu_e$, so that the Newtonian potential of μ is 1 everywhere on K_n. Take now a distribution T supported on K such that $||\frac{1}{|x|} * T||_\infty \leq 1$. We have

$$\langle T, 1 \rangle = \langle T, \frac{1}{|x|} * \mu \rangle = \int (\frac{1}{|x|} * T)(x) \, d\mu(x),$$

and thus

$$|\langle T, 1 \rangle| \leq ||\mu|| = \frac{1}{V_n} = C(K_n).$$

Letting n go to ∞ and then taking supremum on T we get

$$C^*(K) \leq C(K) \quad \square$$

There is, however, a fundamental difference between analytic and Wiener capacities, which we now illustrate. As we said above, the Wiener capacity of a segment is 0. In terms of the La Vallée Poussin expression for Wiener's capacity, this means that a segment does not support a non-zero positive measure with a bounded Newtonian potential. This can be shown as follows.

First of all we remark that for any positive measure μ its Newtonian potential U^μ satisfies the identity

(4) $$U^\mu(a) = \lim_{r \to 0} \frac{1}{|B(a,r)|} \int_{B(a,r)} U^\mu(x) \, dx, \, a \in \mathbb{R}^3,$$

where $|B(a,r)|$ stands for the volume of the ball of center a and radius r. Since $\frac{1}{|x|}$ is super-harmonic, so is $U^\mu = \frac{1}{|x|} * \mu$, and thus

$$U^\mu(a) \geq \frac{1}{|B(a,r)|} \int_{B(a,r)} U^\mu(x) \, dx, \, a \in \mathbb{R}^3,$$

which takes care of one inequality in (4). We obtain the other by observing that

$$\frac{1}{|a-y|} = \lim_{r \to 0} \frac{1}{|B(a,r)|} \int_{B(a,r)} \frac{1}{|x-y|} \, dx, \, a \neq y,$$

and applying Fatou's Lemma.

Assume now that μ is a positive measure supported on the segment $[0,1]$ (naturally identified with a subset of $\mathbb{R}^3$) and that its Newtonian potential is bounded almost everywhere with respect to Lebesgue measure in $\mathbb{R}^3$. Then (4) shows that U^μ is bounded everywhere in $\mathbb{R}^3$. If x is any point in $\mathbb{R}^3$, then

$$\frac{\mu B(x,r)}{r} \leq \int_{B(x,r)} \frac{d\mu(y)}{|x-y|}.$$

Since $\int \frac{d\mu(y)}{|x-y|} < \infty$, the preceding inequality shows that

$$\frac{\mu B(x,r)}{r} \to 0, \text{ as } r \to 0.$$

Therefore μ is absolutely continuous with respect to Lebesgue measure on the segment and its density is identically 0.

On the contrary, the analytic capacity of a segment is positive. This can be shown readily by considering the conformal mapping of the complement of the segment in the Riemann sphere into the unit disc. This function is bounded analytic and non-constant on the complement of the segment.

In fact, one can construct explicit positive measures on the segment with bounded Cauchy potential. To do this first observe that the Cauchy potential of the Lebesgue measure on $[0,1]$ can be computed explicitly:

$$\int_0^1 \frac{dx}{z-x} = \log \frac{z}{z-1}, \; z \neq 0, \; z \neq 1.$$

Hence logarithmic singularities appear at 0 and 1. To get rid of them one can resort to a measure of the type $\mu = \varphi(x)dx$, where φ is a continuous function supported on the interval $[0,1]$, which is positive and linear on each of the intervals $(0,\frac{1}{2})$ and $(\frac{1}{2},1)$.

The reason behind the phenomenon described in the above paragraphs, which makes such a difference between analytic and Wiener capacities, is that the kernel $\frac{1}{|x|}$ is positive and thus only size matters in making the Newtonian potential of a positive measure bounded. Instead, since the Cauchy kernel $\frac{1}{z}$ is odd, cancellation phenomena may intervene in making the Cauchy potential of a positive measure bounded. This is precisely what happens in the case of the segment.

The conclusion is that in analyzing analytic capacity one must take into account cancellation issues. The Calderón-Zygmund theory of singular integrals is an example of how subtle this phenomena may be. The fact that the Cauchy kernel is complex is not as relevant as the fact that it is odd. Indeed, one can work with any of the real kernels $Re\frac{1}{z} = \frac{x}{|z|^2}, -Im\frac{1}{z} = \frac{y}{|z|^2}$, instead of $\frac{1}{z}$ and recover the theory of analytic capacity. In a word, analytic capacity is an object of a real variables nature and complex analysis may be practically ignored in studying it (see [3], [14] and [15]).

3.2. **Old problems on analytic capacity.** The definition of analytic capacity is due to Ahlfors (1947)(see [2]). Analytic capacity did not become an important object of study until Vitushkin showed in the middle sixties its fundamental role in understanding uniform approximation on compact subsets of the plane by rational functions with no poles on the set [16]. The problems raised by Vitushkin in his paper remained open for a long time. As the years passed without much progress, analytic capacity acquired an aura of mystery and was regarded as a difficult object to deal with. One of the reasons is that one is dealing with bounded analytic functions on arbitrary plane domains and bounded analytic functions are already difficult to understand in the unit disc. For the state of the art in 1994 see [13].

One of the problems raised by Vitushkin, known as **the Vitushkin conjecture**, inquired about a geometric description of the sets of zero analytic capacity among those having finite length (i.e., 1-dimensional Hausdorff measure). The conjecture was that a compact subset of the plane with finite positive length has zero analytic

capacity if and only if its projections in almost all directions have zero length. These sets had been studied by Besicovitch in the thirties and are called Besicovitch irregular or purely unrectifiable. The terminology is explained by the fact that they are characterized by the property of having finite length and intersecting any rectifiable curve in a set of zero length. A breakthrough was made in (1996), where a special but significant case of Vitushkin's conjecture was proved [8]. The set was assumed to enjoy a homogeneity property called Ahlfors regularity, which consists in requiring that the length that one finds in the intersection of the set and a disc centered in the set is comparable to the radius of the disc (for discs of radius less than the diameter of the set). The proof made use of the Calderón-Zygmund theory of the Cauchy kernel with respect to the length measure on an Ahlfors regular set. The difficulty in extending the argument to the general case was related to the lack of the standard theory of singular integrals in non-homogeneous situations. This was overcome by G.David who proved the full conjecture in 1998 (see the survey [3]).

Another very well known old open question (this time not formulated by Vitushkin) was the characterization of the planar Cantor sets of zero analytic capacity (see [4]). This was achieved in (2003) (see [7]). In this paper one considered the problem of comparing analytic capacity with positive analytic capacity, which is defined by

$$\gamma_+(K) = \sup\{\mu(K) : \mu \geq 0, \ \mathrm{spt}\mu \subset K \text{ and } ||\frac{1}{z} * \mu||_\infty \leq 1\}.$$

Notice that trivially $\gamma_+(K) \leq \gamma(K)$, for each compact K. One was able to prove that

$$(5) \qquad\qquad\qquad\qquad \gamma(K) \leq C_0 \, \gamma_+(K),$$

for a family of approximations K of the Cantor set under consideration by finite unions of squares. The constant C_0 is independent of K.

In a remarkable article Tolsa proved some months later that (5) holds for all compact subsets of the plane [11]. This showed that analytic capacity is semi-additive, because Tolsa himself had already proved in his thesis that γ_+ is semi-additive [10].

More recent developments concern the bilipschitz invariance of analytic capacity, that is, the inequality

$$C^{-1} \, \gamma(K) \leq \gamma(\Phi(K)) \leq C \, \gamma(K),$$

where Φ is a mapping of the plane onto itself satisfying

$$C^{-1} \, |z - w| \leq |\Phi(z) - \Phi(w)| \leq C \, |z - w|.$$

Analytic capacity has been shown to be bilipschitz invariant in [12]. See also [5], where a partial result for planar Cantor sets was previously found.

3.3. **Some open problems.** There is still at least one apparently difficult open problem on analytic capacity, which consists in deciding whether a compact set of zero analytic capacity projects into zero length in almost all directions. There is a quantitative version of the problem, which we now formulate.

The **Favard length** of a compact set K in the plane (or its integral-geometric measure) is

$$F(K) = \frac{1}{\pi} \int_0^\pi |p_\theta(K)| \, d\theta,$$

where $p_\theta(K)$ stands for the projection of K into the straight line through the origin making an angle θ with the real axis, and $|p_\theta(K)|$ denotes its length.

It has been conjectured that there exists a number C_0 such that

$$F(K) \leq C_0\,\gamma(K)$$

for all compact subsets K of $\mathbb{C}$.

Besides the above inequality, the challenging problem is now extending the semi-additivity inequality and the bilipschitz invariance property to higher dimensional real variables settings. The Cauchy kernel is replaced in $\mathbb{R}^n$ by the vector-valued Riesz kernels

$$R_\alpha(x) = \frac{x}{|x|^{1+\alpha}},\; x \in \mathbb{R}^n,\; 0 < \alpha < n.$$

The natural capacity associated to this kernel is

$$C_\alpha(K) = \sup |\langle T, 1 \rangle|,$$

where the supremum is taken over all distributions T supported on K such that the vector of distributions $R_\alpha * T$ is a measurable bounded vector valued function satisfying $||R_\alpha * T||_\infty \leq 1$. The case $\alpha = n - 1$ is especially interesting, because the well-known formula $(n > 2)$

$$R_{n-1}(x) = c_n\,\nabla\left(\frac{1}{|x|^{n-2}}\right)(x),$$

establishes a relation between C_{n-1} and removable sets for Lipschitz harmonic functions, analogous to the relation between analytic capacity and and removable sets for bounded analytic functions which has been discussed above.

Laura Prat begun a systematic study of the capacities C_α in her thesis [9]. A surprising result is that if $0 < \alpha < 1$, then a compact set of finite α-dimensional Hausdorff measure has zero C_α capacity. The same is true for a non-integer $1 < \alpha < n$ provided the set is Ahlfors regular of dimension α. The result should be true without this additional hypothesis.

In [6] it was shown that for $0 < \alpha < 1$, C_α is comparable to the capacity $C_{s,p}$ of non-linear potential theory, with smoothness index $s = \frac{2}{3}(n - \alpha)$ and L^p-index $p = \frac{3}{2}$. In particular, C_α is semi-additive and bilipschitz invariant for $0 < \alpha < 1$.

Volberg has shown in [17] that C_{n-1} is semi-additive in any dimension.

The two following two problems remain open.

Problem 1. Prove that C_α is semi-additive for all $0 < \alpha < n$. This should be feasible using the methods of [17].

Problem 2. Prove that for a non-integer α between 0 and n one has the inequality

$$C_\alpha^{-1}\,C_{\frac{2}{3}(n-\alpha),\frac{3}{2}}(K) \leq\, C_\alpha(K) \leq\, C_\alpha\,C_{\frac{2}{3}(n-\alpha),\frac{3}{2}}(K),$$

for all compact subsets K of $\mathbb{R}^n$, C_α being a constant independent of K. This seems to be a much more challenging problem, because Riesz kernels R_α with $\alpha > 1$ do not have the convenient subtle positivity properties that one finds in the case $0 < \alpha \leq 1$. See [6] and [9].

Acknowledgments

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