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Extra cancellation of even Calderón–Zygmund operators and quasiconformal mappings

Joan Mateu *, Joan Orobitg, Joan Verdera

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia

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Abstract

In this paper we discuss a special class of Beltrami coefficients whose associated quasiconformal mapping is bilipschitz. A particular example are those of the form $f(z)\chi_\Omega(z)$, where Ω is a bounded domain with boundary of class $C^{1+\varepsilon}$ and f a function in $\mathrm{Lip}(\varepsilon,\Omega)$ satisfying $\|f\|_\infty < 1$. An important point is that there is no restriction whatsoever on the $\mathrm{Lip}(\varepsilon,\Omega)$ norm of f besides the requirement on Beltrami coefficients that the supremum norm be less than 1. The crucial fact in the proof is the extra cancellation enjoyed by even homogeneous Calderón–Zygmund kernels, namely that they have zero integral on half the unit ball. This property is expressed in a particularly suggestive way and is shown to have far reaching consequences.

An application to a Lipschitz regularity result for solutions of second order elliptic equations in divergence form in the plane is presented.

Résumé

Dans cet article nous étudions une classe particulière de coefficients de Beltrami dont la transformation quasiconforme associée est bilipschlitzienne. Un cas particulier correspond à des données de la forme $f(z)\chi_\Omega(z)$, où Ω est un domaine borné de classe $C^{1+\varepsilon}$ et f une fonction de $\mathrm{Lip}(\varepsilon,\Omega)$ vérifiant $\|f\|_\infty < 1$. Un point essentiel est qu'il n'y a aucune restriction sur la norme $\mathrm{Lip}(\varepsilon,\Omega)$ de f en dehors de la condition portant sur les coefficients de Beltrami dont la norme infinie doit être inférieure strictement à 1. Dans la démonstration, le fait fondamental est l'annulation des noyaux de Calderón–Zygmund homogènes d'ordre pair ; plus précisément, ils sont d'intégrales nulles sur une demi-boule unité. Cette propriété est formulée d'un façon particulièrement suggestive et s'avère avoir d'importantes conséquences.

On présente une application à un résultat de régularité lipschitzien des solutions des elliptiques du second ordre écrites sous forme de divergence.

Keywords: Calderón–Zygmund kernels; Beltrami equations; Quasiconformal mappings

* Corresponding author.
E-mail addresses: mateu@mat.uab.cat (J. Mateu), orobitg@mat.uab.cat (J. Orobitg), jvm@mat.uab.cat (J. Verdera).

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1. Introduction

Consider the Beltrami equation:

$$\frac{\partial \Phi}{\partial \bar{z}}(z) = \mu(z)\frac{\partial \Phi}{\partial z}(z), \quad z \in \mathbb{C}, \tag{1}$$

where μ is a Lebesgue measurable function on the complex plane $\mathbb{C}$ satisfying $\|\mu\|_\infty < 1$. According to a remarkable old theorem of Morrey [18] there exists an essentially unique function Φ in the Sobolev space $W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ (functions with first order derivatives locally in L^2) which satisfies (1) almost everywhere and is a *homeomorphism of the plane*. These functions are called quasiconformal. It turns out that Φ may change drastically the Hausdorff dimension of sets. Indeed, sets of arbitrarily small positive Hausdorff dimension may be mapped into sets of Hausdorff dimension as close to 2 as desired (and the other way around by the inverse mapping). There has been during the last decades much hard and penetrating work in understanding how Φ distorts sets (see, for instance [2] and the references given there or [14] for a recent result).

On the other hand, orientation preserving bilipschitz homeomorphisms of the plane are easily seen to satisfy a Beltrami equation for a certain Beltrami coefficient μ. Since bilipschitz mappings preserve all metric properties of sets, in particular Hausdorff dimension, they appear to be a distinguished subclass of particularly simple quasiconformal mappings. In [21] one gives geometric conditions which are necessary and sufficient for Φ being bilipschitz, but which do not involve the Beltrami coefficient μ. In fact, it is widely accepted that the problem of characterizing in an efficient way those μ which determine bilipschitz mappings is hopeless.

A classical result that goes back to Schauder [4] asserts that Φ is of class $C^{1+\varepsilon}$ provided μ is a compactly supported function in $\mathrm{Lip}(\varepsilon, \mathbb{C})$. It is then not difficult to see that Φ is indeed bilipschitz. The main result of this paper identifies a class of non-smooth functions μ which determine bilipschitz quasiconformal mappings Φ.

Theorem. *Let $\{\Omega_j\}$, $1 \leqslant j \leqslant N$, be a finite family of disjoint bounded domains of the plane with boundary of class $C^{1+\varepsilon}$, $0 < \varepsilon < 1$, and let $\mu = \sum_{j=1}^N \mu_j \chi_{\Omega_j}$, where μ_j is of class $\mathrm{Lip}(\varepsilon, \Omega_j)$. Assume in addition that $\|\mu\|_\infty < 1$. Then the associated quasiconformal mapping Φ is bilipschitz.*

Notice that the boundaries of the Ω_j may touch, even on a set of positive length and, of course, μ may have jumps on the boundary of some Ω_j. In particular, if we only have one domain and μ is constant we obtain the following corollary:

Corollary 1. *If Ω is a bounded domain of the plane with boundary of class $C^{1+\varepsilon}$, $0 < \varepsilon < 1$, and $\mu = \lambda\chi_\Omega$, where λ is a complex number such that $|\lambda| < 1$, then the associated quasiconformal mapping Φ is bilipschitz.*

If Ω is a disc then Corollary 1 reduces to the fact that Φ can be computed explicitly and that one can check by direct inspection that is bilipschitz. If Ω is a square Q, then one can show that the mapping Φ associated to $\lambda\chi_Q$ is not Lipschitz for some λ of modulus less than 1, so that the corollary and thus the Theorem are sharp as far as the smoothness of the boundaries of the Ω_j is concerned.

Recall that a μ-quasi-regular function on a domain D is a complex function f in $W^{1,2}_{\mathrm{loc}}(D)$ satisfying (1), with Φ replaced by f, almost everywhere in D. By Stoilow's Factorization Theorem, $f = h \circ \Phi$ for some holomorphic function h on $\Phi(D)$. From the Theorem we then conclude that f is locally Lipschitz on D. This improves on Mori's Theorem, which asserts that, for general μ, f is locally in $\mathrm{Lip}\,\alpha$ for $\alpha = \frac{1-\|\mu\|_\infty}{1+\|\mu\|_\infty} < 1$. Thus, from the perspective of PDE, the Theorem may also be viewed as a regularity result for the Beltrami equation.

The Beltrami equation is intimately related to second order elliptic equations in divergence form of the type:

$$\operatorname{div}(A\nabla u) = 0, \tag{2}$$

where $A = A(z)$ is a 2×2 symmetric elliptic matrix with bounded measurable coefficients and determinant 1 (see [4, Chapter 16]). Indeed, the real and imaginary parts of a solution to the Beltrami equation satisfy (2), where the entries of the matrix A are given explicitly in terms of the Beltrami coefficient. Conversely, given a solution u of (2), one may find a solution of an appropriate Beltrami equation whose real part is u. Thus for regularity issues one can work indistinctly with the Beltrami equation or with Eq. (2). The proof of the Theorem gives, in particular, the following regularity result for solutions of Eq. (2).

Corollary 2. *Let Ω_j, $1 \leqslant j \leqslant N$, be a finite family of disjoint bounded domains of the plane with boundary of class $C^{1+\varepsilon}$, $0<\varepsilon<1$, and assume that each Ω_j is contained in bounded domain D with boundary of class $C^{1+\varepsilon}$. Let $A = A(z)$, $z \in D$, a 2×2 symmetric elliptic matrix with determinant 1 and entries supported in $\bigcup_{j=1}^N \Omega_j$ and belonging to $\mathrm{Lip}(\varepsilon, \Omega_j)$, $1 \leqslant j \leqslant N$. Let u be a solution of Eq. (2) in D. Let D_δ stand for the set of points in D at distance greater than δ from the boundary of D. Then $\nabla u \in \mathrm{Lip}(\varepsilon', \Omega_j \cap D_\delta)$, for $0<\varepsilon'<\varepsilon$, and $1 \leqslant j \leqslant N$. In particular, $\nabla u \in L^\infty(D_\delta)$ and u is a locally Lipschitz function in D.*

The main point of the corollary above is that each solution of (2) is locally Lipschitz in D, while the classical De Giorgi–Nash Theorem gives only that u satisfies locally a Lipschitz condition of order α, for some α satisfying $0<\alpha<1$. See Section 8 for an extension to more general domains, which may have cusps.

There is some overlapping here with previous results by Li and Vogelius [17] and Li and Nirenberg [16]. See at the end of the introduction for more about that.

Another application of our Theorem concerns removability problems. There has recently been a renewed interest in gaining a better understanding of the nature of removable sets for bounded quasi-regular functions (see [3,6] and [7]). Since bilipschitz mappings preserve removable sets for bounded holomorphic functions [24], the Theorem immediately says that the removable sets for bounded μ-quasi-regular functions, with μ as in the Theorem, are exactly the removable sets for bounded holomorphic functions.

If Ω is a domain, the $\mathrm{Lip}\,\varepsilon$ norm of a function f on Ω is:

$$\|f\|_\varepsilon = \|f\|_{\varepsilon,\Omega} = \|f\|_{L^\infty(\Omega)} + \sigma_\varepsilon(f), \tag{3}$$

where

$$\sigma_\varepsilon(f) = \sup\left\{\frac{|f(z)-f(w)|}{|z-w|^\varepsilon}\colon z, w \in \Omega,\ z \neq w\right\}. \tag{4}$$

The main difficulty in proving the Theorem lies in the fact that no smallness assumption is made on $\sup_{1\leqslant j\leqslant N}\|\mu_j\|_{\varepsilon,\Omega_j}$. In the same vein, Corollary 1 is much more difficult to prove if $|\lambda|$ is close to 1. If one assumes that $\|\mu_j\|_{\varepsilon,\Omega_j}$ is small enough (depending on Ω_j) for each j, then the Theorem becomes easier. Similarly, Corollary 1 becomes easier under the assumption that $|\lambda| \leqslant \varepsilon_0(\Omega) \ll 1$. See a sketch of the argument at the end of Section 2.

The scheme for the proof of the Theorem is inspired by a clever idea of Iwaniec [11, pp. 42–43] in the context of L^p spaces, which has been further exploited in [5]. This idea brings into play the index theory of Fredholm operators on Banach spaces and, thus, compact operators. Our underlying Banach space is $\mathrm{Lip}(\varepsilon,\Omega)$, Ω a domain with boundary of class $C^{1+\varepsilon}$, and on this space we estimate the Beurling transform and its powers. We also show that the commutator between the Beurling transform and certain functions is compact on appropriate larger Lipschitz spaces.

The Beurling transform is the principal value convolution operator:

$$Bf(z) = -\frac{1}{\pi} PV \int f(z-w)\frac{1}{w^2}\,dA(w).$$

The Fourier multiplier of B is $\frac{\bar\xi}{\xi}$, or, in other words,

$$\widehat{Bf}(\xi) = \frac{\bar\xi}{\xi}\hat f(\xi).$$

Thus B is an isometry on $L^2(\mathbb{C})$.

Our Main Lemma shows that for each even smooth homogeneous Calderón–Zygmund operator T the mapping,

$$T_\Omega(f)(z) := Tf(z)\chi_\Omega(z),$$

sends continuously $\mathrm{Lip}(\varepsilon,\Omega)$ into itself, where Ω is a bounded domain with boundary of class $C^{1+\varepsilon}$. Throughout the paper we understand that, for $f \in \mathrm{Lip}(\varepsilon,\Omega)$, $Tf = T(f\chi_\Omega)$. The above boundedness result fails if T is not even. As a simple example, one may take as T the Hilbert transform and as Ω the interval $(-1,1)$.

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The even character of T is used in the proof of the Main Lemma in the form,

$$T(\chi_D)\chi_D = 0, \quad \text{for each disc } D,$$

which should be understood as a local version of the global cancellation property $T(1) = 0$ common to all smooth homogeneous Calderón–Zygmund operators. This was proved by Iwaniec for the Beurling transform in [12].

In Section 2 we present a detailed sketch of the proof and we introduce the lemmas required. In Section 3 the Main Lemma is proved in $\mathbb{R}^n$. Section 4 deals with commutators. We compare in Section 5 the operator B^n_Ω with $\chi_\Omega B^n$, Ω a bounded domain with boundary of class $C^{1+\varepsilon}$, and we show that the difference is compact on $\mathrm{Lip}(\varepsilon', \Omega)$, for $0 < \varepsilon' < \varepsilon$. In Section 6 one completes the proof of the Theorem for the case of one domain. Section 7 contains the reduction to the one domain case. In Section 8 we present an extension of the Theorem to what seems to be its more natural setting, namely domains with cusps whose boundary is of class $C^{1+\varepsilon}$ off the set of cusps. Applications to the regularity theory of solutions of Eq. (2) is this setting are also mentioned.

After a first version of the paper was completed, Daniel Faraco brought to our attention the work of Li and Vogelius [17] (and [16]), which deals with Lipschitz regularity for Eq. (2) in $\mathbb{R}^n$ for matrices with entries satisfying a Lipschitz condition of order ε on finitely many disjoint domains with boundary of class $C^{1+\varepsilon}$, but with possible jumps across the boundaries. The closures of the domains were disjoint and a main point was to obtain gradient estimates independent of the mutual distances between the closed domains. This is not an issue for our methods, which even allow touching domains. Moreover, as stated before in Corollary 2, for each solution of (2) we obtain a regularity of class $C^{1+\varepsilon'}$, for each $\varepsilon' < \varepsilon$, in each domain. This is almost the expected best possible result, namely $C^{1+\varepsilon}$ in each domain. In [17] there is a more substantial loss, due to the techniques employed. On the other hand, the setting in [17] and [16] is more general, in the sense that one works in $\mathbb{R}^n$, there is no restriction on the determinant of the matrix and also non-homogeneous terms are considered.

We also learnt from Antonio Córdoba that the regularity theory of the Euler equation in 2D, in particular the regularity theory of vortex patches, makes broad use of even Calderón–Zygmund operators, the Beurling transform in particular. We then became aware of the article [8], in which one also proves the Main Lemma. However, the proof there is different, and certainly not as much in the Calderón–Zygmund tradition as ours.

2. Sketch of the proof

First of all, there is a standard factorization method in quasiconformal mapping theory that reduces the Theorem to the case of only one domain Ω $(N = 1)$. The argument is presented in detail in Section 7. Then, from now on we will assume that μ vanishes off some domain Ω with boundary of class $C^{1+\varepsilon}$ and that $\mu \in \mathrm{Lip}(\varepsilon, \Omega)$.

As is well known, Φ is given explicitly by the formula [1],

$$\Phi(z) = z + C(h)(z),$$

where

$$Ch(z) = \frac{1}{\pi}\int h(z-w)\frac{1}{w}\,dA(w),$$

is the Cauchy transform of h. Recall the important relation between the Cauchy and the Beurling transforms: $\partial C = B$, $\partial = \frac{\partial}{\partial z}$. The function $h = \bar\partial\Phi$ is determined by the equation:

$$(I - \mu B)(h) = \mu.$$

As soon as we can invert the operator $I - \mu B$ on $\mathrm{Lip}(\varepsilon', \Omega)$, for some ε' satisfying $0 < \varepsilon' < \varepsilon$, then

$$h = (I - \mu B)^{-1}(\mu),$$

and thus h is in $\mathrm{Lip}(\varepsilon', \Omega)$ and, in particular, is bounded on Ω. By the Beltrami equation (1), h vanishes on $\mathbb{C} \setminus \overline{\Omega}$ and therefore h is in $L^\infty(\mathbb{C})$. On the other hand, $\partial\Phi = 1 + B(h)$. By the Main Lemma, $B(h)$ is in $L^\infty(\mathbb{C})$ (see (11) below), and so Φ is a Lipschitz function on the plane.

Showing that Φ is bilipschitz still requires an argument. Indeed, we have shown up to now that Φ is of class $C^{1+\varepsilon'}(\Omega)$ and thus its Jacobian is non-zero at each point of Ω [15, Theorem 7.1, p. 233]. On the other hand, Φ is conformal on $\mathbb{C} \setminus \overline{\Omega}$ and thus the Jacobian is also non-zero there. However we cannot infer immediately that the

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Jacobian is bounded below away from zero either on Ω or on $\mathbb{C} \setminus \overline{\Omega}$. This is proved in Section 6 and hence ϕ is bilipschitz.

It remains to prove that $I - \mu B$ is invertible on $\mathrm{Lip}(\varepsilon', \Omega)$ for each ε' with $0 < \varepsilon' < \varepsilon$. For f in $\mathrm{Lip}(\varepsilon, \Omega)$ set:

$$B_\Omega(f)(z) = B(f)(z)\chi_\Omega(z),$$

where, as we said in the introduction, $B(f)$ stands for $B(f\chi_\Omega)$. Following [5, p. 48] we define,

$$P_m = I + \mu B_\Omega + (\mu B_\Omega)^2 + \cdots + (\mu B_\Omega)^m,$$

so that we have,

$$(I - \mu B_\Omega)P_{n-1} = P_{n-1}(I - \mu B_\Omega) = I - (\mu B_\Omega)^n = I - \mu^n B_\Omega^n + R, \tag{5}$$

where $R = \mu^n B_\Omega^n - (\mu B_\Omega)^n$ can be easily seen to be a finite sum of operators that contain as a factor the commutator $K_0 = \mu B_\Omega - B_\Omega \mu$. Lemma 3 in Section 4 asserts that K_0 is compact on $\mathrm{Lip}(\varepsilon', \Omega)$ for each ε' less than ε, so that R is also compact on $\mathrm{Lip}(\varepsilon', \Omega)$. One would like to have now that the operator norm of $\mu^n B_\Omega^n$ on $\mathrm{Lip}(\varepsilon', \Omega)$ is small if n is large. Would this be so, then $I - \mu B_\Omega$ would be a Fredholm operator on $\mathrm{Lip}(\varepsilon', \Omega)$. But it looks like a difficult task to obtain estimates for the operator norm of B_Ω^n better than the obvious exponential upper bound $\|B_\Omega\|^n$. We overcome this difficulty by finding an expression of the form:

$$B_\Omega^n(f) = B^n(f)\chi_\Omega + K_n(f), \tag{6}$$

where K_n is compact on $\mathrm{Lip}(\varepsilon', \Omega)$. This is done in Theorem 1 in Section 5. Incidentally, in turns out that $K_n = 0$ when Ω is a disc, so that in this case $B_\Omega^n(f)$ is exactly $B^n(f)\chi_\Omega$ for each n.

Then (5) can be rewritten as

$$(I - \mu B_\Omega)P_{n-1} = P_{n-1}(I - \mu B_\Omega) = I - \mu^n B^n + S, \tag{7}$$

where S is compact on $\mathrm{Lip}(\varepsilon', \Omega)$.

The kernel of B^n may be computed explicitly, for instance via a Fourier transform argument [23, p. 73], and one obtains:

$$b_n(z) = \frac{(-1)^n n}{\pi} \frac{\bar{z}^{n-1}}{z^{n+1}}.$$

Thus the Calderón–Zygmund constant of b_n, namely,

$$\big\| b_n(z)|z|^2 \big\|_\infty + \big\| \nabla b_n(z)|z|^3 \big\|_\infty,$$

is less than Cn^2, where C is a positive constant. Hence, by the Main Lemma,

$$\big\| \mu^n B^n(f) \big\|_{\varepsilon', \Omega} \leqslant C n^3 \|\mu\|_\infty^n \|\mu\|_{\varepsilon', \Omega} \|f\|_{\varepsilon', \Omega},$$

which tells us that the operator norm of $\mu^n B^n$ as an operator on $\mathrm{Lip}(\varepsilon', \Omega)$ is small for large n. Therefore $I - \mu B_\Omega$ is a Fredholm operator on $\mathrm{Lip}(\varepsilon', \Omega)$.

Clearly $I - t\mu B_\Omega$, $0 \leqslant t \leqslant 1$ is a continuous path from the identity to $I - \mu B_\Omega$. By the index theory of Fredholm operators on Banach spaces (e.g. [22]), the index is a continuous function of the operator. Hence $I - \mu B_\Omega$ has index 0. On the other hand, $I - \mu B_\Omega$ is injective, because if $f = \mu B_\Omega(f)$, then $\|f\|_2 \leqslant \|\mu\|_\infty \|B_\Omega(f)\|_2 \leqslant \|\mu\|_\infty \|B(f)\|_2 = \|\mu\|_\infty \|f\|_2$, which is possible only if $f = 0$. Thus $I - \mu B_\Omega$ is invertible on $\mathrm{Lip}(\varepsilon', \Omega)$.

As we mentioned before, the proof of the Theorem simplifies if $\|\mu\|_{\varepsilon, \Omega}$ is assumed to be less than a small number $\delta_0 = \delta_0(\Omega)$. In this case one can invert $I - \mu B$ by a Neumann series and get $h = \sum_{n=0}^\infty (\mu B)^n(\mu)$. By the Main Lemma B_Ω is bounded on $\mathrm{Lip}(\varepsilon, \Omega)$. Denote by $\|B_\Omega\|$ its operator norm and assume that $\|\mu\|_{\varepsilon, \Omega} < (2\|B_\Omega\|)^{-1}$. Then

$$\|h\|_{\varepsilon, \Omega} \leqslant \sum_{n=0}^\infty \|\mu\|_{\varepsilon, \Omega}^{n+1} \|B_\Omega\|^n \leqslant 2\|\mu\|_{\varepsilon, \Omega} < \|B_\Omega\|^{-1}.$$

But is also part of the Main Lemma that

$$\big\| B(h) \big\|_{L^\infty(\mathbb{C})} \leqslant C(\Omega) \|h\|_{\varepsilon, \Omega}.$$

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Hence, if we also assume that $2\|\mu\|_{\varepsilon,\Omega}C(\Omega) < 1$, we have $\|B(h)\|_{L^\infty(\mathbb{C})} < 1$. Thus

$$\|\bar\partial\Phi\|_{L^\infty(\mathbb{C})} < \|B_\Omega\|^{-1} \quad \text{and} \quad \|\partial\Phi\|_{L^\infty(\mathbb{C})} \leqslant 1 + \big\|B(h)\big\|_{L^\infty(\mathbb{C})} \leqslant 2,$$

and so Φ is a Lipschitz function. That Φ is bilipschitz follows from,

$$\big|\partial\Phi(z)\big| = \big|1 + B(h)(z)\big| \geqslant 1 - \big\|B(h)\big\|_{L^\infty(\mathbb{C})} > 0, \quad z \in \mathbb{C}\setminus\partial\Omega.$$

3. The Main Lemma

In this section we move to $\mathbb{R}^n$. We say that a bounded domain $\Omega \subset \mathbb{R}^n$ has a boundary of class $C^{1+\varepsilon}$ if $\partial\Omega$ is a C^1 hyper-surface whose unit normal vector satisfies a Lipschitz condition of order ε as a function on the surface. To state an alternative condition, for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ we use the notation $x = (x', x_n)$, where $x' = (x_1, \dots, x_{n-1})$. Then Ω has a boundary of class $C^{1+\varepsilon}$ if for each point $a \in \partial\Omega$ one may find a ball $B(a, r)$ and a function $x_n = \varphi(x')$, of class $C^{1+\varepsilon}$, such that, after a rotation if necessary, $\Omega \cap B(a, r)$ is the part of $B(a, r)$ lying below the graph of φ. Thus we get:

$$\Omega \cap B(a, r) = \big\{x \in B(a, r)\colon\ x_n < \varphi(x_1, \dots, x_{n-1})\big\}. \tag{8}$$

A smooth (of class C^1) homogeneous Calderón–Zygmund operator is a principal value convolution operator of type

$$T(f)(x) = PV\int f(x-y)K(y)\,dy, \tag{9}$$

where

$$K(x) = \frac{\omega(x)}{|x|^n}, \quad x \neq 0,$$

$\omega(x)$ being a homogeneous function of degree 0, continuously differentiable on $\mathbb{R}^n \setminus \{0\}$ and with zero integral on the unit sphere. The maximal singular integral associated to T is:

$$T^\star f(x) = \sup_{\delta>0}\big|T^\delta f(x)\big|, \quad x \in \mathbb{R}^n,$$

where

$$T^\delta f(x) = \int_{|y-x|>\delta} f(x-y)K(y)\,dy.$$

The Calderón–Zygmund constant of the kernel of T is defined as

$$\|T\|_{CZ} = \big\|K(x)|x|^n\big\|_\infty + \big\|\nabla K(x)|x|^{n+1}\big\|_\infty.$$

The operator T is said to be even if the kernel is even, namely, if $\omega(-x) = \omega(x)$, for all $x \neq 0$.

We are now ready to state our main lemma. The definition of the norm in $\operatorname{Lip}(\varepsilon, \Omega)$ is as in (3). As we explained in the previous section, we need the precise form of the constant in the inequality below.

Main Lemma. *Let Ω be a bounded domain with boundary of class $C^{1+\varepsilon}$, $0 < \varepsilon < 1$, and let T be an even smooth homogeneous Calderón–Zygmund operator. Then T maps* $\operatorname{Lip}(\varepsilon, \Omega)$ *into* $\operatorname{Lip}(\varepsilon, \Omega)$*, and T also maps* $\operatorname{Lip}(\varepsilon, \Omega)$ *into* $\operatorname{Lip}(\varepsilon, \Omega^c)$*. In fact, one has the inequalities*:

$$\|Tf\|_{\varepsilon,\Omega} \leqslant C\|T\|_{CZ}\|f\|_{\varepsilon,\Omega},$$

and

$$\|Tf\|_{\varepsilon,\Omega^c} \leqslant C\|T\|_{CZ}\|f\|_{\varepsilon,\Omega},$$

where C is a constant depending only on n, ε and Ω.

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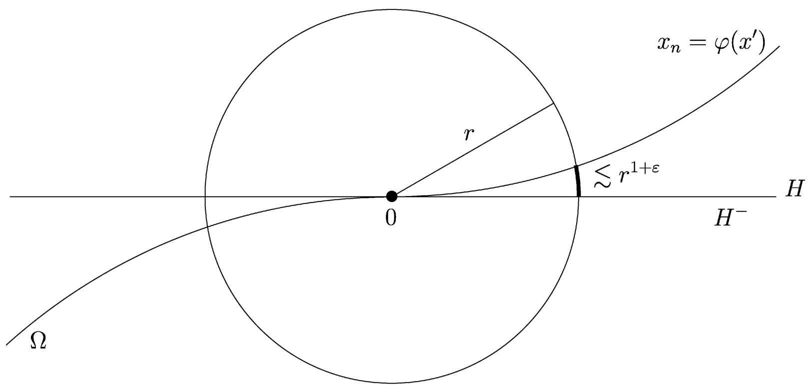

Fig. 1.

Proof. We choose a positive $r_0 = r_0(\Omega)$ small enough so that a series of properties that will be needed along the proof are satisfied. The first one is that for each $a \in \partial\Omega$, which we can assume to be $a = 0$, we have (8). After a rotation we may assume that the tangent hyperplane to $\partial\Omega$ at 0 is $x_n = 0$. We take r_0 so small that

$$\left|\varphi(x')\right| \leqslant C|x'|^{1+\varepsilon}, \quad |x'| < r_0, \tag{10}$$

for some positive constant C depending only on Ω. We claim that

$$T^* f(x) \leqslant C\|T\|_{CZ}\|f\|_\varepsilon, \quad x \in \mathbb{R}^n. \tag{11}$$

The proof of (11) is a technical variation of the proof of Lemma 5 in [19]. We have:

$$T^\delta(f)(x) = \int\limits_{\delta<|y-x|<r_0} f(y)K(x-y)\,dy + \int\limits_{r_0<|y-x|} \cdots = \mathrm{I}_\delta + \mathrm{II}.$$

Clearly,

$$|\mathrm{II}| \leqslant \int\limits_{r_0<|y-x|} \left|f(y)\right|\left|K(x-y)\right| dy \leqslant r_0^{-n}|\Omega|\|T\|_{CZ}\|f\|_\infty.$$

To deal with the term I_δ we write,

$$\begin{aligned}\mathrm{I}_\delta &= \int\limits_{\delta<|y-x|<r_0} \chi_\Omega(y)\big(f(y)-f(x)\big)K(x-y)\,dy + f(x)\int\limits_{\delta<|y-x|<r_0} \chi_\Omega(y)K(x-y)\,dy \\ &= \mathrm{III}_\delta + f(x)\mathrm{IV}_\delta,\end{aligned}$$

and we remark that III_δ can easily be estimated as follows:

$$\begin{aligned}|\mathrm{III}_\delta| &\leqslant \|f\|_\varepsilon \int\limits_\Omega |y-x|^\varepsilon\left|K(x-y)\right| dy \leqslant C\|f\|_\varepsilon\|T\|_{CZ}\int\limits_\Omega |y-x|^{-n+\varepsilon}\,dy \\ &\leqslant C(\varepsilon)(\operatorname{diam}\Omega)^\varepsilon\|f\|_\varepsilon\|T\|_{CZ}.\end{aligned}$$

Taking care of IV_δ is not so easy. Assume first that $x = 0$ is in $\partial\Omega$. Without loss of generality we may also assume that the tangent hyperplane to $\partial\Omega$ at 0 is $\{x_n = 0\}$ (see Fig. 1).

Let H_- be the half space $\{x_n < 0\}$. Take spherical coordinates $y = r\xi$ with $0 \leqslant r$ and $|\xi| = 1$. Then

$$\mathrm{IV}_\delta = \int_\delta^{r_0}\left(\int\limits_{A(r)} \omega(\xi)\,d\sigma(\xi)\right)\frac{dr}{r}, \tag{12}$$

where

$$A(r) = \left\{\xi\colon\ |\xi| = 1 \text{ and } r\xi \in \Omega\right\},$$

and σ is the surface measure on the unit sphere U. Since K is even,

$$0=\int_U \omega(\xi)\,d\sigma(\xi)=2\int_{U\cap H_-}\omega(\xi)\,d\sigma(\xi).$$

Thus

$$\int_{A(r)}\omega(\xi)\,d\sigma(\xi)=\int_{A(r)\setminus(U\cap H_-)}\omega(\xi)\,d\sigma(\xi)-\int_{(U\cap H_-)\setminus A(r)}\omega(\xi)\,d\sigma(\xi),$$

and so

$$\biggl|\int_{A(r)}\omega(\xi)\,d\sigma(\xi)\biggr|\leqslant C\|T\|_{CZ}\bigl(\sigma\bigl(A(r)\setminus(U\cap H_-)\bigr)+\sigma\bigl((U\cap H_-)\setminus A(r)\bigr)\bigr).$$

By (10), we obtain:

$$\sigma\bigl(A(r)\setminus(U\cap H_-)\bigr)+\sigma\bigl((U\cap H_-)\setminus A(r)\bigr)\leqslant Cr^{\varepsilon},$$

which yields, by (12),

$$|\mathrm{IV}_\delta|\leqslant C\|T\|_{CZ}.$$

Take now $x\in\mathbb{R}^n\setminus\partial\Omega$. Denote by δ_0 the distance from x to $\partial\Omega$ and let x_0 be a point in $\partial\Omega$ where such distance is attained. Set:

$$A=\bigl\{y\in\Omega:\ \delta_0<|y-x|<r_0\bigr\}$$

and

$$A_0=\bigl\{y\in\Omega:\ \delta_0<|y-x_0|<r_0\bigr\}.$$

We compare IV_δ to the expression we get replacing x by x_0 and δ by δ_0 in the definition of IV_δ. For $\delta\leqslant\delta_0$ we have, by the standard cancellation property of the kernel,

$$\int_{\delta<|y-x|<r_0}\chi_\Omega(y)K(x-y)\,dy=\int_{\delta_0<|y-x|<r_0}\chi_\Omega(y)\,K(x-y)\,dy,$$

and then

$$\begin{aligned}&\biggl|\int_{\delta<|y-x|<r_0}\chi_\Omega(y)K(x-y)\,dy-\int_{\delta_0<|y-x_0|<r_0}\chi_\Omega(y)K(x_0-y)\,dy\biggr|\\&\quad=\biggl|\int_A K(x-y)\,dy-\int_{A_0}K(x_0-y)\,dy\biggr|\\&\quad\leqslant\int_{A\cap A_0}\bigl|K(x-y)-K(x_0-y)\bigr|\,dy+\biggl|\int_{A\setminus A_0}\chi_\Omega(y)K(x-y)\,dy\biggr|+\biggl|\int_{A_0\setminus A}\chi_\Omega(y)K(x_0-y)\,dy\biggr|\\&\quad=J_1+J_2+J_3.\end{aligned}$$

If $y\in A\cap A_0$, then

$$\bigl|K(x-y)-K(x_0-y)\bigr|\leqslant C\|T\|_{CZ}\frac{|x-x_0|}{|y-x|^{n+1}}.$$

Hence

$$J_1\leqslant C\|T\|_{CZ}|x-x_0|\int_{|y-x|>\delta_0}\frac{dy}{|y-x|^{n+1}}\leqslant C\|T\|_{CZ}.$$

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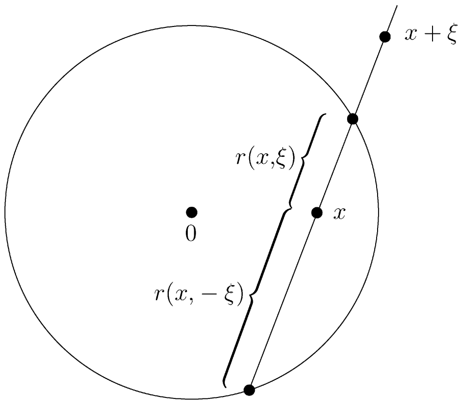

Fig. 2.

To estimate J_2 observe that

$$A \setminus A_0 = \big(A \cap B(x_0, \delta_0)\big) \cup \big(A \cap \big(\mathbb{R}^n \setminus B(x_0, r_0)\big)\big).$$

Assume for the moment that $\delta_0 \leqslant r_0/2$. Now, it is obvious that if $|y - x_0| \geqslant r_0$, then $|y - x| \geqslant r_0/2$, and so

$$J_2 \leqslant \|T\|_{CZ}\Bigg(\int\limits_{|y-x_0|<\delta_0} \frac{dy}{\delta_0^n} + \int\limits_{\Omega} \frac{2^n}{r_0^n}\, dy\Bigg) \leqslant C\|T\|_{CZ}.$$

A similar argument does the job for J_3.

If $\delta_0 \geqslant r_0/2$, then the estimate of $T^\delta(f)(x)$ is straightforward. Indeed, by the cancellation of the kernel we may assume that $\delta \geqslant \delta_0$ and so

$$\big|T^\delta(f)(x)\big| \leqslant \|f\|_\infty \|T\|_{CZ} \int\limits_{|y-x|>\delta} \chi_\Omega(y) \frac{1}{|y-x|^n}\, dy \leqslant \frac{2^n}{r_0^n} |\Omega| \|f\|_\infty \|T\|_{CZ}.$$

This completes the proof of (11). □

Our next task is to estimate the semi-norm $\sigma_\varepsilon(T(f))$ on Ω. For this we need a lemma, which should be viewed as a manifestation of the extra cancellation enjoyed by even kernels. Notice that no smoothness assumptions on the kernel are required. The lemma is known for the Beurling transform [12, p. 389].

Lemma 3. *Let $K(x) = \frac{\omega(x)}{|x|^n}$, where ω is an even homogeneous function of degree* 0*, integrable on the unit sphere and with vanishing integral there. Let T be the associated Calderón–Zygmund operator defined by* (9)*. Then*

$$T(\chi_B)\chi_B = 0, \quad \text{for each ball } B.$$

Proof. Assume, without loss of generality that B is the unit ball. Fix a point x in B. Then

$$T(\chi_B)(x) = \lim_{\varepsilon \to 0} \int\limits_{B \cap B^c(x,\varepsilon)} K(x-y)\, dy = \int\limits_{B \cap B^c(x, 1-|x|)} K(x-y)\, dy = \int\limits_{B^c \cap B(x, 1+|x|)} K(x-y)\, dy.$$

Expressing the latest integral above in polar coordinates $y = x + r\xi$ centered at x, we get:

$$T(\chi_B)(x) = \int\limits_{|\xi|=1} \int\limits_{r(x,\xi)}^{1+|x|} \frac{dr}{r} \omega(\xi)\, d\sigma(\xi) = \int\limits_{|\xi|=1} \log\bigg(\frac{1+|x|}{r(x,\xi)}\bigg) \omega(\xi)\, d\sigma(\xi).$$

The lower value $r(x, \xi)$ is determined as shown in Fig. 2.

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Since ω has zero integral on the unit sphere,

$$T(\chi_B)(x) = \int_{|\xi|=1} \log\left(\frac{1}{r(x,\xi)}\right)\omega(\xi)\,d\sigma(\xi).$$

Set U^+ be the half of the unit sphere above the hyperplane $\{x_n = 0\}$. Since ω is even,

$$T(\chi_B)(x) = \int_{U^+} \log\left(\frac{1}{r(x,\xi)r(x,-\xi)}\right)\omega(\xi)\,d\sigma(\xi).$$

Now, the points 0, x, $x+\xi$ and $x-\xi$ lie in a plane that intersects the unit sphere in a circumference. It is clear from Fig. 2 that the product $r(x,\xi)r(x,-\xi)$ is the power of x with respect to that circumference and thus it does not depend on ξ (in fact it is exactly $1-|x|^2$). Since the integral of ω on each semi-sphere is zero the proof is complete. □

We are now ready to estimate the semi-norm $\sigma_\varepsilon(T(f))$. We deal first with the case $f = 1$. The general case follows from this by the $T(1)$-Theorem for Lipschitz spaces on spaces of homogeneous type [26] (see also [10] and [9] for the non-doubling case). The conditions on the kernel required in [26] (and in [10,9]) are implied by the fact that $T^*(\chi_\Omega) \in L^\infty(\Omega)$, which we proved before. However, in our particular setting the reduction to $f = 1$ is elementary and will be discussed afterwards for the sake of completeness.

We want to prove that

$$\left|T(\chi_\Omega)(x) - T(\chi_\Omega)(y)\right| \leqslant C|x-y|^\varepsilon, \quad x, y \in \Omega. \tag{13}$$

Fix x and y in Ω. Changing notation if necessary, we can assume that $\operatorname{dist}(x,\partial\Omega) \leqslant \operatorname{dist}(y,\partial\Omega)$. We may also assume, without loss of generality, that $\operatorname{dist}(x,\partial\Omega) \leqslant r_0/4$. Otherwise we have:

$$\left|T(\chi_\Omega)(x) - T(\chi_\Omega)(y)\right| \leqslant C|x-y|\left\|\nabla T(\chi_\Omega)\right\|_{L^\infty(\Omega_0)},$$

where $\Omega_0 = \{z \in \Omega\colon \operatorname{dist}(x,\partial\Omega) \geqslant r_0/4\}$ and C depends only on Ω. Notice that $T(\chi_\Omega) \in C^1(\Omega)$, because

$$T(\chi_\Omega) = T(1-\chi_{\mathbb{C}\setminus\Omega}) = -T(\chi_{\mathbb{C}\setminus\Omega}),$$

and the kernel of T is continuously differentiable off the origin. Indeed, for some constant depending only on n, r_0 and Ω, we have:

$$\left\|\nabla T(\chi_{\mathbb{C}\setminus\Omega})\right\|_{L^\infty(\Omega_0)} \leqslant C\|T\|_{CZ}.$$

We may also assume, without loss of generality, that $|x-y| \leqslant r_0/4$, because, otherwise,

$$\left|T(\chi_\Omega)(x) - T(\chi_\Omega)(y)\right| \leqslant \frac{8}{r_0}\left\|T(\chi_\Omega)\right\|_\infty |x-y|.$$

Having settled these preliminaries we proceed to the core of the proof of (13). We may assume that the point of $\partial\Omega$ nearest to x is the origin. Let B be the ball with center $(0,\dots,0,-r_0)$ and radius r_0, so that ∂B is tangent to $\partial\Omega$ at 0. Let S stand for the set $(\Omega\setminus B)\cup(B\setminus\Omega)$. The central idea in the proof of the Main Lemma is to use the extra cancellation of even Calderón–Zygmund operators via Lemma 1 to write:

$$\left|T(\chi_\Omega)(x) - T(\chi_\Omega)(y)\right| = \left|T(\chi_S)(x) - T(\chi_S)(y)\right|.$$

The obvious advantage is that S is a region which is "tangential" to $\partial\Omega$ at 0, and hence small. By (10) we may take r_0 so small that for some constant C depending only on Ω,

$$\left|(z,\vec{n})\right| \leqslant C|z'|^{1+\varepsilon}, \quad z \in S \cap B(0,r_0), \tag{14}$$

where $\vec{n}$ stands for the inward unit normal vector to $\partial\Omega$ at 0 and $(\,,)$ denotes the scalar product in $\mathbb{R}^n$. Thus, if r_0 is small enough,

$$\left|(z,\vec{n})\right| < \frac{1}{\sqrt{2}}|z|, \quad z \in S \cap B(0,r_0).$$

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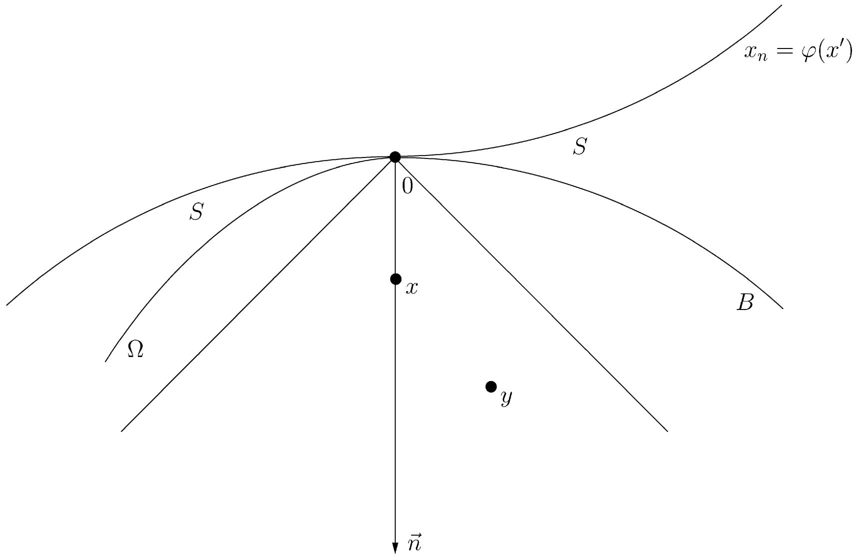

Fig. 3.

We distinguish two cases according to whether the position of x and y relative to $\partial\Omega$ is non-tangential or tangential. To make this precise we introduce the cone Γ with vertex 0 and amplitude $\pi/2$, namely,

$$\Gamma = \left\{ z \in \mathbb{C}\colon\ (z, \vec{n}) \geqslant \frac{1}{\sqrt{2}}|z| \right\}. \tag{15}$$

Clearly, if r_0 is chosen small enough, then the part of the cone near 0 is contained in Ω and in B. More precisely,

$$\Gamma \setminus \{0\} \cap B(0, r_0) \subset \Omega \cap B \cap B(0, r_0).$$

See Fig. 3.

Case 1: x and y are in non-tangential position, that is, x and y belong to Γ. We have:

$$\begin{aligned}\big|T(\chi_S)(x) - T(\chi_S)(y)\big| &\leqslant \left| \int\limits_{S\cap B^c(x,r_0)} K(x-z)\,dz - \int\limits_{S\cap B^c(y,r_0)} K(y-z)\,dz \right| \\ &\quad + \left| \int\limits_{S\cap B(x,r_0)} K(x-z)\,dz - \int\limits_{S\cap B(y,r_0)} K(y-z)\,dz \right| \\ &= \mathrm{I} + \mathrm{II}.\end{aligned}$$

Split I into three terms as follows:

$$\begin{aligned}\mathrm{I} &\leqslant \left| \int\limits_{S\cap B^c(x,r_0)\cap B(y,r_0)} K(x-z)\,dz \right| + \left| \int\limits_{S\cap B^c(y,r_0)\cap B(x,r_0)} K(y-z)\,dz \right| \\ &\quad + \left| \int\limits_{S\cap B^c(x,r_0)\cap B^c(y,r_0)} \big(K(x-z) - K(y-z)\big)\,dz \right| \\ &= \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3.\end{aligned}$$

The terms I_1 and I_2 are estimated in the same way. For instance, for I_1, we get:

$$|\mathrm{I}_1| \leqslant \int\limits_{B^c(x,r_0)\cap B(y,r_0)} \frac{C}{|x-z|^n}\,dz \leqslant \frac{C}{r_0^n}\big|B^c(x,r_0) \cap B(y,r_0)\big| \leqslant \frac{C}{r_0^n}|x-y| r_0^{n-1} = \frac{C}{r_0}|x-y|,$$

where in the latest inequality we used that $|x-y| \leqslant r_0$.

The term I_3 is controlled by a gradient estimate, namely,

$$\mathrm{I}_3 \leqslant \int_{B^c(x,r_0)} C\frac{|x-y|}{|x-z|^{n+1}}\,dz \leqslant \frac{C}{r_0}|x-y|.$$

The more difficult term II is not greater than

$$\begin{aligned}&\left|\int_{S\cap B(x,2|x-y|)} K(x-z)\,dz\right| + \left|\int_{S\cap B(y,2|x-y|)} K(y-z)\,dz\right| + \left|\int_{S\cap B(x,r_0)\cap B^c(x,2|x-y|)} K(x-z)\,dz\right.\\ &\quad\left.-\int_{S\cap B(y,r_0)\cap B^c(y,2\,|x-y|)} K(y-z)\,dz\right|\\ &= \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{III}.\end{aligned}$$

Estimating the three terms above requires a simple lemma.

Lemma 4. *If r_0 is small enough, then one has*:

$$|w-z| \geqslant C|z|, \quad w \in \Gamma \cap B(0,r_0),\ z \in S \cap B(0,r_0),$$

for $C = (2(1+\sqrt{2}))^{-1}$.

Proof. According to the definition of the cone Γ and by (10),

$$\begin{aligned}|z| &\leqslant |z-w| + |w| \leqslant |z-w| + \sqrt{2}(w,\vec{n}) \leqslant (1+\sqrt{2})|z-w| + \sqrt{2}\big|(z,\vec{n})\big|\\ &\leqslant (1+\sqrt{2})|z-w| + \sqrt{2}C|z|^{1+\varepsilon} \leqslant (1+\sqrt{2})|z-w| + \sqrt{2}Cr_0^{\varepsilon}|z|.\end{aligned}$$

If r_0 satisfies $\sqrt{2}\,Cr_0^{\varepsilon} \leqslant 1/2$, then $|z| \leqslant 2(1+\sqrt{2})|z-w|$, which proves the lemma. □

To estimate the term II_1 we apply Lemma 2 to $w = x$ to obtain,

$$\mathrm{II}_1 \leqslant \int_{S\cap B(x,2\,|x-y|)} \frac{C}{|x-z|^n}\,dz \leqslant C \int_{S\cap B(x,2\,|x-y|)} \frac{dz}{|z|^n}.$$

Changing to polar coordinates we get:

$$\mathrm{II}_1 \leqslant C \int_0^{2\,|x-y|} \sigma\big(\big\{\xi \in S^{n-1}:\ r\xi \in S\big\}\big)\frac{dr}{r}.$$

By (10)

$$\sigma\big(\big\{\xi \in S^{n-1}:\ r\xi \in S\big\}\big) \leqslant Cr^{\varepsilon} \tag{16}$$

and hence

$$\mathrm{II}_1 \leqslant C|x-y|^{\varepsilon}.$$

One estimates II_2 likewise, so we turn our attention to III. The method is similar to what we have done before with other terms: in the intersection of the domains of integration of the two integrals in III we apply a gradient estimate and in the complement, which we split in four terms, we resort to the smallness of the resulting domain of integration. Performing the plan just sketched we get:

$$\begin{aligned}\mathrm{III} &\leqslant \left|\int_{S\cap B(x,r_0)\cap B^c(x,2|x-y|)\cap B(y,r_0)\cap B^c(y,2\,|x-y|)} \big(K(x-z)-K(y-z)\big)\,dz\right|\\ &\quad + \left|\int_{S\cap B(x,r_0)\cap B^c(x,2|x-y|)\cap B^c(y,r_0)} K(x-z)\,dz\right| + \left|\int_{S\cap B(x,r_0)\cap B^c(x,2|x-y|)\cap B(y,2|x-y|)} K(x-z)\,dz\right|\end{aligned}$$

$$+\left|\int\limits_{S\cap B(y,r_0)\cap B^c(y,2|x-y|)\cap B^c(x,r_0)} K(y-z)\,dz\right| + \left|\int\limits_{S\cap B(y,r_0)\cap B^c(y,2|x-y|)\cap B(x,2|x-y|)} K(y-z)\,dz\right|$$
$$= \mathrm{III}_1 + \mathrm{III}_2 + \mathrm{III}_3 + \mathrm{III}_4 + \mathrm{III}_5.$$

By a gradient estimate III_1 is not greater than

$$\left|\int\limits_{S\cap B(x,r_0)\cap B^c(x,2|x-y|)} C\frac{|x-y|}{|x-z|^{n+1}}\,dz\right| \leqslant C|x-y|\int\limits_{2|x-y|}^{r_0} r^{-2+\varepsilon}\,dr = C|x-y|^{\varepsilon},$$

where (16) has been used in the first inequality.

The terms III_2 and III_4 are estimated in the same way. For instance, for III_2 we have:

$$\mathrm{III}_2 \leqslant C\int\limits_{B(x,r_0)\cap B^c(y,r_0)} \frac{1}{|x-z|^n}\,dz \leqslant \frac{C}{r_0^n}\left|B(x,r_0)\cap B^c(y,r_0)\right| \leqslant \frac{C}{r_0}|x-y|.$$

The terms III_3 and III_5 are also estimated in the same way. For instance, for III_3 we have:

$$\mathrm{III}_3 \leqslant \int\limits_{S\cap B^c(x,2|x-y|)\cap B(x,3|x-y|)} \frac{C}{|x-z|^n}\,dz.$$

Since $x\in\Gamma$, for $z\in S\cap B(x,3|x-y|)$, we get by Lemma 2

$$|z| \leqslant 6(1+\sqrt{2})|x-y| \leqslant 18|x-y|,$$

and so, making use of (16),

$$\mathrm{III}_3 \leqslant C\int\limits_{S\cap B(0,18|x-y|)} \frac{dz}{|z|^n} \leqslant C|x-y|^{\varepsilon}.$$

Case 2: x and y are in tangential position, that is, $y\in\Omega\setminus\Gamma$. We intend to perform a reduction to the non-tangential case. With this in mind take the point p in $\partial\Omega$ nearest to y and let $\vec{N}$ be the inner unit normal vector to $\partial\Omega$ at the point p. Consider the ray $y+t\vec{N}$, $t>0$. See Fig. 4.

The condition on t for $y+t\vec{N}\in\Gamma$ is

$$(y+t\vec{N},\vec{n}) \geqslant \frac{1}{\sqrt{2}}|y+t\vec{N}|. \tag{17}$$

We clearly have:

$$(y+t\vec{N},\vec{n}) \geqslant t(\vec{N},\vec{n}) - |y|,$$

and

$$|y+t\vec{N}| \leqslant |y| + t.$$

A sufficient condition for (17) is then

$$t \geqslant \frac{(1+\frac{1}{\sqrt{2}})|y|}{(\vec{N},\vec{n})-\frac{1}{\sqrt{2}}}.$$

If r_0 is small enough, then $(\vec{N},\vec{n}) \geqslant \frac{3}{4}\frac{1}{\sqrt{2}}$, and thus we obtain a simpler sufficient condition for (17) namely,

$$t \geqslant 3(1+\sqrt{2})|y| \equiv t_0.$$

Set $y_0 = y + t_0\vec{N}$, so that $y_0\in\Gamma$. The reduction will be completed if we show that

$$|y-y_0| \leqslant C|x-y|, \tag{18}$$

because x and y_0 on one hand, and y and y_0 on the other, are in non-tangential position.

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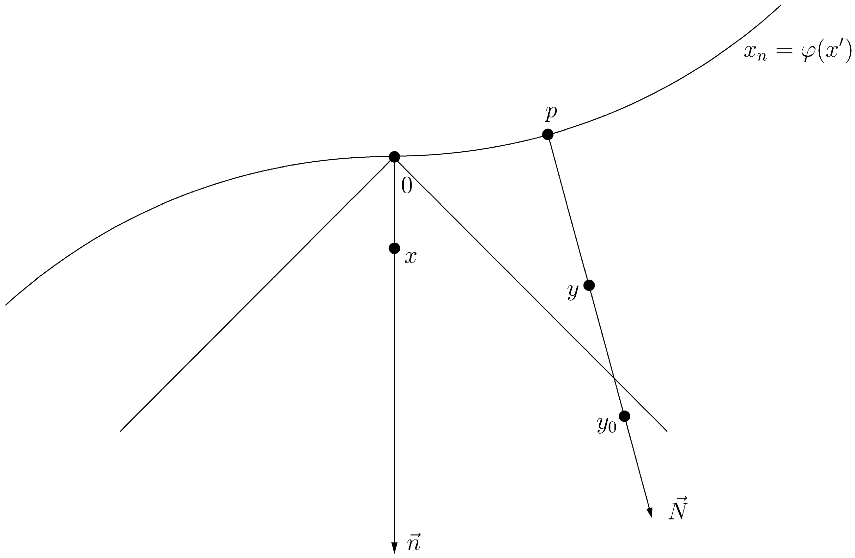

Fig. 4.

Clearly $|y - y_0| = t_0 = C|y|$. Since $y \in \Omega \setminus \Gamma$, $|(y, \vec{n})| < \frac{1}{\sqrt{2}}|y|$, and hence

$$|y| \leqslant \left|(y, \vec{n})\right| + |y'| \leqslant \frac{1}{\sqrt{2}}|y| + |y'|,$$

which yields $(1 - \frac{1}{\sqrt{2}})|y| \leqslant |y'|$. Therefore

$$|x - y| = \left||x|\vec{n} - (y, \vec{n})\vec{n} - y'\right| \geqslant |y'| \geqslant \left(1 - \frac{1}{\sqrt{2}}\right)|y| = C|y - y_0|,$$

which is (18).

This completes the proof that $T(\chi_\Omega) \in \mathrm{Lip}(\varepsilon, \Omega)$.

We now proceed to prove that for an arbitrary $f \in \mathrm{Lip}(\varepsilon, \Omega)$ one has that $T(f) \in \mathrm{Lip}(\varepsilon, \Omega)$. Recall that we already know that $T(f) \in L^\infty(\Omega)$ (see (11)). To estimate the semi-norm $\sigma_\varepsilon(T(f))$ we start with the obvious decomposition of $T(f)$, namely,

$$T(f)(x) = \int_\Omega \big(f(y) - f(x)\big) K(x - y)\, dy + f(x)T(\chi_\Omega)(x),$$

so that only the first term, which we denote by $S(f)(x)$, is still a problem. Let A stand for $\Omega \cap B(x_1, 2|x_1 - x_2|)$ and set $B = \Omega \setminus A$. Then

$$\begin{aligned} S(f)(x_1) - S(f)(x_2) &= \int_A \big[\big(f(y) - f(x_1)\big)K(x_1 - y) - \big(f(y) - f(x_2)\big)K(x_2 - y)\big]\, dy \\ &\quad + \int_B \big[\big(f(y) - f(x_1)\big)K(x_1 - y) - \big(f(y) - f(x_2)\big)K(x_2 - y)\big]\, dy \\ &= \mathrm{I} + \mathrm{II}. \end{aligned}$$

Set $A' = \Omega \cap B(x_2, 3|x_1 - x_2|)$. Clearly $A \subset A'$. The term I is easy to estimate as indicated below,

$$\begin{aligned} |\mathrm{I}| &\leqslant C\left(\int_A \frac{|f(y) - f(x_1)|}{|y - x_1|^n}\, dy + \int_{A'} \frac{|f(y) - f(x_2)|}{|y - x_2|^n}\, dy\right) \leqslant C\|f\|_\varepsilon \int_0^{3|x_1 - x_2|} r^{-1+\varepsilon}\, dr \\ &= C\|f\|_\varepsilon |x_1 - x_2|^\varepsilon. \end{aligned}$$

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For the term II we have:

$$\begin{aligned}\mathrm{II} &= \int_B \big(f(y)-f(x_2)\big)\big(K(x_1-y)-K(x_2-y)\big)\,dy + \big(f(x_2)-f(x_1)\big)\int_B K(x_1-y)\,dy\\ &= \mathrm{III}+\mathrm{IV}.\end{aligned}$$

On one hand, by a gradient estimate we get:

$$\begin{aligned}|\mathrm{III}| &\leqslant \|f\|_\varepsilon |x_1-x_2| \int_B \frac{1}{|y-x_1|^{n+1-\varepsilon}}\,dy = \|f\|_\varepsilon |x_1-x_2| \int_{2|x_1-x_2|}^{\infty} r^{-2+\varepsilon}\,dr\\ &= C\|f\|_\varepsilon |x_1-x_2|^\varepsilon,\end{aligned}$$

and on the other hand, we clearly have by (11),

$$|\mathrm{IV}| \leqslant \|f\|_\varepsilon |x_1-x_2|^\varepsilon T^*(\chi_\Omega)(x_1) \leqslant C\|f\|_\varepsilon |x_1-x_2|^\varepsilon.$$

Finally, one can check without pain that the arguments above may be adapted to yield the boundedness of T as a map from $\mathrm{Lip}(\varepsilon,\Omega)$ into $\mathrm{Lip}(\varepsilon,\Omega^c)$.

4. Estimates for commutators

In this section we consider the commutator between the smooth homogeneous even Calderón–Zygmund operator T (see (9)) and the multiplication operator by a function $a \in \mathrm{Lip}(\alpha,\Omega)$, $0<\alpha<1$,

$$[T,a](f)(x) = \int_\Omega \big(a(x)-a(y)\big)K(x-y)f(y)\,dy, \quad x\in\Omega, \tag{19}$$

where $K(x)$ is the kernel of T and $f\in \mathrm{Lip}(\beta,\Omega)$, $0<\beta<1$. As in the previous section, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary of class $C^{1+\varepsilon}$, $0<\varepsilon<1$.

Lemma 5. *For $0<\alpha<1$ and $0<\beta\leqslant\varepsilon$ we have the estimate*:

$$\big\|[T,a](f)\big\|_\alpha \leqslant C\sigma_\alpha(a)\|f\|_\beta, \quad f\in\mathrm{Lip}(\beta,\Omega), \tag{20}$$

where C is a constant depending only on n, Ω, ε, α and β.

Recall that for $0<\alpha<1$,

$$\|g\|_\alpha = \|g\|_\infty + \sigma_\alpha(g),$$

where $\|g\|_\infty$ is the supremum norm of g on Ω, and

$$\sigma_\alpha(g) = \sup\left\{\frac{|g(x)-g(y)|}{|x-y|^\alpha}\colon x,y\in\Omega,\ x\neq y\right\}.$$

A consequence of the preceding lemma is that if $\beta\leqslant\varepsilon$ and $\beta<\alpha$ then the commutator $[T,a]$ is compact as an operator from $\mathrm{Lip}(\beta,\Omega)$ into itself. This follows from the fact that each ball of $\mathrm{Lip}(\alpha,\Omega)$ is relatively compact in $\mathrm{Lip}(\beta,\Omega)$ [13, Corollary 3.3, p. 154]. The lemma is applied to the Beurling transform and the function $a=\mu$. Then $\alpha=\varepsilon$ and $\beta=\varepsilon'$, where ε' is any number with $0<\varepsilon'<\varepsilon$.

Proof of Lemma 5. We first estimate $\|[T,a](f)\|_\infty$. For each $x\in\Omega$,

$$\big|[T,a](f)(x)\big| \leqslant C\sigma_\alpha(a)\|f\|_\infty \int_\Omega |x-y|^{-n+\alpha}\,dy \leqslant C\sigma_\alpha(a)\|f\|_\infty \int_0^d r^{-1+\alpha}\,dr = C\sigma_\alpha(a)\|f\|_\infty d^\alpha,$$

where d is the diameter of Ω.

We turn now to the more difficult task of estimating $\sigma_\alpha([T,a](f))$. Fix x_1 and x_2 in Ω. Then

$$\begin{aligned}\bigl|[T,a](f)(x_1)-[T,a](f)(x_2)\bigr| &\leqslant \bigl|a(x_1)-a(x_2)\bigr|\biggl|\int_\Omega K(x_1-y)f(y)\,dy\biggr| \\ &\quad+\biggl|\int_\Omega \bigl(a(x_2)-a(y)\bigr)\bigl(K(x_1-y)-K(x_2-y)\bigr)f(y)\,dy\biggr| \\ &= \mathrm{I}+\mathrm{II},\end{aligned}$$

and clearly, by the Main Lemma,

$$\mathrm{I}\leqslant C|x_1-x_2|^\alpha\sigma_\alpha(a)\bigl\|T(f)\bigr\|_\infty\leqslant C|x_1-x_2|^\alpha\sigma_\alpha(a)\|f\|_\beta.$$

To estimate II we introduce the sets:

$$A=\bigl\{y\in\Omega:\ |y-x_1|>2|x_1-x_2|\bigr\}$$

and

$$B=\bigl\{y\in\Omega:\ |y-x_1|\leqslant 2|x_1-x_2|\bigr\}.$$

Notice that $|y-x_2|>|x_1-x_2|$, $y\in A$ and $|y-x_2|\leqslant 3|x_1-x_2|$, $y\in B$. Let II_A (respectively II_B) denote the absolute value of the integral in II with domain of integration restricted to A (respectively to B).

By a gradient estimate:

$$\begin{aligned}\mathrm{II}_A &\leqslant \int_A \bigl|a(x_2)-a(y)\bigr|\frac{|x_1-x_2|}{|x_2-y|^{n+1}}\bigl|f(y)\bigr|\,dy\leqslant C|x_1-x_2|\sigma_\alpha(a)\|f\|_\infty\int_A |x_2-y|^{-(n+1)+\alpha}\,dy \\ &\leqslant C|x_1-x_2|\sigma_\alpha(a)\|f\|_\infty\int_{|x_1-x_2|}^{\infty} r^{-2+\alpha}\,dr\leqslant C\sigma_\alpha(a)\|f\|_\infty|x_1-x_2|^\alpha.\end{aligned}$$

For the term II_B we have:

$$\mathrm{II}_B\leqslant\biggl|\int_B\bigl(a(x_2)-a(y)\bigr)K(x_1-y)f(y)\,dy\biggr|+\biggl|\int_B\bigl(a(x_2)-a(y)\bigr)K(x_2-y)f(y)\,dy\biggr|=\mathrm{III}+\mathrm{IV},$$

and IV can be estimated directly as follows,

$$\mathrm{IV}\leqslant\sigma_\alpha(a)\|f\|_\infty\int_B\frac{|x_2-y|^\alpha}{|x_2-y|^n}\,dy\leqslant\sigma_\alpha(a)\|f\|_\infty\int_0^{3|x_1-x_2|} r^{-1+\alpha}\,dr=C\sigma_\alpha(a)\|f\|_\infty|x_1-x_2|^\alpha.$$

The term III needs an additional manoeuvre, which consists is bringing back $a(x_1)$:

$$\mathrm{III}\leqslant\biggl|\int_B\bigl(a(x_1)-a(y)\bigr)K(x_1-y)f(y)\,dy\biggr|+\bigl|a(x_2)-a(x_1)\bigr|\biggl|\int_B K(x_1-y)f(y)\,dy\biggr|=\mathrm{IV}'+\mathrm{V},$$

and IV' can be treated as IV. Now

$$\int_B K(x_1-y)f(y)\,dy=\int_\Omega K(x_1-y)f(y)\,dy-\int_{\Omega\cap B^c(x_1,2|x_1-x_2|)} K(x_1-y)f(y)\,dy$$

and thus, by (11),

$$\biggl|\int_B K(x_1-y)f(y)\,dy\biggr|\leqslant 2T^*(f)(x_1)\leqslant C\|f\|_\beta.$$

Therefore

$$V\leqslant C\sigma_\alpha(a)\|f\|_\beta|x_1-x_2|^\alpha.\qquad\square$$

5. Relationship between B^n_Ω and B^n

Recall that if B is the Beurling transform then $B_\Omega(f) := B(f)\chi_\Omega$. The main goal of this section is to prove the following result.

Theorem 1. *Let $\Omega \subset \mathbb{C}$ be a bounded domain with boundary of class $C^{1+\varepsilon}$, $0 < \varepsilon < 1$. Then, for each positive integer n, we have*:

$$B^n_\Omega(f)(z) = B^n(f)(z)\chi_\Omega(z) + K_n(f)(z),$$

where K_n is a compact operator from $\mathrm{Lip}(\varepsilon', \Omega)$ *into itself,* $0 < \varepsilon' < \varepsilon$.

Proof. For $n \geqslant 2$, we obtain, proceeding by induction,

$$\begin{aligned} B^n_\Omega(f) &= B\big(B^{n-1}_\Omega(f)\big)\chi_\Omega = B\big(B^{n-1}(f)\chi_\Omega + K_{n-1}(f)\big)\chi_\Omega \\ &= B\big(B^{n-1}(f) - B^{n-1}(f)\chi_{\Omega^c} + K_{n-1}(f)\big)\chi_\Omega = B^n(f)\chi_\Omega - B\big(B^{n-1}(f)\chi_{\Omega^c}\big)\chi_\Omega + B\big(K_{n-1}(f)\big)\chi_\Omega. \end{aligned}$$

It is then enough to prove that, for $n \geqslant 1$, the operator,

$$B\big(B^n(f)\chi_{\Omega^c}\big)\chi_\Omega,$$

is compact from $\mathrm{Lip}(\varepsilon', \Omega)$ into itself.

Let dA stand for area measure in the plane and take a function $f \in L^\infty(\Omega)$. Then, for $z \in \Omega$,

$$\begin{aligned} B\big(B^n(f)\chi_{\Omega^c}\big)(z) &= -\frac{1}{\pi}\int_{\Omega^c} \frac{B^n(f)(w)}{(z-w)^2}\, dA(w) \\ &= -\frac{1}{\pi}\int_{\Omega^c} \frac{1}{(z-w)^2}\frac{(-1)^n n}{\pi}\int_{\Omega} \frac{(\overline{w-\zeta})^{n-1}}{(w-\zeta)^{n+1}} f(\zeta)\, dA(\zeta)\, dA(w) \\ &= c_n \int_\Omega K(z,\zeta) f(\zeta)\, dA(\zeta), \end{aligned}$$

where

$$K(z,\zeta) = K_n(z,\zeta) := \int_{\Omega^c} \frac{1}{(z-w)^2}\frac{n(\overline{w-\zeta})^{n-1}}{(w-\zeta)^{n+1}}\, dA(w)$$

and $c_n = \frac{(-1)^{n+1}}{\pi^2}$.

Notice that if Ω is a disc, say the unit disc, then $K(z,\zeta) = 0$, $z, \zeta \in \Omega$. To see this readily, apply Green–Stokes' Theorem to the complement of the unit disc to obtain:

$$K(z,\zeta) = \frac{-1}{2\imath}\int_{\partial\Omega} \frac{1}{(z-w)^2}\frac{(\overline{w-\zeta})^n}{(w-\zeta)^{n+1}}\, dw.$$

Expand $(\overline{w-\zeta})^n$ by Newton's formula and then use $\overline{w} = \frac{1}{w}$, $|w| = 1$. Thus $K(z,\zeta)$ is a finite sum of integrals over the unit cercle of rational functions with all poles in the open unit disc. Hence each of these integrals is zero.

We claim that if Ω is not a disc, then the operator,

$$P(f)(z) = \int_\Omega K(z,\zeta) f(\zeta)\, dA(\zeta), \quad z \in \Omega,$$

which may be non-zero, is a smoothing operator. By this we mean that

$$\big\|P(f)\big\|_\alpha \leqslant C\|f\|_\infty, \quad 0 < \alpha < \varepsilon, \tag{21}$$

where C depends only on α, ε and Ω.

Of course (21) completes the proof of Theorem 1, because then P maps the unit ball of $\mathrm{Lip}(\varepsilon',\Omega)$ into a ball of $\mathrm{Lip}(\alpha,\Omega)$, for $\alpha<\varepsilon$, which is relatively compact in $\mathrm{Lip}(\varepsilon',\Omega)$ provided $\varepsilon'<\alpha$.

Our next goal is to show that (21) is a consequence of the properties of the kernel $K(z,\zeta)$ described in the following lemma.

Lemma 6. *The kernel $K(z,\zeta)$ satisfies the following*:

(i) $$\bigl|K(z,\zeta)\bigr|\leqslant C\frac{1}{|z-\zeta|^{2-\varepsilon}},\quad z,\zeta\in\Omega;$$

(ii) $$\bigl|K(z_1,\zeta)-K(z_2,\zeta)\bigr|\leqslant C\frac{|z_1-z_2|^{\varepsilon}}{|\zeta-z_1|^2},\quad z_1,z_2\in\Omega,\ |\zeta-z_1|\geqslant 2\,|z_1-z_2|.$$

Before discussing the proof of Lemma 6 we show how it yields (21).

We first prove that $P(f)$ is bounded on Ω. Denoting by d the diameter of Ω, we obtain, by Lemma 6(i),

$$\bigl|P(f)(z)\bigr|\leqslant\int_\Omega\bigl|K(z,\zeta)\bigr|\bigl|f(\zeta)\bigr|\,dA(\zeta)\leqslant C\|f\|_\infty\int_\Omega\frac{dA(\zeta)}{|z-\zeta|^{2-\varepsilon}}\leqslant C\|f\|_\infty\int_0^d r^{-1+\varepsilon}\,dr=Cd^{\varepsilon}\|f\|_\infty.$$

Next we claim that

$$\bigl|P(f)(z_1)-P(f)(z_2)\bigr|\leqslant C|z_1-z_2|^{\varepsilon}\left(1+\log\frac{d}{|z_1-z_2|}\right)\|f\|_\infty,\quad z_1,z_2\in\Omega. \tag{22}$$

Clearly (21) follows from (22). To prove (22) take $z_1,z_2\in\Omega$. Define $A=\{\zeta\in\Omega:\ |z_1-\zeta|<2|z_1-z_2|\}$ and $B=\Omega\setminus A$. Therefore

$$\begin{aligned}\bigl|P(f)(z_1)-P(f)(z_2)\bigr|&\leqslant\int_A\bigl|K(z_1,\zeta)\bigr|\bigl|f(\zeta)\bigr|\,dA(\zeta)+\int_A\bigl|K(z_2,\zeta)\bigr|\bigl|f(\zeta)\bigr|\,dA(\zeta)\\&\quad+\int_B\bigl|K(z_1,\zeta)-K(z_2,\zeta)\bigr|\bigl|f(\zeta)\bigr|\,dA(\zeta)\\&=\mathrm{I}+\mathrm{II}+\mathrm{III}.\end{aligned}$$

Applying Lemma 6(i), the terms I and II can be estimated by:

$$C\|f\|_\infty\int_0^{3|z_1-z_2|}r^{-1+\varepsilon}\,dr\leqslant C|z_1-z_2|^{\varepsilon}\|f\|_\infty.$$

Applying Lemma 6(ii), the term III can be estimated by:

$$\begin{aligned}\mathrm{III}&\leqslant C\|f\|_\infty|z_1-z_2|^{\varepsilon}\left(\int_B\frac{dA(\zeta)}{|\zeta-z_1|^2}\right)\leqslant C\|f\|_\infty|z_1-z_2|^{\varepsilon}\int_{2|z_1-z_2|}^{d}\frac{dr}{r}\\&=C\|f\|_\infty|z_1-z_2|^{\varepsilon}\log\frac{d}{2|z_1-z_2|},\end{aligned}$$

which completes the proof of Theorem 1. □

Proof of Lemma 6. For each $\zeta\in\Omega$ consider the Cauchy integral of the function $(\overline{w-\zeta})^n$ on $\partial\Omega$, that is,

$$H_\zeta(w)=\frac{1}{2\pi i}\int_{\partial\Omega}\frac{(\overline{t-\zeta})^n}{t-w}\,dt,\quad w\in\mathbb{C}\setminus\partial\Omega.$$

For $w \in \partial\Omega$ let $H_\zeta(w)$ be the non-tangential limit of H_ζ from Ω, that is, the limit of $H_\zeta(w')$ as $w' \in \Omega$ tends to w non-tangentially. Similarly, denote by $H_\zeta^c(w)$ the non-tangential limit of H_ζ from $\mathbb{C} \setminus \Omega$. These limits exist a.e. on $\partial\Omega$ with respect to arc-length and one has the Plemelj formula (e.g., [25, p. 143]):

$$(\overline{w-\zeta})^n = H_\zeta(w) - H_\zeta^c(w), \quad w \text{ a.e. on } \partial\Omega.$$

Indeed, it can be shown that H_ζ is of class $C^{1+\varepsilon}$ in Ω and in $\mathbb{C} \setminus \overline{\Omega}$, so that the above limits exist everywhere on $\partial\Omega$ and without the non-tangential approach restriction. We do not need, however, such fact.

Applying the Green–Stokes Theorem to the form,

$$\frac{(\overline{w-\zeta})^n + H_\zeta^c(w)}{(z-w)^2(w-\zeta)^{n+1}}\, dw,$$

and the domain Ω^c, we get:

$$K(z,\zeta) = -\int_{\partial\Omega} \frac{H_\zeta(w)}{(z-w)^2(w-\zeta)^{n+1}}\, dw,$$

which by the Residue Theorem is

$$-2\pi i \left\{ \frac{d}{dw} \frac{H_\zeta(w)}{(w-\zeta)^{n+1}} \bigg|_{w=z} + \frac{1}{n!} \frac{d^n}{dw^n} \frac{H_\zeta(w)}{(w-z)^2} \bigg|_{w=\zeta} \right\}.$$

A straightforward computation of the residues yields:

$$\begin{aligned} K(z,\zeta) = -2\pi i \Bigg\{ & \frac{H_\zeta'(z)}{(z-\zeta)^{n+1}} - (n+1) \frac{H_\zeta(z)}{(z-\zeta)^{n+2}} \\ & + \sum_{\ell=0}^{n} (-1)^{n-\ell} \frac{(n-\ell+1)}{\ell!} \frac{d^\ell}{d\zeta^\ell} H_\zeta(\zeta) \frac{1}{(\zeta-z)^{n+2-\ell}} \Bigg\}. \end{aligned} \tag{23}$$

In the expression above for the kernel $K(z,\zeta)$ one may divine the presence of non-obvious cancellation properties (consider the case $n=1$). The strategy to unravel them is to bring into the scene the function:

$$h(z) = 2\pi i H_z(z) = \int_{\partial\Omega} \frac{(\overline{t-z})^n}{t-z}\, dt,$$

and express $K(z,\zeta)$ in terms of h and its derivatives. Taylor's expansions of h and its derivatives will then help in understanding cancellations. The derivatives of h are given by:

$$\frac{\partial^\ell}{\partial z^\ell} \frac{\partial^k}{\partial \bar{z}^k} h(z) = (-1)^k \frac{\ell! n!}{(n-k)!} \int_{\partial\Omega} \frac{(\overline{t-z})^{n-k}}{(t-z)^{1+\ell}}\, dt. \tag{24}$$

On the other hand, by the binomial formula,

$$2\pi i H_\zeta(z) = \int_{\partial\Omega} \frac{(\overline{t-\zeta})^n}{t-z}\, dt = \sum_{\ell=0}^{n} \binom{n}{\ell} (\overline{z-\zeta})^\ell \int_{\partial\Omega} \frac{(\overline{t-z})^{n-\ell}}{t-z}\, dt = \sum_{\ell=0}^{n} \frac{(-1)^\ell}{\ell!} \frac{\partial^\ell h}{\partial \bar{z}^\ell}(z) (\overline{z-\zeta})^\ell.$$

Differentiating the preceding identity with respect to z,

$$2\pi i\, H_\zeta'(z) = \sum_{\ell=0}^{n} \frac{(-1)^\ell}{\ell!} \frac{\partial^{\ell+1}}{\partial \bar{z}^\ell \partial z} h(z) (\overline{z-\zeta})^\ell. \tag{25}$$

Therefore

$$\begin{aligned} -(\zeta-z)^{n+2} K(z,\zeta) = & \sum_{\ell=0}^{n} \frac{(-1)^\ell}{\ell!} \frac{\partial^{\ell+1}}{\partial \bar{z}^\ell \partial z} h(z) (\overline{z-\zeta})^\ell (z-\zeta) - (n+1) \sum_{\ell=0}^{n} \frac{(-1)^\ell}{\ell!} \frac{\partial^\ell h}{\partial \bar{z}^\ell}(z) (\overline{z-\zeta})^\ell \\ & + \sum_{\ell=0}^{n} \frac{(n+1-\ell)}{\ell!} \frac{\partial^\ell h(\zeta)}{\partial \zeta^\ell} (z-\zeta)^\ell. \end{aligned} \tag{26}$$

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In each of the terms of the last sum it will be convenient to write a Taylor expansion of $\frac{\partial^\ell h(\zeta)}{\partial\zeta^\ell}$ up to order $n-\ell$ around the point z. Doing so we obtain:

$$\begin{aligned}(z-\zeta)^{n+2}K(z,\zeta) &= \sum_{\ell=0}^{n}\frac{1}{\ell!}\frac{\partial^\ell}{\partial\bar z^\ell}\frac{\partial}{\partial z}h(z)(\overline{\zeta-z})^\ell(\zeta-z)+(n+1)\sum_{\ell=0}^{n}\frac{1}{\ell!}\frac{\partial^\ell h}{\partial\bar z^\ell}(z)(\overline{\zeta-z})^\ell\\ &\quad-\sum_{\ell=0}^{n}\frac{n+1-\ell}{\ell!}\sum_{j=0}^{n-\ell}\sum_{k=0}^{j}\frac{(-1)^\ell}{k!(j-k)!}\frac{\partial^{\ell+k}}{\partial z^{\ell+k}}\frac{\partial^{j-k}}{\partial\bar z^{j-k}}h(z)(\zeta-z)^{k+\ell}(\overline{\zeta-z})^{j-k}+R(z,\zeta)\\ &\equiv S(z,\zeta)+R(z,\zeta).\end{aligned}$$

A cumbersome but easy computation shows now that

$$S(z,\zeta)=0,\quad z,\zeta\in\Omega.$$

The most direct way to ascertain this is to check that the coefficient of $S(z,\zeta)$ in the monomial $(\zeta-z)^{m_0}(\overline{\zeta-z})^{p_0}$ vanishes for all non-negative exponents m_0 and p_0. For this we distinguish four cases.

Case 1: Assume that $m_0\geqslant 2$. Only in the third sum may appear terms of this type and they must cancel out by themselves. This can be shown using the identities:

$$\sum_{\ell=0}^{m_0}\binom{m_0}{\ell}(-1)^\ell=\sum_{\ell=0}^{m_0}\ell\binom{m_0}{\ell}(-1)^\ell=0.$$

Case 2: Take $m_0=1$ and $0\leqslant p_0\leqslant n-1$. Two terms appear in the third sum and one in the first, and they cancel.

Case 3: Take $m_0=1$ and $p_0=n$. There is only one term of this type, which corresponds to letting $l=n$ in the first sum. To show that this term vanishes we resort to (24) for $l=1$ and $k=n$ and then we apply Cauchy's Theorem.

Case 4: Take $m_0=0$ and $0\leqslant p_0\leqslant n$. One term in the second sum cancels with a term in the third sum.

We turn now to the analysis of the kernel $K(z,\zeta)$. Since $S(z,\zeta)$ vanishes identically we get:

$$-(\zeta-z)^{n+2}K(z,\zeta)=\sum_{\ell=0}^{n}\frac{(n+1-\ell)}{\ell!}R_{n-l}(z,\zeta)\,(z-\zeta)^\ell, \tag{27}$$

where $R_{n-l}(z,\zeta)$ is the remainder of the Taylor expansion of $\frac{\partial^\ell h(\zeta)}{\partial\zeta^\ell}$ up to order $n-\ell$ around the point z.

A key fact in the present proof is that the remainder $R_{n-l}(z,\zeta)$ is $O(|z-\zeta|^{n-l+\varepsilon})$, because the nth order derivatives of $h(z)$ are in $\mathrm{Lip}(\varepsilon,\Omega)$. To show this we resort to (24) to get:

$$\frac{\partial^k}{\partial z^k}\frac{\partial^{n-k}}{\partial\bar z^{n-k}}h(z)=(-1)^{n-k}n!\int_{\partial\Omega}\frac{(\overline{t-z})^k}{(t-z)^{1+k}}\,dt.$$

If $k=0$, then the above expression is

$$(-1)^n n!\int_{\partial\Omega}\frac{dt}{(t-z)}=(-1)^n n!2\pi i.$$

If $1\leqslant k\leqslant n$, then we obtain, by Green–Stokes and for some constant $c_{n,k}$,

$$(-1)^{n-k}n!k2i\int_{\Omega}\frac{(\overline{t-z})^{k-1}}{(t-z)^{k+1}}\,dA(t)=c_{n,k}B^k(\chi_\Omega)(z),$$

which is in $\mathrm{Lip}(\varepsilon,\Omega)$ owing to the Main Lemma. Here B^k is the kth iteration of the Beurling transform.

Part (i) of the lemma is a straightforward consequence of (27) and the size estimate on the remainder $R_{n-l}(z,\zeta)$ we have just proved.

We are left with part (ii). Take points z_1, z_2 and ζ in Ω with $|\zeta-z_1|\geqslant 2\,|z_1-z_2|$. From (27) we obtain:

$$K(z_1,\zeta)-K(z_2,\zeta)=\frac{(-1)^{n+1}}{2\pi\imath}\sum_{\ell=0}^{n}\frac{(n+1-\ell)}{\ell!}\left(\frac{R_{n-l}(z_1,\zeta)}{(z_1-\zeta)^{n+2-l}}-\frac{R_{n-l}(z_2,\zeta)}{(z_2-\zeta)^{n+2-l}}\right).$$

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Add and subtract $R_{n-l}(z_1,\zeta)$ in the numerator of the second fraction above to get:

$$K(z_1,\zeta)-K(z_2,\zeta)=\mathrm{I}+\mathrm{II},$$

where

$$\mathrm{I}=\frac{(-1)^{n+1}}{2\pi\imath}\sum_{\ell=0}^{n}\frac{(n+1-\ell)}{\ell!}R_{n-l}(z_1,\zeta)\left(\frac{1}{(z_1-\zeta)^{n+2-l}}-\frac{1}{(z_2-\zeta)^{n+2-l}}\right),$$

and

$$\mathrm{II}=\frac{(-1)^{n+1}}{2\pi\imath}\sum_{\ell=0}^{n}\frac{(n+1-\ell)}{\ell!}\frac{R_{n-l}(z_1,\zeta)-R_{n-l}(z_2,\zeta)}{(z_2-\zeta)^{n+2-l}}.$$

Controlling I is easy via an obvious gradient estimate, which yields:

$$\mathrm{I}\leqslant C|z_1-\zeta|^{n-l+\varepsilon}\frac{|z_1-z_2|}{|z_1-\zeta|^{n+3-l}}=C\frac{|z_1-z_2|}{|z_1-\zeta|^{3-\varepsilon}}\leqslant C\frac{|z_1-z_2|^{\varepsilon}}{|z_1-\zeta|^{2}}.$$

To estimate the term II we need a sublemma.

Sublemma. *We have the identity*:

$$R_{n-l}(z_1,\zeta)-R_{n-l}(z_2,\zeta)=\sum_{j+k=n-l}c_{j,k}\big(B^{l+j}(\chi_\Omega)(z_1)-B^{l+j}(\chi_\Omega)(z_2)\big)(\zeta-z_2)^j\,\overline{(\zeta-z_2)}^k$$
$$+O\big(|z_1-z_2|^{1+\varepsilon}|\zeta-z_2|^{n-l-1}\big).$$

Since $B^m(\chi_\Omega)$ is in $\mathrm{Lip}(\varepsilon,\Omega)$ for each non-negative number m, the Sublemma immediately provides the right control on the term II, namely,

$$\mathrm{II}\leqslant\frac{C}{|\zeta-z_2|^{n+2-l}}\big(|z_1-z_2|^{\varepsilon}|\zeta-z_2|^{n-l}+|z_1-z_2|^{1+\varepsilon}|\zeta-z_2|^{n-l-1}\big)\leqslant C\frac{|z_1-z_2|^{\varepsilon}}{|\zeta-z_2|^{2}},$$

and this completes the proof of Lemma 6. □

Proof of the Sublemma. The most convenient way of proving the Sublemma is to place ourselves in a real variables context. Given a smooth function f on $\mathbb{R}^d$ let,

$$T_m(f,a)(x)=\sum_{|\alpha|\leqslant m}\frac{\partial^\alpha f(a)}{\alpha!}(x-a)^\alpha,$$

be its Taylor polynomial of degree m around the point a. Then, clearly,

$$R_{n-l}(z_1,\zeta)-R_{n-l}(z_2,\zeta)=T_{n-l}\left(\frac{\partial^\ell h}{\partial\zeta^\ell},z_2\right)(\zeta)-T_{n-l}\left(\frac{\partial^\ell h}{\partial\zeta^\ell},z_1\right)(\zeta),$$

and so the Sublemma is an easy consequence of the fact that each nth order derivative of h is a constant times $B^m(\chi_\Omega)$, for an appropriate exponent m, and the following elementary calculus lemma. □

Lemma 7. *If f is a m times continuously differentiable function on $\mathbb{R}^d$, then*

$$T_m(f,a_1)(x)-T_m(f,a_2)(x)=\sum_{|\alpha|=m}\partial^\alpha f(a_1)-\partial^\alpha f(a_2)$$
$$-\sum_{|\alpha|<m}\frac{1}{\alpha!}\big(\partial^\alpha f(a_2)-T_{m-|\alpha|}\big(\partial^\alpha f,a_1\big)(a_2)\big)(x-a_2)^\alpha.$$

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Proof. Let $P(x)$ stand for the polynomial $T_m(f,a_1)(x) - T_m(f,a_2)(x)$, so that

$$P(x) = \sum_{|\alpha|\leqslant m} \frac{\partial^\alpha P(a_2)}{\alpha!}(x-a_2)^\alpha.$$

A straightforward computation yields

$$\partial^\alpha P(a_2) = T_{m-|\alpha|}\big(\partial^\alpha f, a_1\big)(a_2) - \partial^\alpha f(a_2),$$

which completes the proof of Lemma 7. □

6. ϕ is bilipschitz

In the preceding sections we have proved that ϕ is a Lipschitz function on $\mathbb{C}$. Moreover,

$$\bar\partial\phi = h = (I-\mu B)^{-1}(\mu) \in \operatorname{Lip}(\varepsilon', \Omega), \quad 0<\varepsilon'<\varepsilon,$$

and so, by the Main Lemma,

$$\partial\phi = 1 + B(h) \in \operatorname{Lip}(\varepsilon', \Omega) \cap \operatorname{Lip}\big(\varepsilon', (\overline{\Omega})^c\big), \quad 0<\varepsilon'<\varepsilon.$$

Since ϕ is holomorphic on $(\overline{\Omega})^c$, $\phi'(z) = \partial\phi(z)$ extends continuously to Ω^c, but nothing excludes that this extension might vanish somewhere on $\partial\Omega$. The functions $\bar\partial\phi$ and $\partial\phi$ also extend continuously from Ω to $\overline{\Omega}$, but again it could well happen that both vanish at some point of $\partial\Omega$. We will show now that this is not possible. Indeed, we claim that for some positive number ε_0 we have:

$$\big|\partial\phi(z)\big| \geqslant \varepsilon_0, \quad z \in \Omega \cap (\overline{\Omega})^c. \tag{28}$$

This implies that the Jacobian of ϕ is bounded from below by $(1-\|\mu\|_\infty^2)\,\varepsilon_0$ at z almost all points of $\mathbb{C}$. Thus the inverse mapping ϕ^{-1} has gradient in $L^\infty(\mathbb{C})$ and hence ϕ is bilipschitz.

Proof of (28). For $a \in \partial\Omega$ denote by $\phi'(a)$ the limit of $\phi'(z)$ as $z \in (\overline{\Omega})^c$ tends to a. We claim that (28) follows if we can show that

$$\phi'(a) \neq 0, \quad a \in \partial\Omega. \tag{29}$$

Indeed, this clearly implies $\inf_{z\in(\overline{\Omega})^c} |\partial\phi(z)| > 0$. Now denote by $\partial\phi(a)$ and $\bar\partial\phi(a)$, $a \in \partial\Omega$, the limits of $\partial\phi(z)$ and $\bar\partial\phi(z)$ as $z \in \Omega$ tends to a. Take a parametrization $z(t)$ of $\partial\Omega$ of class C^1, such that $z'(t) \neq 0$ for all t. Computing $\frac{d}{dt}\phi(z(t))$ in two different ways,

$$\phi'\big(z(t)\big)z'(t) = \frac{d}{dt}\phi\big(z(t)\big) = \frac{\partial\phi}{\partial z}\big(z(t)\big)z'(t) + \frac{\partial\phi}{\partial\bar z}\big(z(t)\big)\overline{z'(t)}$$

and so

$$\phi'\big(z(t)\big) = \left(1 + \mu\big(z(t)\big)\frac{\overline{z'(t)}}{z'(t)}\right)\frac{\partial\phi}{\partial z}\big(z(t)\big).$$

Thus, by (29), $\frac{\partial\phi}{\partial z}(a) \neq 0$, $a \in \partial\Omega$, which yields $\inf_{z\in\Omega} |\frac{\partial\phi}{\partial z}(z)| > 0$.

We turn now to the proof of (29). Assume that $0 = a \in \partial\Omega$. Performing a rotation before applying ϕ we may assume that $\lambda = \mu(0)$ is a non-negative real number ($\mu(0)$ is the limit of $\mu(z)$ as $z \in \Omega$ tends to 0). Performing a rotation after applying ϕ we may also assume that the tangent plane to $\partial\Omega$ at the origin is the real axis. Denote by H^+ and H^- the upper and lower half planes, respectively. Consider the continuous piecewise linear mapping:

$$z = L(w) = (w - \lambda\overline{w})\chi_{H^-}(w) + (1-\lambda)w\chi_{H^+}(w).$$

Then, by [15, (5.6), p. 83], the Beltrami coefficient $\nu(w)$ of the mapping $\phi \circ L$ is:

$$\nu(w) = \frac{\bar\partial(\phi\circ L)(w)}{\partial(\phi\circ L)(w)} = \frac{-\lambda\chi_{H^-}(w) + \mu(L(w))}{1 - \lambda\chi_{H^-}(w)\mu(L(w))}. \tag{30}$$

Since ν vanishes on $H^+ \cap L^{-1}((\overline{\Omega})^c)$,

$$\int_{|w|<r_0} \frac{|\nu(w)|}{|w|^2}\, dA(w) = \int_{H^+\cap L^{-1}(\Omega)\cap B(0,r_0)} \cdots + \int_{H^-\cap L^{-1}(\Omega^c)\cap B(0,r_0)} \cdots - \int_{H^-\cap L^{-1}(\Omega)\cap B(0,r_0)} \cdots = \mathrm{I} + \mathrm{II} + \mathrm{III},$$

where r_0 is the small number introduced in the proof of the Main Lemma (see Fig. 1). By (30),

$$|\mathrm{I}| \leqslant \int_{H^+\cap L^{-1}(\Omega)\cap B(0,r_0)} \frac{|\mu(L(w))|}{|w|^2}\, dA(w).$$

Since $L(w) = (1-\lambda)w$ on H^+, making the change of variables $z = L(w)$ gives:

$$|\mathrm{I}| \leqslant \int_{H^+\cap\Omega\cap B(0,r_0)} \frac{dA(z)}{|z|^2} \leqslant \int_0^{r_0} r^{-1+\varepsilon}\, dr < \infty,$$

where in the next to the last inequality we used (16).

For II we begin by remarking that

$$|\mathrm{II}| = \lambda \int_{H^-\cap L^{-1}(\Omega^c)\cap B(0,r_0)} \frac{dA(w)}{|w|^2},$$

and making the change of variables $z = w - \lambda\overline{w}$, we get:

$$|\mathrm{II}| \leqslant \lambda \int_{H\cap\Omega^c\cap B(0,r_0)} \frac{1}{|z|^2}\frac{1+\lambda}{1-\lambda}\, dA(z),$$

which can be shown to be finite as before (in particular, using again (16)). To take care of III we make the same change of variables and we obtain:

$$\begin{aligned}|\mathrm{III}| &\leqslant \int_{H^-\cap L^{-1}(\Omega)\cap B(0,r_0)} \left|\frac{-\lambda+\mu(L(w))}{1-\lambda\mu(L(w))}\right| \frac{dA(w)}{|w|^2} \leqslant \frac{1}{1-\lambda}\int_{H^-\cap L^{-1}(\Omega)\cap B(0,r_0)} \frac{|\mu(L(w))-\mu(0)|}{|w|^2}\, dA(w)\\ &\leqslant \frac{1+\lambda}{(1-\lambda)^2}\int_{H^-\cap\Omega\cap B(0,r_0)} \frac{|\mu(z)-\mu(0)|}{|z|^2}\, dA(z) \leqslant C\int_{B(0,r_0)} \frac{dA(z)}{|z|^{2-\varepsilon}} < \infty.\end{aligned}$$

Therefore

$$\int_{|w|<r_0} \frac{|\nu(w)|}{|w|^2}\, dA(w) < \infty,$$

and so, by [15, p. 232], $H = \phi \circ L$ is conformal at the origin, in the sense that the limit,

$$H'(0) = \lim_{z\to 0} \frac{H(z)-H(0)}{z},$$

exists and $H'(0) \neq 0$. The part of the imaginary positive axis close to the origin is included in $(\overline{\Omega})^c$ (see Fig. 1), and thus $L^{-1}(iy) = \frac{iy}{1-\lambda}$ if $y > 0$ is small. Hence

$$\phi'(0) = \lim_{0<y\to 0} \phi'(iy) = \lim_{0<y\to 0} \frac{\phi(iy)-\phi(0)}{iy} = \lim_{0<y\to 0} \frac{H(L^{-1}(iy))-H(0)}{iy} = \frac{H'(0)}{(1-\lambda)}.$$

This completes the proof of (29). □

7. Reduction to the one domain case

Suppose, as in the statement of the Theorem, that $\Omega_1, \ldots, \Omega_N$ are bounded disjoint domains with boundary of class $C^{1+\varepsilon}$, for some ε with $0 < \varepsilon < 1$, and that $\mu = \sum_{j=1}^N \mu_j \chi_{\Omega_j}$, where μ_j is of class $\mathrm{Lip}(\varepsilon, \Omega_j)$, and $\|\mu\|_\infty < 1$. Let Φ^μ

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be the quasiconformal mapping associated with μ. Assume that $N > 1$ and set $\nu_1 = \mu_1 \chi_{\Omega_1}$ and $\nu_2 = \sum_{j=2}^{N} \mu_j \chi_{\Omega_j}$. By [1, (10), p. 9],

$$\Phi^{\mu} = \Phi^{\lambda} \circ \Phi^{\nu_1},$$

where

$$\lambda(w) = \nu_2(z) \frac{\partial \Phi^{\nu_1}(z)}{\overline{\partial \Phi^{\nu_1}(z)}},$$

and, for each $w \in \mathbb{C}$ the point z is defined by $w = \Phi^{\nu_1}(z)$. In particular, λ is supported on $\bigcup_{j=2}^{N} \Phi^{\nu_1}(\Omega_j)$. Recall from the previous section that Φ^{ν_1} is bilipschitz, of class $C^{1+\varepsilon'}$, $0 < \varepsilon' < \varepsilon$, on Ω_1 and $(\overline{\Omega_1})^c$, and conformal on Ω_1^c. In particular, Φ^{ν_1} is holomorphic on $(\overline{\Omega_1})^c$, and

$$\frac{d\Phi^{\nu_1}}{dz}(z) \neq 0, \quad z \in \Omega_1^c.$$

Thus the bounded domains $\Phi^{\nu_1}(\Omega_j)$, $1 \leqslant j \leqslant N$, have boundaries of class $C^{1+\varepsilon'}$, $0 < \varepsilon' < \varepsilon$. On the other hand, the function λ satisfies a Lipschitz condition of order ε', for $0 < \varepsilon' < \varepsilon$, in each domain $\Phi^{\nu_1}(\Omega_j)$, $2 \leqslant j \leqslant N$. Proceeding by induction we conclude now that Φ^{μ} is bilipschitz and of class $C^{1+\varepsilon'}$, $0 < \varepsilon' < \varepsilon$, in each domain Ω_j, $1 \leqslant j \leqslant N$.

One can prove now Corollary 2. Let D be a bounded planar domain and let f be a function in $W^{1,2}_{\mathrm{loc}}(D)$ satisfying the Beltrami equation:

$$\frac{\partial f}{\partial \bar z}(z) = \mu(z) \frac{\partial f}{\partial z}(z), \quad z \in D,$$

where μ is as in the statement of the theorem. By Stoilow's Factorization Theorem, $f = h \circ \Phi$, where Φ is the quasiconformal mapping associated with μ and h is a holomorphic function on $\Phi(D)$. Let D_δ stand for the set of points in D whose distance to the boundary of D is larger than δ. Thus h has bounded derivatives of all orders on $\Phi(D_\delta)$ and consequently f is as smooth as Φ on D_δ. Hence f is Lipschitz on D_δ and of class $C^{1+\varepsilon'}$, $0 < \varepsilon' < \varepsilon$, in each open set $D_\delta \cap \Omega_j$, $1 \leqslant j \leqslant N$. If D contains the closure of each Ω_j, then f is of class $C^{1+\varepsilon'}$, $0 < \varepsilon' < \varepsilon$, on each Ω_j.

Recalling the relation between the Beltrami equation and second order elliptic equations in divergence form, as explained in the introduction, Corollary 2 follows immediately from the above argument.

8. Cuspidal domains

As we remarked in the introduction, the conclusion of the Theorem fails for domains with corners; for instance, for a square. However, the class of domains with boundary of class $C^{1+\varepsilon}$ is not optimal for the Theorem. There is a heuristic argument that points out at a more general class of domains, which, at least in a first approximation, may be viewed as optimal.

First of all we recall that a central point in the proof of the Theorem was the fact, which is part of the Main Lemma, that each power of the Beurling transform sends the characteristic function of the domain into a bounded function. Let us concentrate on the Beurling transform B and find a simple condition on a bounded domain Ω with rectifiable boundary so that $B(\chi_\Omega)$ is bounded. Our first remark is that $B(\chi_\Omega)$ can be written as the Cauchy transform of a boundary measure. For this we use, on one hand, the basic property of B,

$$\partial \varphi = B(\bar\partial \varphi),$$

which holds for all compactly supported smooth functions φ and extends to a variety of situations by regularization. On the other hand, we use the elementary identity,

$$\partial \chi_\Omega = \frac{1}{2\imath} d\bar z_{\partial\Omega},$$

which holds at least for bounded domains with rectifiable boundary. Combining the above two identities we get:

$$\bar\partial B(\chi_\Omega) = B(\bar\partial \chi_\Omega) = \partial \chi_\Omega = \frac{1}{2\imath} d\bar z_{\partial\Omega},$$

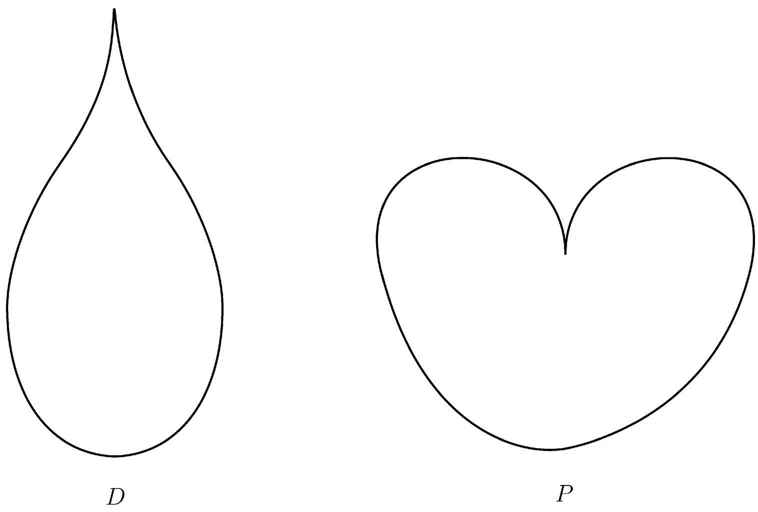

Fig. 5.

which yields

$$B(\chi_\Omega) = \frac{1}{2\imath} C(d\bar{z}_{\partial\Omega}).$$

Now, $d\bar{z}_{\partial\Omega} = \bar{\tau}^2(z)\,dz_{\partial\Omega}$, $\tau(z)$ being the unit tangent vector to $\partial\Omega$ at z. Assume now that the arc-length measure on the boundary of Ω satisfies the Ahlfors condition length$(\partial\Omega \cap D(z,r)) \leqslant Cr$, for each $z \in \partial\Omega$ and $r > 0$, where $D(z,r)$ stands for the open disc with center z and radius r. Then a simple estimate shows that the Cauchy integral of a function f on $\partial\Omega$, that is,

$$\frac{1}{2\pi\imath} \int_{\partial\Omega} \frac{f(w)}{z-w}\,dw, \quad z \in \mathbb{C} \setminus \partial\Omega,$$

is bounded provided f satisfies a Lipschitz condition of some positive order on $\partial\Omega$. Therefore, to get boundedness of $B(\chi_\Omega)$ one has to require that the square of the tangent unit vector satisfies a Lipschitz condition of some positive order on $\partial\Omega$. This is weaker than requiring the Lipschitz condition on the tangent unit vector itself, because it allows jumps of 180 degrees on the argument of the tangent unit vector. In other words, cusps are allowed.

We want now to define formally *cuspidal domains of class* $C^{1+\varepsilon}$. Given a planar domain Ω we say that $\partial\Omega$ is $C^{1+\varepsilon}$-smooth at a boundary point z_0 if there is a positive r_0 such that $\Omega \cap D(z_0,r_0)$ is, after possibly a rotation, the part of $D(z_0,r_0)$ lying below the graph of a function of class $C^{1+\varepsilon}$.

We say that Ω has an *interior cusp of class* $C^{1+\varepsilon}$ at $z_0 = x_0 + \imath y_0 \in \partial\Omega$ provided there is a positive r_0 and functions $y = a(x)$, $y = b(x)$, of class $C^{1+\varepsilon}$ on the interval $(x_0 - r_0, x_0 + r_0)$, such that $a(x_0) = a'(x_0) = b(x_0) = b'(x_0) = 0$, and, after possibly a rotation, a point $z = x + \imath y$ is in $\Omega \cap D(z_0,r_0)$ if and only if $z \in D(z_0,r_0)$ and $b(x) < y < a(x)$.

We say that Ω has an *exterior cusp of class* $C^{1+\varepsilon}$ at $z_0 \in \partial\Omega$ provided $\overline{\Omega}^c$ has an interior cusp of class $C^{1+\varepsilon}$ at z_0.

A planar domain Ω is a *cuspidal domain of class* $C^{1+\varepsilon}$ if $\partial\Omega$ is $C^{1+\varepsilon}$-smooth at all boundary points, except possibly at finitely many boundary points where Ω has a cusp of class $C^{1+\varepsilon}$ (either interior or exterior).

The simplest examples of non-smooth cuspidal domains are the drop like domain D and the peach like domain P shown in Fig. 5. The reader may easily imagine more complicated cuspidal domains with lots of cusps of both types (see Fig. 7).

With appropriate formulations the Theorem and Corollaries 1 and 2 hold true for cuspidal domains of class $C^{1+\varepsilon}$. The right statements involve the notion of geodesic distance in the domain Ω, which we discuss now. Given two points z and w in Ω their geodesic distance is defined by:

$$d(z,w) = d_\Omega(z,w) = \inf_\gamma l(\gamma),$$

where the infimum is taken over all rectifiable curves γ in Ω joining z and w. Here $l(\gamma)$ stands for the length of γ. Notice that if Ω has only interior cusps, then the geodesic and the Euclidean distances are comparable and that this

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is not the case in the proximity of an exterior cusp. The Lipschitz norm $\| \cdot \|_{\varepsilon,\Omega,d_\Omega}$ of order ε and the corresponding Lipschitz spaces $\operatorname{Lip}(\varepsilon, \Omega, d_\Omega)$ with respect to the distance d are defined is the usual way, with the Euclidean distance replaced by d in (3) and (4).

The Theorem for cuspidal domains reads as follows:

Theorem′. *Let $\{\Omega_j\}$, $1 \leqslant j \leqslant N$, be a finite family of disjoint bounded cuspidal domains of class $C^{1+\varepsilon}$, $0 < \varepsilon < 1$, and let $\mu = \sum_{j=1}^N \mu_j \chi_{\Omega_j}$, where μ_j is of class $\operatorname{Lip}(\varepsilon, \Omega_j, d_{\Omega_j})$. Assume in addition that $\|\mu\|_\infty < 1$. Then the associated quasiconformal mapping Φ is bilipschitz.*

Corollary 1 remains true without any change.

Corollary 1′. *If Ω is a bounded cuspidal domain of class $C^{1+\varepsilon}$, $0 < \varepsilon < 1$, and $\mu = \lambda \chi_\Omega$, where λ is a complex number such that $|\lambda| < 1$, then the associated quasiconformal mapping Φ is bilipschitz.*

Corollary 2′. *Let Ω_j, $1 \leqslant j \leqslant N$, be a finite family of disjoint bounded cuspidal domains of class $C^{1+\varepsilon}$, $0 < \varepsilon < 1$, and assume that all domains Ω_j are contained in a bounded domain D with boundary of class $C^{1+\varepsilon}$. Let $A = A(z)$, $z \in D$, a 2×2 symmetric elliptic matrix with determinant 1 and entries supported in $\bigcup_{j=1}^N \Omega_j$ and belonging to $\operatorname{Lip}(\varepsilon, \Omega_j, d_{\Omega_j})$, $1 \leqslant j \leqslant N$. Let u be a solution of Eq.* (2) *in D. Let D_δ stand for the set of points in D at distance greater than δ from the boundary of D. Then $\nabla u \in \operatorname{Lip}(\varepsilon', \Omega_j \cap D_\delta, d_{\Omega_j})$, for $0 < \varepsilon' < \varepsilon$ and $1 \leqslant j \leqslant N$. In particular, $\nabla u \in L^\infty(D_\delta)$ and u is a locally Lipschitz function in D.*

We proceed now to sketch the proof of Theorem′. The modifications needed are minor and fortunately one can reduce without much pain the cuspidal case to the smooth case.

We start by discussing the proof in the one domain case ($N = 1$). The difficulty is that the Main Lemma does not hold for cuspidal domains with exterior cusps and the usual Euclidean Lipschitz spaces. We present an example to make the difficulty clear.

Example. Let Ω be the union of the open discs of radius 1 centered at 1 and -1. Thus Ω has an exterior cusp at the origin. We claim that $B(\chi_\Omega)$ does not satisfy a Lipschitz condition on Ω of any order ε such that $1/2 < \varepsilon \leqslant 1$. The point is that we can explicitly calculate $B(\chi_\Omega)$. If $D(a,r)$ stands for the open disc centered at a of radius r, then a simple argument (see, for example, [20, p. 965]) shows that

$$B(\chi_{D(a,r)})(z) = -\frac{r^2}{(z-a)^2}\, \chi_{D^c(a,r)}(z), \quad z \in \mathbb{C}.$$

Thus

$$B(\chi_\Omega)(z) = \frac{-1}{(z-1)^2}, \quad z \in D(-1,1)$$

and

$$B(\chi_\Omega)(z) = \frac{-1}{(z+1)^2}, \quad z \in D(1,1).$$

Set $z_1 = -x + \imath y$ and $z_2 = x + \imath y$, where x and y are positive real numbers such that z_1 and z_2 are in Ω. Then $|z_1 - z_2| = 2x$. On the other hand, a simple computation yields $|B(\chi_\Omega)(z_1) - B(\chi_\Omega)(z_2)| \simeq y$ as x and y tend to 0. Choosing $y \simeq \sqrt{x}$ we conclude that $B(\chi_\Omega)$ does not satisfy a Lipschitz condition on Ω of any order ε with $1/2 < \varepsilon \leqslant 1$.

The Main Lemma for cuspidal domains reads as follows.

Main Lemma. *Let Ω be a bounded cuspidal domain of class $C^{1+\varepsilon}$, $0 < \varepsilon < 1$, and let T be an even smooth homogeneous Calderón–Zygmund operator. Then T maps $\operatorname{Lip}(\varepsilon, \Omega, d_\Omega)$ into itself, and T also maps $\operatorname{Lip}(\varepsilon, \Omega, d_\Omega)$ into $\operatorname{Lip}(\varepsilon, \overline{\Omega}^c, d_{\overline{\Omega}^c})$. In fact, one has the inequalities*:

$$\|Tf\|_{\varepsilon,\Omega,d_\Omega} \leqslant C \|T\|_{CZ} \|f\|_{\varepsilon,\Omega,d_\Omega},$$

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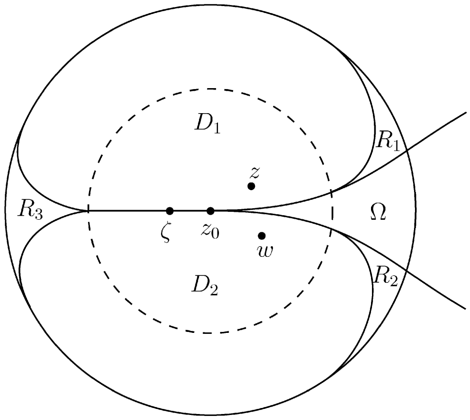

Fig. 6.

and

$$\|Tf\|_{\varepsilon,\overline{\Omega}^c,d_{\overline{\Omega}^c}} \leqslant C\|T\|_{CZ}\|f\|_{\varepsilon,\Omega,d_\Omega}$$

where C is a constant depending only on ε and Ω.

Proof. We first prove that $T^*(f) \in L^\infty(\mathbb{C})$ for each $f \in \operatorname{Lip}(\varepsilon, \Omega, d_\Omega)$. Take a point $z \in \mathbb{C}$. Clearly we may assume that $z \in D(z_0, \frac{1}{3}r_0)$, where z_0 is a cuspidal point and r_0 is as in the definition of cusp. Otherwise z is far from all cusps and thus we may apply the arguments of the smooth case. Since there are only finitely many cusps we may also assume that there is a positive number r_0 which works in the definition of cusp simultaneously for all cusps. Clearly

$$T^*(f)(z) \leqslant T^*(f\chi_{\Omega\cap D(z_0,r_0)})(z) + T^*(f\chi_{\Omega\cap D^c(z_0,r_0)})(z),$$

and the second term is bounded by $C\log\frac{d}{r_0}\|f\|_\infty$, where d stands for the diameter of Ω. We are therefore left with the first term.

Assume for the moment that Ω has an interior cusp at z_0. We connect the cercle of center z_0 and radius $\frac{2}{3}r_0$ with the concentric cercle of radius r_0 to produce domains D_1 and D_2 with boundary of class $C^{1+\varepsilon}$, as shown in Fig. 6.

Notice that three other "residual" domains R_j, $1 \leqslant j \leqslant 3$, have been formed. The domains R_j are not smooth but they are far from $D(z_0, \frac{1}{3}r_0)$. Since we are assuming that Ω has an interior cusp at z_0, the restriction of f to $\Omega \cap D(z_0, r_0)$ satisfies an Euclidean Lipschitz condition of order ε. By the well-known extension theorem for Lipschitz functions [23, Chapter VI] we may extend the restriction of f to $\Omega \cap D(z_0, r_0)$ to a function $g \in \operatorname{Lip}(\varepsilon, \mathbb{C})$ such that

$$\|g\|_{\varepsilon,\mathbb{C}} \leqslant C\|f\|_{\varepsilon,\Omega\cap D(z_0,r_0)}. \tag{31}$$

Hence

$$f\chi_{\Omega\cap D(z_0,r_0)} = g\chi_{D(z_0,r_0)} - g\chi_{D_1} - g\chi_{D_2} - \sum_{j=1}^{3} g\chi_{R_j}$$

and so

$$T^*(f\chi_{\Omega\cap D(z_0,r_0)}) \leqslant T^*(g\chi_{D(z_0,r_0)}) + T^*(g\chi_{D_1}) + T^*(g\chi_{D_2}) + \sum_{j=1}^{3} T^*(g\chi_{R_j}).$$

The last three terms are controlled by $C\|g\|_\infty$, where C is a constant depending on r_0, because the domains of the functions to which T^* is applied are far from $D(z_0, \frac{1}{3}r_0)$. By (31) this gives the correct bound $C\|f\|_{\varepsilon,\Omega}$.

To estimate the first three terms we remark that D_1, D_2 and $D(z_0, r_0)$ are domains with boundary of class $C^{1+\varepsilon}$. Thus the proof of the Main Lemma for smooth domains of Section 3 yields, by (31),

$$\|T^*(f)\|_\infty \leqslant C\|f\|_{\varepsilon,\Omega}. \tag{32}$$

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Assume now that Ω has an exterior cusp at z_0. Then

$$f\chi_{\Omega\cap D(z_0,r_0)} = f\chi_{D_1} + f\chi_{D_2} + \sum_{j=1}^{3} f\chi_{R_j}$$

and so

$$T^*(f\chi_{\Omega\cap D(z_0,r_0)}) \leqslant T^*(f\chi_{D_1}) + T^*(f\chi_{D_2}) + \sum_{j=1}^{3} T^*(f\chi_{R_j}).$$

The proof of the estimate (32) proceeds as before.

Let us turn to prove the Lipschitz condition on $T(f)$. Since $T(f)$ is bounded, to estimate $|T(f)(z) - T(f)(w)|$ we may restrict our attention to the case in which z and w are very close to each other. We may also assume that z and w are close to a cusp, say, $z, w \in D(z_0, \frac{1}{3}r_0)$. From this point on the proof is very similar to what we did before, with T^* replaced by T. The first step is to restrict our attention to $f\chi_{\Omega\cap D(z_0,r_0)}$, which may be achieved by the identity:

$$T(f) = T(f\chi_{\Omega\cap D(z_0,r_0)}) + T(f\chi_{\Omega\cap D^c(z_0,r_0)}).$$

Assume first that Ω has an interior cusp at z_0 and consider again the extension g of $f\chi_{\Omega\cap D(z_0,r_0)}$ satisfying (31). We clearly have

$$T(f\chi_{\Omega\cap D(z_0,r_0)}) = T(g\chi_{D(z_0,r_0)}) - T(g\chi_{D_1}) - T(g\chi_{D_2}) - \sum_{j=1}^{3} T(g\chi_{R_j}).$$

By the smooth version of the Main Lemma $T(g\chi_{D_1})$ and $T(g\chi_{D_2})$ satisfy an Euclidean Lipschitz condition of order ε on the complement of D_1 and D_2 respectively. Clearly, again by the smooth Main Lemma, $T(g\chi_{D(z_0,r_0)})$ satisfies an Euclidean Lipschitz condition of order ε on $D(z_0,r_0)$. Finally $T(g\chi_{R_j})$, $1 \leqslant j \leqslant 3$, satisfy an Euclidean Lipschitz condition of order ε on $D(z_0, \frac{1}{3}r_0)$, because the domains R_j are far from $D(z_0, \frac{1}{3}r_0)$. If both z and w belong to Ω we then get an estimate of the form,

$$\left|T(f)(z) - T(f)(w)\right| \leqslant C\|f\|_{\varepsilon,\Omega}|z-w|^{\varepsilon},$$

which completes the proof, because in the case at hand the Euclidean distance between z and w is comparable to their geodesic distance. If z and w are in $\overline{\Omega}^c$, then we may assume that $z \in D_1$ and $w \in D_2$. Otherwise z and w belong both to either D_1 or D_2 and thus we obtain the above Euclidean Lipschitz estimate of order ε. Choose then a point $\xi \in \partial D_1 \cap \partial D_2$ such that $\max\{|z-\xi|, |w-\xi|\} \simeq d_{\overline{\Omega}^c}(z,w)$ (see Fig. 6). Since $T(g\chi_{D_1})$ and $T(g\chi_{D_2})$ satisfy an Euclidean Lipschitz condition of order ε on D_1 and D_2 respectively, and since $T(f)$ is continuous on the complement of Ω, because f is supported on Ω, we get:

$$\left|T(f)(z) - T(f)(w)\right| \leqslant C\max\left\{|z-\xi|^{\varepsilon}, |w-\xi|^{\varepsilon}\right\} \simeq d_{\overline{\Omega}^c}(z,w)^{\varepsilon}.$$

If Ω has an exterior cusp at z_0, we argue similarly, using the identity,

$$T(f\chi_{\Omega\cap D(z_0,r_0)}) = T(f\chi_{D_1}) + T(f\chi_{D_2}) + \sum_{j=1}^{3} T(f\chi_{R_j}).$$

The details are left to the reader. □

We continue now the proof of Theorem′ in the one domain case. If the domain has only interior cusps the argument we described to prove the Theorem goes through without any change. The reason is that, since the geodesic distance in Ω is comparable to the Euclidean distance Sections 4 and 5 hold true, because of the cuspidal version of the Main Lemma. For Section 6 one has to remark that $\nabla\Phi$ is now only in $\mathrm{Lip}(\varepsilon', \overline{\Omega}^c, d_{\overline{\Omega}^c})$, $0 < \varepsilon' < \varepsilon$, but that this still implies that $\nabla\Phi$ extends continuously from $\overline{\Omega}^c$ to $\partial\Omega$. The conformality of Φ on Ω^c is proved as in Section 6, after remarking that cusps do not create any problem because they are perfectly suited for an appeal to [15, p. 232].

Assume now that our domain has exterior cusps. In Fig. 7 it is shown how to subdivide Ω in finitely many subdomains Ω_j, $1 \leqslant j \leqslant M$, which are cuspidal domains of class $C^{1+\varepsilon}$ *with only interior cusps*. Since $\mu \in \mathrm{Lip}(\varepsilon, \Omega, d_\Omega)$,

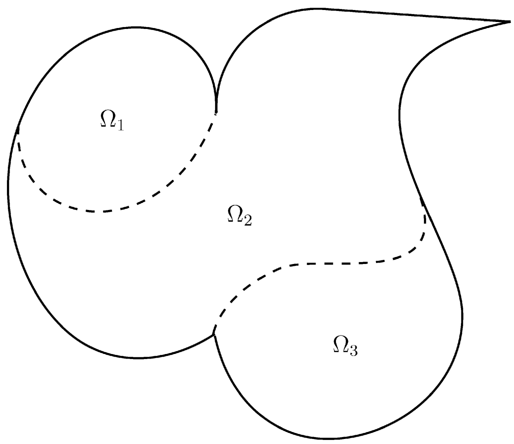

Fig. 7.

$\mu_j = \mu\chi_{\Omega_j}$ is in $\mathrm{Lip}(\varepsilon, \Omega_j)$. We can then apply the factorization method of Section 7 to get that $\nabla\Phi \in \mathrm{Lip}(\varepsilon', \Omega, d_\Omega)$ and $\nabla\Phi \in \mathrm{Lip}(\varepsilon', \overline{\Omega}^c, d_{\overline{\Omega}^c})$, $0 < \varepsilon' < \varepsilon$.

The reduction to the one domain case needs only one comment. Using the notation of Section 7, we use the fact that Φ^{ν_1} is conformal on Ω_1^c to ascertain that the image of each Ω_j under Φ^{ν_1} is again a cuspidal domain of class $C^{1+\varepsilon'}$, $0 < \varepsilon' < \varepsilon$. We repeat for emphasis that the conformality of Φ^{ν_1} is proved at a cusp appealing, as in Section 6, to [15, p. 232].

9. Final comments

Very likely the restriction on the determinant of the matrix A in Corollaries 2 and 2′ is superfluous. This would follow if Lipschitz regularity results should hold for the general elliptic system,

$$\bar\partial\Phi(z) = \mu(z)\partial\Phi(z) + \nu(z)\overline{\partial\Phi(z)},$$

where $|\mu(z)| + |\nu(z)| \leqslant k < 1$, a.e. on $\mathbb{C}$.

It seems also rather clear that the right conclusion in Corollary 2 (and analogously in Corollary 2′) should be that the solution u is of class $\mathrm{Lip}(\varepsilon, \Omega_j)$, $1 \leqslant j \leqslant N$, in all dimensions (and without any restriction on the determinant of A). Evidence for this conjecture is provided by the fact that it is true in the plane whenever the Lipschitz norm of the coefficients of the matrix A are small enough. See the argument for the Beltrami equation at the end of Section 2. We acknowledge some useful correspondence with L. Escauriaza and D. Faraco on that issue.

Apparently it is not known what is the best exponent p such that $\nabla\Phi^\mu \in L^p_{\mathrm{loc}}(\mathbb{C})$ for $\mu = \lambda\chi_Q$, where λ is a complex number such that $|\lambda| < 1$ and Q is a square. This looks extremely surprising to the authors, who would very much appreciate knowing the precise regularity properties of Φ^μ in the scale of the local Sobolev spaces $W^{1,p}_{\mathrm{loc}}(\mathbb{C})$.

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References

[1] L. Ahlfors, Lectures on Quasiconformal Mappings, second ed., University Lecture Series, vol. 38, American Mathematical Society, Providence, RI, 2006.

[2] K. Astala, Area distortion of quasiconformal mappings, Acta Math. 173 (1994) 37–60.

[3] K. Astala, A. Clop, J. Mateu, J. Orobitg, I. Uriarte-Tuero, Distortion of Hausdorff measures and improved Painlevé removability for quasiregular mappings, Duke Math. J. 141 (3) (2008) 539–571.

[4] K. Astala, T. Iwaniec, G. Martin, Elliptic Equations and Quasiconformal Mappings in the Plane, Princeton Mathematical Series, vol. 47, Princeton University Press, 2009.
[5] K. Astala, T. Iwaniec, E. Saksman, Beltrami operators in the plane, Duke Math. J. 107 (2001) 27–56.
[6] A. Clop, D. Faraco, J. Mateu, J. Orobitg, X. Zhong, Beltrami equations with coefficient in the Sobolev space $W^{1,p}$, Publ. Mat. 53 (2009) 197–230.
[7] A. Clop, X. Tolsa, Analytic capacity and quasiconformal mappings with $W^{1,2}$-Beltrami coefficients, Math. Res. Lett. 15 (4) (2008) 779–793.
[8] N. Depauw, Poche de tourbillon pour Euler 2D dans un ouvert à bord, J. Math. Pures Appl. 78 (3) (1999) 313–351.
[9] J.L. García-Cuerva, A.E. Gatto, Lipschitz spaces and Calderón–Zygmund operators associated to non-doubling measures, Publ. Mat. 49 (2005) 285–296.
[10] A.E. Gatto, On the boundedness on inhomogeneous Lipschitz spaces of fractional integrals, singular integrals and hypersingular integrals associated to non-doubling measures on metric spaces, Collect. Math. 60 (2009) 101–114.
[11] T. Iwaniec, L^p-Theory of Quasiregular Mappings, Lecture Notes in Math., vol. 1508, Springer, Berlin, 1992.
[12] T. Iwaniec, The best constant in a BMO-inequality for the Beurling–Ahlfors transform, Mich. Math. J. 34 (1987) 407–434.
[13] J.A. Johnson, Banach spaces of Lipschitz functions and vector-valued Lipschitz functions, Trans. Amer. Math. Soc. 148 (1970) 147–169.
[14] M.T. Lacey, E.T. Sawyer, I. Uriarte-Tuero, Astala's conjecture on distortion of Hausdorff measures under quasiconformal maps in the plane, arXiv:0805.4711.
[15] O. Lehto, K.I. Virtanen, Quasiconformal Mappings in the Plane, second ed., Springer-Verlag, New York, 1973.
[16] Y.Y. Li, L. Nirenberg, Estimates for elliptic systems from composite material, Comm. Pure Appl. Math. 56 (7) (2003) 892–925.
[17] Y.Y. Li, M.S. Vogelius, Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients, Arch. Rational Mech. Anal. 153 (2) (2000) 91–151.
[18] C.B. Morrey, On the solutions of quasi-linear elliptic partial differential equations, Trans. Amer. Math. Soc. 43 (1) (1938) 126–166.
[19] J. Mateu, J. Orobitg, J. Verdera, Estimates for the maximal singular integral in terms of the singular integral: The case of even kernels, arXiv:0707.4610v1.
[20] J. Mateu, J. Verdera, L^p and weak L^1 estimates for the maximal Riesz transform and the maximal Beurling transform, Math. Res. Lett. 13 (5–6) (2006) 957–966.
[21] S. Rohde, Bi-Lipschitz maps and the modulus of rings, Ann. Acad. Sci. Fenn. Math. 22 (2) (1997) 465–474.
[22] M. Schechter, Principles of Functional Analysis, student ed., Academic Press/Harcourt Brace Jovanovich Publishers, New York/London, 1973.
[23] E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, Princeton, 1970.
[24] X. Tolsa, Bilipschitz maps, analytic capacity, and the Cauchy integral, Ann. of Math. 162 (2005) 1243–1304.
[25] J. Verdera, L^2 boundedness of the Cauchy integral and Menger curvature, Contemp. Math. 277 (2001) 139–158.
[26] R. Wittmann, Application of a theorem of M.G. Krein to singular integrals, Trans. Amer. Math. Soc. 299 (2) (1987) 581–599.

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