Capacities associated with scalar signed Riesz kernels, and analytic capacity

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Abstract

Analytic capacity is associated with the Cauchy kernel $1/z$ and the space L^∞. One has likewise capacities associated with the real and imaginary parts of the Cauchy kernel and L^∞. Striking results of Tolsa and a simple remark show that these three capacities are comparable. We present an extension of this fact to $\mathbb{R}^n$, $n \geq 3$, involving the vector valued Riesz kernel of homogeneity -1 and $n-1$ of its components.

1 Introduction

The analytic capacity of a compact subset E of the plane is defined by

$$\gamma(E) = \sup |f'(\infty)|$$

where the supremum is taken over those analytic functions on $\mathbb{C} \setminus E$ such that $|f(z)| \leq 1$, $z \in \mathbb{C} \setminus E$. Sets of zero analytic capacity are exactly the removable sets for bounded analytic functions, as it is easily seen, and thus $\gamma(E)$ quantifies the non-removability of E. Early work on analytic capacity used basically one complex variable methods (see, e.g., [A], [Ga1] and [Vi]). Analytic capacity may be written as

$$\gamma(E) = \sup |\langle T, 1 \rangle| \tag{1}$$

where the supremum is taken over all complex distributions T supported on E whose Cauchy potential $f = 1/z * T$ is in the closed unit ball of $L^\infty(\mathbb{C})$. The transition from f to T and viceversa is performed through the formulae $T = \dfrac{1}{\pi}\overline{\partial} f$ and $f = 1/z * T$.

Expression (1) shows that analytic capacity is formally an analogue of classical logarithmic capacity, in which the logarithmic kernel has been replaced by the complex kernel $1/z$. This suggests that real variables techniques could help in studying analytic capacity, in spite of the fact that the Cauchy kernel is complex. In fact, significant progress in the understanding of analytic capacity was achieved when real variables methods, in particular the Calderón-Zygmund theory of the Cauchy singular integral, were systematically used ([C], [Da], [MaMeV], [MTV], [T2] and [T4]). A striking result of Tolsa [T2] asserts that analytic capacity is comparable to a smaller quantity, called positive analytic capacity, which is defined on compact sets E by

$$\gamma_+(E) = \sup \mu(E)$$

where the supremum is taken over those positive measures supported on E whose Cauchy potential $1/z * \mu$ is in the closed unit ball of $L^\infty(\mathbb{C})$. In other words, there exists a positive constant C such that

$$\gamma(E) \leq C\, \gamma_+(E), \tag{2}$$

for each compact subset E of the plane. This implies, in particular, that analytic capacity is comparable to planar Lipschitz harmonic capacity. The Lipschitz harmonic capacity of a compact subset of $\mathbb{R}^n$ is defined by

$$\kappa(E) = \sup |\langle T, 1 \rangle| \tag{3}$$

where the supremum is taken over those real distributions T supported on E such that the vector field $\dfrac{x}{|x|^n} * T$ is in the unit ball of $L^\infty(\mathbb{R}^n, \mathbb{R}^n)$. The terminology stems from the fact that $\kappa(E)$ vanishes if and only if E is removable for harmonic functions on $\mathbb{R}^n \setminus E$ satisfying a global Lipschitz condition. Notice that the fact that analytic capacity and Lipschitz harmonic capacity in the plane are comparable cannot be deduced just by inspection from (1) and (3). The reason is that the distributions involved in the supremum in (1) are complex.

For a compact subset E of $\mathbb{R}^n$ and $1 \leq i \leq n$ set

$$\kappa_i(E) = \sup |\langle T, 1 \rangle| \tag{4}$$

where the supremum is taken over those real distributions T such that the scalar signed i-th Riesz potential

$$\frac{x_i}{|x|^2} * T \tag{5}$$

is in the unit ball of $L^\infty(\mathbb{R}^n)$.

In the plane, in spite of what has been said before, it is precisely a simple complex analytic argument that provides a complete characterization of the capacities κ_1 and κ_2. For some positive constant C and for each compact subset E of the plane, we have

$$C^{-1}\, \kappa_i(E) \leq \gamma(E) \leq C\, \kappa_i(E), \;\; i = 1, 2. \tag{6}$$

Indeed, if T is a real distribution supported on E such that $\dfrac{x_1}{|x|^2} * T$ is in the unit ball of $L^\infty(\mathbb{R}^2)$, then $\dfrac{1}{z} * T$ is an analytic function on $\mathbb{C} \setminus E$ whose real part is bounded in absolute value by 1. Mapping conformally the strip $\{z \in \mathbb{C} : |Re(z)| \leq 1\}$

onto the unit disk we get a function f, bounded and analytic on $\mathbb{C} \setminus E$, such that $|\langle T, 1\rangle| \leq C\,|f'(\infty)|$ ([G]). Hence $C^{-1}\,\kappa_i(E) \leq \gamma(E)$. The second inequality in (6) is an immediate consequence of the striking inequality (2), because the real part of the Cauchy potential of a positive measure μ is precisely $x_1/|x|^2 * \mu$. Notice that this is not the case if μ is a complex measure.

Although there are obvious formal similarities between the definitions of the set functions in (1) and (4), very little is known about κ_i for $n \geq 3$. The reader will find in section 6.3 a proof of the elementary fact that $\kappa_i(E)$ is finite for each compact subset E of $\mathbb{R}^n$. The reason why κ_i is difficult to understand in higher dimensions is that boundedness of the potential (5) does not provide any linear growth condition on T in dimensions $n \geq 2$ (then, even in dimension 2). Concretely, it is not true that boundedness of (5) implies that for each cube Q one has

$$|\langle T, \varphi_Q\rangle| \leq C\,l(Q), \tag{7}$$

for each test function $\varphi_Q \in \mathcal{C}_0^\infty(Q)$ satisfying $\|\partial^s \varphi_Q\|_\infty \leq l(Q)^{-|s|}$ for all multi-indexes s of length not greater than some positive integer N. Here $l(Q)$ stands for the side length of Q and we are adopting the standard notation related to multi-indexes, that is, $s = (s_1, \ldots, s_n)$, where each coordinate s_j is a non-negative integer and $|s| = s_1 + \cdots + s_n$ is the length of s. The reader will find in section 5 three exemples of such phenomenon. The fact that this examples exist also in dimension 2, makes the first inequality in (6) very surprising. Indeed, the natural conjecture that the capacities κ_i, $1 \leq i \leq n$, $n \geq 3$ are semiadditive seems presently completely out of reach. The reason is that one should develop real variables techniques which replace the simple minded but extremely powerful complex variable argument described above.

On the other hand, recall that if T is a compactly supported distribution with bounded Cauchy potential then

$$\begin{aligned} |\langle T, \varphi_Q\rangle| &= \left|\left\langle T, \frac{1}{\pi z} * \overline{\partial}\varphi_Q \right\rangle\right| = \left|\left\langle \frac{1}{\pi z} * T, \overline{\partial}\varphi_Q \right\rangle\right| \\ &\leq \frac{1}{\pi}\left\|\frac{1}{z} * T\right\|_\infty \|\overline{\partial}\varphi_Q\|_{L^1(Q)} \leq \frac{1}{\pi}\left\|\frac{1}{z} * T\right\|_\infty l(Q), \end{aligned} \tag{8}$$

whenever φ_Q is normalized by $\|\overline{\partial}\varphi_Q\|_{L^1(Q)} \leq l(Q)$. The preceding argument extends to $\mathbb{R}^n$ for even dimensions $n = 2N$ as follows. A standard Fourier transform computation shows that, for some constant c_n and each test function φ, one has

$$\varphi = c_n \sum_{j=1}^n \frac{x_j}{|x|^2} * \partial_j(\triangle^{N-1})\varphi \equiv c_n\,\frac{x}{|x|^2} * \nabla(\triangle^{N-1})\varphi. \tag{9}$$

Let T be a compactly supported real distribution with bounded vector valued Riesz potential $x/|x|^2 * T$ and let φ_Q a function in $\mathcal{C}^{n-1}(Q)$. Then

$$\begin{aligned} |\langle T, \varphi_Q\rangle| &= \left|\left\langle T, c_n \frac{x}{|x|^2} * \nabla(\triangle^{N-1})\varphi_Q\right\rangle\right| = \left|\left\langle c_n \frac{x}{|x|^2} * T, \nabla(\triangle^{N-1})\varphi_Q\right\rangle\right| \\ &\leq C \, \|\frac{x}{|x|^2} * T\|_\infty \, \|\nabla^{n-1}\varphi_Q\|_{L^1(Q)} \leq C \, \|\frac{x}{|x|^2} * T\|_\infty \, l(Q), \end{aligned} \tag{10}$$

provided φ_Q is normalized by $\|\nabla^{n-1}\varphi_Q\|_{L^1(Q)} \leq l(Q)$. Here we are adopting the standard convention of denoting by $\nabla^m \varphi$ the vector $(\partial^s \varphi)_{|s|=m}$ of all m-th order partial derivatives of φ and by $|\nabla^m \varphi|$ its Euclidean norm.

For odd dimensions one has to require a stronger normalization condition. The first remark is that (9) can be rewritten as

$$\varphi = c_n \frac{x}{|x|^2} * \nabla(-\Delta)^{(n-2)/2}\varphi, \tag{11}$$

which makes sense for all dimensions. Since

$$(-\Delta)^{\frac{1}{2}}\varphi = d_n \sum_{j=1}^{n} R_j \partial_j \varphi$$

for some dimensional constant d_n, the R_j being the Riesz transforms (the Calderón-Zygmund operators with Fourier multiplier $\xi_j/|\xi|$), we have

$$\nabla(-\Delta)^{(n-2)/2}\varphi = c_n \left(\partial_j \left((\sum_{k=1}^{n} R_k \partial_k)^{n-2} \varphi \right)\right)_{j=1}^{n}.$$

Each component of the vector in the right hand side above is a sum of terms of the form $T\,\partial^s \varphi$, where s is a multi-index of length $n-1$ and T is a product of $n-2$ Riesz transforms. Hence, denoting by $\|\cdot\|_{H^1(\mathbb{R}^n)}$ the norm of the real Hardy space $H^1(\mathbb{R}^n)$, we get

$$\|\nabla(-\Delta)^{(n-2)/2}\varphi\|_{L^1(\mathbb{R}^n)} \leq C \, \|\nabla^{n-1}\varphi\|_{H^1(\mathbb{R}^n)}, \tag{12}$$

where we have set

$$\|\nabla^{n-1}\varphi\|_{H^1(\mathbb{R}^n)} = \sum_{|s|=n-1} \|\partial^s \varphi\|_{H^1(\mathbb{R}^n)}.$$

Recall that a function $f \in H^1(\mathbb{R}^n)$ if and only if $f \in L^1(\mathbb{R}^n)$ and all its Riesz transforms are also in $L^1(\mathbb{R}^n)$. The norm of f in $H^1(\mathbb{R}^n)$ is defined as

$$\|f\|_{H^1(\mathbb{R}^n)} = \|f\|_{L^1(\mathbb{R}^n)} + \sum_{j=1}^{n} \|R_j(f)\|_{L^1(\mathbb{R}^n)}.$$

A basic result is that the Riesz transforms send continuously $H^1(\mathbb{R}^n)$ into itself, and this is what we used in (12).

For even dimensions, as we have seen before, the Riesz transforms disappear from the reproducing formula (11) and we get the better estimate

$$\|\nabla(-\Delta)^{(n-2)/2}\varphi\|_{L^1(\mathbb{R}^n)} \le C\,\|\nabla^{n-1}\varphi\|_{L^1(\mathbb{R}^n)}.$$

This accounts for the difference between even and odd dimensions.

Let T be a compactly supported real distribution with bounded vector valued Riesz potential $x/|x|^2 * T$ and let φ_Q a function in $C^{n-1}(Q)$. Therefore

$$\begin{aligned}|\langle T,\varphi_Q\rangle| &= \left|\left\langle T, c_n\,\frac{x}{|x|^2} * \nabla(-\Delta)^{(n-2)/2}\varphi_Q\right\rangle\right| = \left|\left\langle c_n\,\frac{x}{|x|^2} * T, \nabla(-\Delta)^{(n-2)/2}\varphi_Q\right\rangle\right| \\ &\le C\,\|\frac{x}{|x|^2} * T\|_\infty\,\|\nabla^{n-1}\varphi_Q\|_{H^1(\mathbb{R}^n)} \le C\,\|\frac{x}{|x|^2} * T\|_\infty\,l(Q),\end{aligned} \tag{13}$$

provided φ_Q is normalized by $\|\nabla^{n-1}\varphi_Q\|_{H^1(\mathbb{R}^n)} \le l(Q)$.

We say that a distribution T has linear growth if

$$G(T) = \sup_{\varphi_Q} \frac{|\langle T,\varphi_Q\rangle|}{l(Q)} < \infty, \tag{14}$$

where the supremum is taken over all $\varphi_Q \in \mathcal{C}_0^\infty(Q)$ satisfying the normalization inequalities

$$\|\partial^s\varphi_Q\|_{H^1(\mathbb{R}^n)} \le l(Q), \quad |s| = n-1. \tag{15}$$

Notice that no distinction has been made between even or odd dimensions in the preceding definition and that we have chosen the stronger Hardy space normalization. This is due to the fact that, since we will assume in our main result that the distributions we deal with satisfy the linear growth condition, the stronger the normalization we require the weaker the assumption we get.

The normalization in the H^1 norm is the right condition to impose, as will become clear later on. For positive Radon measures μ in $\mathbb{R}^n$ the preceding notion of linear growth is equivalent to the usual one (see (20) below). In subsection 6.5 complete details on this fact are provided.

For a compact set E in $\mathbb{R}^n$ we define $g(E)$ as the set of all distributions supported on E having linear growth with constant $G(T)$ at most 1.

Our main result is a higher dimensional version of (6). For a compact $E \subset \mathbb{R}^n$ set

$$\Gamma(E) = \sup\{|\langle T,1\rangle|\}$$

where the supremum is taken over those real distributions T supported on E such that the vector field $\dfrac{x}{|x|^n} * T$ is in the unit ball of $L^\infty(\mathbb{R}^n,\mathbb{R}^n)$. Hence $\Gamma(E) = \kappa(E)$ for $n=2$. Finally, for $1 \le k \le n$, set

$$\Gamma_{\hat{k}}(E) = \sup\left\{|\langle T,1\rangle| : T \in g(E) \quad \text{and} \quad \left\|\frac{x_i}{|x|^2} * T\right\|_\infty \le 1,\, 1 \le i \le n,\, i \ne k\right\}.$$

Thus we require the boundedness of $n-1$ components of the vector valued potential $x/|x|^2 * T$ with Riesz kernel of homogeneity -1.

The requirement of the growth condition in the preceding definition is vital in obtaining the localization result (24). In subsection 6.4 we show that a growth condition is necessary for a localization estimate in L^∞.
Our extension of (6) to $\mathbb{R}^n$ is the following.

Theorem. *There exists a positive constant C such that for each compact set $E \subset \mathbb{R}^n$ and $1 \le k \le n$*

$$C^{-1}\,\Gamma_{\hat{k}}(E) \le \Gamma(E) \le C\,\Gamma_{\hat{k}}(E). \tag{16}$$

The second inequality in (16) follows immediately from the definitions of Γ and $\Gamma_{\hat{k}}$, because any real distribution T with bounded vector valued Riesz potential has linear growth as shown in (10) and (13).

The paper is organized as follows. In section 2 we present a sketch of the proof of the Theorem. It becomes clear that the proof depends on two facts: the close relationship between the quantities one obtains after symmetrization of the kernels $x/|x|^2$ and $x_i/|x|^2$ and a localization L^∞ estimate for the scalar kernels $x_i/|x|^2$. In section 3 we deal with the symmetrization issue and in section 4 with the localization estimate. In section 5 we discuss three examples showing that boundedness of $(n-1)$-scalar signed Riesz potentials $x_i/|x|^2 * T$ does not imply a linear growth estimate on T. In section 6 we present various additional results and examples. We show that $\kappa_i(E)$ is finite for each compact E. We present counter-examples to two natural inequalities. The first shows that the obvious extension of the Theorem to the vector valued Riesz kernels $x/|x|^{1+\alpha}$ and scalar kernels $x_i/|x|^{1+\alpha}$ of homogeneity α, $0 < \alpha < 1$, fails. The second counter-example shows that the obvious extension of (6) to kernels of homogeneity $-d$, where d is an integer greater than 1, also fails. Finally we point out that a growth condition is necessary to have localization inequalities in L^∞.

Our notation and terminology are standard. For instance, $\mathcal{C}_0^m(E)$, $0 \le m \le \infty$, denotes the set of all functions with compact support contained in the set E and with continuous partial derivatives up to order m. Cubes will always be supposed to have sides parallel to the coordinate axis, $l(Q)$ is the side length of the cube Q and $|Q| = l(Q)^n$ its volume. A good reference for the theory of the real Hardy space $H^1(\mathbb{R}^n)$ is [St2, Chapters 3 and 4].

We remind the reader that the convolution of two distributions T and S is well defined if T has compact support. In this case the action of $T * S$ on the test function φ is

$$\langle T * S, \varphi\rangle = \langle T, S * \varphi\rangle,$$

which makes sense because $S * \varphi$ is an infinitely differentiable function on $\mathbb{R}^n$.

2 Sketch of the proof of the Theorem

As we remarked before, one only has to prove that

$$\Gamma_{\hat{k}}(E) \le C\,\Gamma(E). \tag{17}$$

Clearly $\Gamma(E)$ is larger than or equal to

$$\Gamma_+(E) = \sup \mu(E) \tag{18}$$

where the supremum is taken over those positive measures μ supported on E whose vector valued Riesz potential $x/|x|^2 * \mu$ lies in the closed unit ball of $L^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Now, $\Gamma_+(E)$ is comparable to yet another quantity $\Gamma_{\text{op}}(E)$, that is, for some positive constant C one has

$$C^{-1}\,\Gamma_{\text{op}}(E) \le \Gamma_+(E) \le C\,\Gamma_{\text{op}}(E), \tag{19}$$

for each compact set $E \subset \mathbb{R}^n$ (see [T1]). Before giving the definition of $\Gamma_{\text{op}}(E)$ we need to introduce the Riesz transform with respect to an underlying positive Radon measure μ satisfying the linear growth condition

$$\mu(B(x,r)) \le C\,r, \quad x \in \mathbb{R}^n, \quad r \ge 0. \tag{20}$$

Given $\epsilon > 0$ we define the truncated Riesz transform at level ϵ as

$$R_\epsilon(f\,\mu)(x) = \int_{|y-x|>\epsilon} \frac{x-y}{|x-y|^2} f(y)\,d\mu(y), \quad x \in \mathbb{R}^n, \tag{21}$$

for $f \in L^2(\mu)$. The growth condition on μ insures that each R_ϵ is a bounded operator on $L^2(\mu)$ with operator norm $\|R_\epsilon\|_{L^2(\mu)}$ possibly depending on ϵ. We say that the Riesz transform is bounded on $L^2(\mu)$ when

$$\|R\|_{L^2(\mu)} = \sup_{\epsilon>0} \|R_\epsilon\|_{L^2(\mu)} < \infty,$$

or, in other words, when the truncated Riesz transforms are uniformly bounded on $L^2(\mu)$. Call $L(E)$ the set of positive Radon measures supported on E which satisfy (20) with $C = 1$. One defines $\Gamma_{\text{op}}(E)$ by

$$\Gamma_{\text{op}}(E) = \sup\{\mu(E) : \mu \in L(E) \quad \text{and} \quad \|R\|_{L^2(\mu)} \le 1\}.$$

From the first inequality in (19) we get that, for some constant C and all compact sets E,

$$\Gamma_{\text{op}}(E) \le C\,\Gamma(E).$$

We remind the reader that the first inequality in (19) depends on a simple but ingenious duality argument due to Davie and Oksendal (see [DO, p.139], [Ch, Theorem 23, p.107] and [V3, Lemma 4.2]). To prove (17) we have to estimate $\Gamma_{\hat{k}}(E)$ by a constant times $\Gamma_{\rm op}(E)$. The natural way to perform that is to introduce the quantity $\Gamma_{\hat{k},{\rm op}}(E)$ and try the two estimates

$$\Gamma_{\hat{k}}(E) \leq C\,\Gamma_{\hat{k},{\rm op}}(E) \tag{22}$$

and

$$\Gamma_{\hat{k},{\rm op}}(E) \leq C\,\Gamma_{\rm op}(E). \tag{23}$$

We define the truncated scalar Riesz transform $R^i_\varepsilon(f\,\mu)(x)$ associated with the i-th coordinate as in (21) with the vector valued Riesz kernel replaced by the scalar Riesz kernel $\dfrac{x_i - y_i}{|x-y|^2}$. We also set

$$\|R^i\|_{L^2(\mu)} = \sup_{\epsilon>0} \|R^i_\varepsilon\|_{L^2(\mu)},$$

and

$$\Gamma_{\hat{k},{\rm op}}(E) = \sup\{\mu(E) : \mu \in L(E) \quad \text{and} \quad \|R^i\|_{L^2(\mu)} \leq 1,\, 1 \leq i \leq n,\, i \neq k\}.$$

One proves (23) by checking that symmetrization of a scalar Riesz kernel is controlled by the symmetrization of the scalar Riesz kernels associated with all other variables. This result was known to Stephen Semmes many years ago [S]. Here the fact that we are dealing with kernels of homogeneity -1 plays a key role, because, as it is well-known, they enjoy a special positivity property which is missing in general. See section 3 for complete details. For other homogeneities, either the corresponding statements are false or open (see section 6).

The proof of (22) depends on Tolsa's approach to the proof of (2), which extends without any significant change to the higher dimensional setting to give

$$\Gamma(E) \leq C\,\Gamma_+(E).$$

The main technical point missing in our setting is a localization result for scalar Riesz potentials. This turns out to be a delicate issue, which we deal with in section 4. Specifically, we prove that there exists a positive constant C such that, for each compactly supported distribution T and for each coordinate i, we have

$$\left\| \frac{x_i}{|x|^2} * \varphi_Q T \right\|_\infty \leq C \left(\left\| \frac{x_i}{|x|^2} * T \right\|_\infty + G(T) \right) \tag{24}$$

for each cube Q and each $\varphi_Q \in \mathcal{C}_0^\infty(Q)$ satisfying $\|\partial^s \varphi_Q\|_\infty \leq l(Q)^{-|s|}$, $0 \leq |s| \leq n-1$.

This improves significantly a previous localization result in [MPrVe], which, in particular, yields

$$\left\| \frac{x}{|x|^2} * \varphi_Q T \right\|_\infty \leq C \left\| \frac{x}{|x|^2} * T \right\|_\infty, \tag{25}$$

for φ_Q as above. Inequality (24) implies (25) because boudedness of the vector valued potential $x/|x|^2 * T$ provides a growth condition on T. Indeed one has (see Lemma 3.2 in [Pr1] or (10) and (13))

$$G(T) \leq C \left\| \frac{x}{|x|^2} * T \right\|_\infty .$$

Once (24) is at our disposition Tolsa's machinery applies straightforwardly as was already explained in [MPrVe, Section 2.2]. However we will again describe the main steps in the proof of inequality (22) at the end of section 4.

3 Proof of $\Gamma_{\hat{k},\mathrm{op}}(E) \leq C\,\Gamma_{\mathrm{op}}(E)$

The symmetrization process for the Cauchy kernel introduced in [Me] has been succesfully applied to many problems of analytic capacity and L^2 boundedness of the Cauchy integral operator (see [MeV], [MaMeV] and the book [P], for example) and also to problems concerning the capacities, γ_α, $0 < \alpha < 1$, (which are related to the vector valued Riesz kernels $x/|x|^{1+\alpha}$) and the L^2 boundedness of the α-Riesz transforms (see [Pr1], [MPrVe], [Pr2] and [Pr3]). Given 3 distinct points in the plane, z_1, z_2 and z_3, one finds out, by an elementary computation that

$$c(z_1, z_2, z_3)^2 = \sum_\sigma \frac{1}{(z_{\sigma(1)} - z_{\sigma(3)})\overline{(z_{\sigma(2)} - z_{\sigma(3)})}} \tag{26}$$

where the sum is taken over the permutations of the set $\{1, 2, 3\}$ and $c(z_1, z_2, z_3)$ is *Menger curvature*, that is, the inverse of the radius of the circle through z_1, z_2 and z_3. In particular (26) shows that the sum on the right hand side is a non-negative quantity.

In $\mathbb{R}^n$ and for $1 \leq i \leq n$ the quantity

$$\sum_\sigma \frac{x^i_{\sigma(2)} - x^i_{\sigma(1)}}{|x_{\sigma(2)} - x_{\sigma(1)}|^2} \frac{x^i_{\sigma(3)} - x^i_{\sigma(1)}}{|x_{\sigma(3)} - x_{\sigma(1)}|^2} \tag{27}$$

where the sum is taken over the permutations of the set $\{1, 2, 3\}$, is the obvious analogue of the right hand side of (26) for the i-th coordinate of the Riesz kernel $x/|x|^2$. Notice that (27) is exactly

$$2\, p_i(x_1, x_2, x_3),$$

where $p_i(x_1, x_2, x_3)$ is defined as the sum in (27) taken only on the three permutations $(1, 2, 3)$, $(3, 1, 2)$ and $(2, 1, 3)$.

In Lemma 1, we will show that, given three distinct points $x_1, x_2, x_3 \in \mathbb{R}^n$, the quantity $p_i(x_1, x_2, x_3)$, $1 \leq i \leq n$, is also non-negative. We will use this remarkable fact to study the L^2 boundedness of the operators associated with the scalar Riesz kernels $x_i/|x|^2$.

The relationship between the quantity $p_i(x_1,x_2,x_3)$, $1\le i\le n$, and the L^2 estimates of the operator with kernel $x_i/|x|^2$ is as follows. Take a positive finite Radon measure μ in $\mathbb{R}^n$ with linear growth. Given $\varepsilon>0$ consider the truncated scalar Riesz transform $R^i_\varepsilon(\mu)$ of μ associated with the kernel $x_i/|x|^2$, as in section 2. Then we have (see in [MeV] the argument for the Cauchy integral operator)

$$\left|\int |R^i_\varepsilon(\mu)(x)|^2\,d\mu(x)-\frac{1}{3}p_{i,\varepsilon}(\mu)\right|\le C\|\mu\|, \tag{28}$$

C being a positive constant depending only on n, and

$$p_{i,\varepsilon}(\mu)=\iiint_{S_\varepsilon} p_i(x,y,z)\,d\mu(x)\,d\mu(y)\,d\mu(z),$$

with

$$S_\varepsilon=\{(x,y,z):|x-y|>\varepsilon,\ |x-z|>\varepsilon \text{ and } |y-z|>\varepsilon\}.$$

Lemma 1. *For $1\le i\le n$, and any three distinct points $x_1,x_2,x_3\in\mathbb{R}^n$ we have*

$$p_i(x_1,x_2,x_3)\ge 0.$$

Moreover,

1. *If $p_i(x_1,x_2,x_3)=0$ for $n-1$ values of $i\in\{1,2,\dots,n\}$, then x_1, x_2, x_3 are aligned.*
2. *If the three points x_1, x_2, x_3 are aligned, then $p_i(x_1,x_2,x_3)=0$ for $1\le i\le n$.*

Proof. Write $a=x_2-x_1$ and $b=x_3-x_2$. Then

$$\begin{aligned}p_i(x_1,x_2,x_3)&=\frac{a_i(a_i+b_i)|b|^2-b_ia_i|a+b|^2+b_i(a_i+b_i)|a|^2}{|a|^2|b|^2|a+b|^2}\\&=\frac{a_ib_i\left(-2\sum_{j=1}^n a_jb_j\right)+\sum_{j=1}^n a_i^2b_j^2+b_i^2a_j^2}{|a|^2|b|^2|a+b|^2}\\&=\frac{\sum_{j=1}^n(a_ib_j-b_ia_j)^2}{|a|^2|b|^2|a+b|^2}=\frac{\sum_{j:j\neq i}(a_ib_j-b_ia_j)^2}{|a|^2|b|^2|a+b|^2}\ge 0.\end{aligned}$$

Therefore, given three pairwise distinct points x_1, x_2, x_3, the permutations $p_i(x_1,x_2,x_3)$ vanish if and only if $a_ib_j=b_ia_j$ for all $1\le j\le n$.

Without loss of generality, assume that $p_i(x_1,x_2,x_3)=0$ for $1\le i\le n-1$. Then the following $n(n-1)/2$ conditions hold

$$a_ib_j=a_jb_i\quad 1\le i\le n-1,\quad i+1\le j\le n.$$

These conditions imply that $a = \lambda b$, for some $\lambda \in \mathbb{R}$, which means the three points x_1, x_2, x_3 lie on the same line.

Assume now that the three points are aligned. Without loss of generality set $x_1 = 0$, $x_2 = y$ and $x_3 = \lambda y$ for some $\lambda > 0$, and $y \in \mathbb{R}^n$. Then for $i, j \in \{1, 2, \dots, n\}$, we have

$$a_i b_j = y_i(\lambda - 1)y_j = (\lambda - 1)y_i y_j = b_i a_j,$$

hence $p_i(x_1, x_2, x_3) = 0$ for $1 \le i \le n$. □

If we are in the plane, then Menger curvature can be written as

$$c(x_1, x_2, x_3) = \frac{4A}{|x_1 - x_2||x_1 - x_3||x_3 - x_2|},$$

where A denotes the area of the triangle determined by the points x_1, x_2, x_3. A consequence of Lemma 1 and its proof is the following.

Corollary 2. *Given three different points $x_1, x_2, x_3 \in \mathbb{R}^2$, we have*

$$p_1(x_1, x_2, x_3) = p_2(x_1, x_2, x_3) = \frac{1}{4} c(x_1, x_2, x_3)^2.$$

Hence, the quantities $p_1(x_1, x_2, x_3)$ and $p_2(x_1, x_2, x_3)$ are non-negative, and vanish if and only if x_1, x_2, x_3 are aligned.

In the plane the singular Cauchy transform C with respect to the underlying measure μ may be written as $C(f\mu) = R^1(f\mu) - iR^2(f\mu)$. By Corollary 2 and the $T(1)$-Theorem , we see that C is bounded on $L^2(\mu)$ if and only if one of its real components, no matter which one, is bounded on $L^2(\mu)$. We state this, for emphasis, as a corollary.

Corollary 3. *If μ is a compactly supported positive measure in the plane having linear growth, the Cauchy transform of μ is bounded on $L^2(\mu)$ if and only if R^i is bounded on $L^2(\mu)$ for one $i \in \{1, 2\}$.*

For a positive measure μ with linear growth we have, by (28),

$$\begin{aligned}
\|R_\varepsilon(\mu)\|^2_{L^2(\mu)} &= \sum_{j=1}^{n} \int |R^j_\varepsilon(\mu)(x)|^2 \, d\mu(x) \\
&= \frac{1}{3} \sum_{j=1}^{n} \iiint\limits_{S_\varepsilon} p_j(x, y, z) \, d\mu(x) \, d\mu(y) \, d\mu(z) + O(\|\mu\|) \\
&\le \frac{2}{3} \sum_{\substack{j=1 \\ j \ne i}}^{n} \iiint\limits_{S_\varepsilon} p_j(x, y, z) \, d\mu(x) \, d\mu(y) \, d\mu(z) + O(\|\mu\|),
\end{aligned}$$

where the last inequality follows easily from the formula

$$p_i(x_1,x_2,x_3) = \frac{\sum_{j\neq i}\left((x_2^i-x_1^i)(x_3^j-x_2^j)-(x_2^j-x_1^j)(x_3^i-x_2^i)\right)^2}{|x_2-x_1|^2|x_3-x_2|^2|x_3-x_1|^2}, \quad 1\leq i\leq n.$$

The above estimate can be localized replacing μ by $\chi_B\mu$ for each ball B. Therefore, appealing to the $T(1)$-Theorem for non necessarily doubling measures [NTV1], if $n-1$ components R^j are bounded on $L^2(\mu)$ (no matter which $n-1$ components), then the whole vector valued operator R is bounded on $L^2(\mu)$.

Theorem 4. *Let μ be a non-negative measure with compact support in $\mathbb{R}^n$ and linear growth. Then the vector valued Riesz operator R associated with the kernel $x/|x|^2$ is bounded on $L^2(\mu)$ provided any set of $n-1$ components R^j of R are bounded on $L^2(\mu)$.*

The inequality (23) is an immediate consequence of Theorem 4.

4 Proof of $\Gamma_{\hat{k}}(E) \leq C\,\Gamma_{\hat{k},\mathrm{op}}(E)$

The proof of the inequality $\Gamma_{\hat{k}}(E) \leq C\,\Gamma_{\hat{k},\mathrm{op}}(E)$ is based in two ingredients, the localization of scalar Riesz potentials and the exterior regularity of $\Gamma_{\hat{k}}$, which we discuss below.

4.1 Localization of scalar Riesz potentials

When analyzing the argument for the proof of (2) (see Theorem 1.1 in [T2]) one realizes that one of the technical tools used is the fact that the Cauchy kernel $1/z$ localizes in the uniform norm. By this we mean that if T is a compactly supported distribution such that $1/z * T$ is a bounded measurable function, then $1/z*(\varphi T)$ is also bounded measurable for each compactly supported $\mathcal{C}^1$ function φ. This is an old result, which is simple to prove because $1/z$ is related to the differential operator $\overline{\partial}$ (see [Ga1, Chapter V]). The same localization result can be proved easily in any dimension for the kernel $x/|x|^n$, which is, modulo a multiplicative constant, the gradient of the fundamental solution of the Laplacian. Again the proof is reasonably straightforward because the kernel is related to a differential operator (see [Pa] and [V1]).

In [MPrVe, Lemma 3.1] we were concerned with the localization of the vector valued α-Riesz kernel $x/|x|^{1+\alpha}$, $0<\alpha<n$. For general values of α there is no differential operator in the background and consequently the corresponding localization result becomes far from obvious (see Lemma 3.1 in [MPrVe]).

We now state the new localization lemma we need.

Lemma 5. *Let T be a compactly supported distribution in $\mathbb{R}^n$, with linear growth, such that $(x_i/|x|^2) * T$ is in $L^\infty(\mathbb{R}^n)$ for some i, $1 \le i \le n$. Let Q be a cube and assume that $\varphi_Q \in \mathcal{C}_0^\infty(Q)$ satisfies $\|\partial^s \varphi_Q\|_\infty \le l(Q)^{-|s|}$, $0 \le |s| \le n-1$. Then $(x_i/|x|^2) * \varphi_Q T$ is in $L^\infty(\mathbb{R}^n)$ and*

$$\left\| \frac{x_i}{|x|^2} * \varphi_Q T \right\|_\infty \le C \left(\left\| \frac{x_i}{|x|^2} * T \right\|_\infty + G(T) \right),$$

for some positive constant $C = C(n)$ depending only on n.

With analogous techniques and replacing $G(T)$ by $G_\alpha(T)$ (see section 6 for a definition) one can prove that the above lemma also holds in $\mathbb{R}^n$ for the scalar α-Riesz potentials

$$\frac{x_i}{|x|^{1+\alpha}} * T, \quad 0 < \alpha < n, \ \alpha \in \mathbb{Z}.$$

For the proof of Lemma 5 we need the following.

Lemma 6. *Let T be a compactly supported distribution in $\mathbb{R}^n$ with linear growth and assume that Q is a cube and $\varphi_Q \in \mathcal{C}_0^\infty(Q)$ satisfies $\|\partial^s \varphi_Q\|_\infty \le l(Q)^{-|s|}$, $0 \le |s| \le n-1$. Then, for each coordinate i, the distribution $(x_i/|x|^2) * \varphi_Q T$ is a locally integrable function and there exists a point $x_0 \in \frac{1}{4}Q$ such that*

$$\left| \left(\frac{x_i}{|x|^2} * \varphi_Q T \right)(x_0) \right| \le C\, G(T),$$

where $C = C(n)$ is a positive constant depending only on n.

Remark. Since the function $f = (x_i/|x|^2) * \varphi_Q T$ is only locally integrable, it may look strange to evaluate f at a point. Indeed we show that the mean of f on $\dfrac{1}{4}Q$ is bounded by $C\, G(T)$ and then at many Lebesgue points of f the above stated inequality holds, doubling the constant if necessary.

Proof of Lemma 6. Without loss of generality set $i = 1$ and write $k^1(x) = x_1/|x|^2$. Since $k^1 * \varphi_Q T$ is infinitely differentiable off the closure of Q, we only need to show that $k^1 * \varphi_Q T$ is integrable on $2Q$. We will actually prove a stronger statement, namely, that $k^1 * \varphi_Q T$ is in $L^p(2Q)$ for each p in the interval $1 \le p < \frac{n}{n-1}$. Indeed, fix any q satisfying $n < q < \infty$ and call p the dual exponent, so that $1 < p < \frac{n}{n-1}$. We need to estimate the action of $k^1 * \varphi_Q T$ on functions $\psi \in \mathcal{C}_0^\infty(2Q)$ in terms of $\|\psi\|_q$. We clearly have

$$\langle k^1 * \varphi_Q T, \psi \rangle = \langle T, \varphi_Q (k^1 * \psi) \rangle.$$

We claim that, for an appropriate dimensional constant C, the test function

$$\frac{\varphi_Q (k^1 * \psi)}{C\, l(Q)^{\frac{n}{p}-1} \|\psi\|_q} \tag{29}$$

satisfies the normalization inequalities (15) in the definition of $G(T)$. Once this is proved, by the definition of $G(T)$ we get

$$|\langle k^1 * \varphi_Q T, \psi\rangle| \leq C\, l(Q)^{\frac{n}{p}} \|\psi\|_q\, G(T),$$

and so

$$\|k^1 * \varphi_Q T\|_{L^p(2Q)} \leq C\, l(Q)^{\frac{n}{p}} G(T).$$

Hence

$$\begin{aligned}
\frac{1}{|\frac{1}{4}Q|}\int_{\frac{1}{4}Q} |(k^1 * \varphi_Q T)(x)|\, dx &\leq 4^n \frac{1}{|Q|}\int_Q |(k^1 * \varphi_Q T)(x)|\, dx \\
&\leq 4^n \left(\frac{1}{|Q|}\int_Q |(k^1 * \varphi_Q T)(x)|^p\, dx\right)^{\frac{1}{p}} \\
&\leq C\, G(T),
\end{aligned}$$

which completes the proof of Lemma 6.

To prove the claim we need the following auxiliary remark. We let R_i stand for the Riesz transforms, that is, the Calderón-Zygmund operators with kernel $c_n\, x_i/|x|^{n+1}$ and multiplier $\xi_i/|\xi|$.

Sublemma. *Assume that f_Q is a test function supported on the square Q satisfying*

$$\|\partial^s f_Q\|_{L^1(Q)} \leq l(Q), \quad |s| = n-1,$$

and

$$\|R_i(\partial^s f_Q)\|_{L^1(2Q)} \leq Cl(Q). \tag{30}$$

Then

$$\|R_i(\partial^s f_Q)\|_{L^1(\mathbb{R}^n)} \leq Cl(Q) \quad \text{for } |s| = n-1 \quad \text{and } 1 \leq i \leq n.$$

Proof. For any multi-index s with $|s| = n-1$, integrating by parts to take one derivative on the Riesz kernel we obtain

$$\begin{aligned}
\|R_i(\partial^s f_Q)\|_{L^1((2Q)^c)} &= c_n \int_{(2Q)^c} |\int_Q \partial^s f_Q(z) \frac{z_i - y_i}{|z-y|^{n+1}}\, dz|\, dy \\
&\leq C\|\nabla^{n-2} f_Q\|_{L^1(Q)}\, l(Q)^{-1} \\
&\leq C\, \|\nabla^{n-2} f_Q\|_{L^{n/n-1}(Q)},
\end{aligned}$$

where the last estimate follows from Hölder's inequality.

A well known result of Maz'ya (see [Mz, 1.1.4, p. 15] and [Mz, 1.2.2, p. 24]) states that

$$\|\nabla^m f_Q\|_{\frac{n}{1+m}} \leq C \int |\nabla^{n-1} f_Q|, \quad 0 \leq m \leq n-1. \tag{31}$$

Applying this for $m = n-2$ we get

$$\begin{aligned}\|R_i(\partial^s f_Q)\|_{L^1((2Q)^c)} &\le C\|\nabla^{n-2} f_Q\|_{n/n-1}\\ &\le C\|\nabla^{n-1} f_Q\|_1 \le C\, l(Q).\end{aligned}$$

□

By the Sublemma, to prove the claim we only have to show that for $|s| = n-1$,

$$\|\partial^s\left(\varphi_Q\,(k^1 * \psi)\right)\|_{L^1(Q)} \le C\, l(Q)^{\frac{n}{p}}\|\psi\|_q. \tag{32}$$

and

$$\|R_i(\partial^s(\varphi_Q(k^1 * \psi)))\|_{L^1(2Q)} \le C\, l(Q)^{\frac{n}{p}}\|\psi\|_q, \quad 1 \le i \le n. \tag{33}$$

By Hölder's inequality and the fact that the Riesz transforms preserve $L^q(\mathbb{R}^n)$, $1 < q < \infty$, we get

$$\begin{aligned}\|R_i(\partial^s(\varphi_Q(k^1 * \psi)))\|_{L^1(2Q)} &\le Cl(Q)^{\frac{n}{p}}\|R_i(\partial^s(\varphi_Q(k^1 * \psi)))\|_{L^q(\mathbb{R}^n)}\\ &\le Cl(Q)^{\frac{n}{p}}\|\partial^s(\varphi_Q(k^1 * \psi))\|_{L^q(Q)}.\end{aligned}$$

Hence (33) and (32) follow from

$$\|\partial^s(\varphi_Q(k^1 * \psi))\|_{L^q(Q)} \le C\|\psi\|_q. \tag{34}$$

By Leibnitz formula

$$\begin{aligned}\partial^s\left(\varphi_Q\,(k^1 * \psi)\right) &= \varphi_Q\,\partial^s(k^1 * \psi) + \sum_{|r|=1}^{n-1} c_{s,r}\,\partial^r\varphi_Q\,\partial^{s-r}(k^1 * \psi)\\ &\equiv A + B,\end{aligned}$$

where the last identity is a definition of A and B.

To estimate the L^q-norm of the function in B we remark that, since $|s| = n-1$,

$$|\partial^{s-r}k^1(x)| \le C\,|x|^{-(n-|r|)}, \quad 1 \le |r| \le n-1,$$

and then, by Hölder's inequality and $\|\partial^r\varphi_Q\|_\infty \le l(Q)^{-|r|}$, $1 \le r \le n-1$, we see that

$$\begin{aligned}\|\partial^r\varphi_Q\,\partial^{s-r}(k^1 * \psi)\|_{L^q(Q)} &\le C\,\|\partial^r\varphi_Q\|_\infty\left(\int_Q\left(\int_{2Q}\frac{|\psi(y)|}{|x-y|^{n-|r|}}dy\right)^q dx\right)^{1/q}\\ &\le C\|\partial^r\varphi_Q\|_\infty\|\psi\|_q\left(\int_Q\left(\int_{2Q}\frac{dy}{|y-x|^{p(n-|r|)}}\right)^{\frac{q}{p}} dx\right)^{\frac{1}{q}}\\ &\le C\,l(Q)^{-|r|}\|\psi\|_q l(Q)^{\frac{n}{q}} l(Q)^{\frac{n}{p}-n+|r|}\\ &= C\|\psi\|_q,\end{aligned}$$

for each $1 \leq |r| \leq n-1$. We therefore conclude that

$$\|B\|_q \leq C \sum_{|r|=1}^{n-1} \|\partial^r \varphi_Q \, \partial^{s-r}(k^1 * \psi)\|_q \leq C \, \|\psi\|_q.$$

We turn now to the term A. We remark that, for $|s| = n-1$,

$$\partial^s k^1 * \psi = c\,\psi + S(\psi), \tag{35}$$

where S is a smooth homogeneous convolution Calderón-Zygmund operator and c a constant depending on s. This can be seen by computing the Fourier transform of $\partial^s k^1$ and then using that each homogeneous polynomial can be decomposed in terms of homogeneous harmonic polynomials of lower degrees (see [St, 3.1.2 p. 69]). Since Calderón-Zygmund operators preserve $L^q(\mathbb{R}^n)$, $1 < q < \infty$, we get, using that $\|\varphi_Q\|_\infty \leq C$,

$$\|A\|_q \leq C \, \|\psi\|_q.$$

This finishes the estimate of term A and the proof of (34). $\square$

Proof of Lemma 5. Without loss of generality take $i = 1$. Consider first a point $x \in \mathbb{R}^n \setminus \frac{3}{2}Q$. Then $k^1(x-y)\,\varphi_Q(y)$ is in $\mathcal{C}_0^\infty(Q)$ as a function of y. We claim that $c\, l(Q)\, k^1(x-y)\,\varphi_Q(y)$ satisfies the normalization condition (15) for some small constant c depending only on n. Once the claim is proved we get

$$|(k^1 * \varphi_Q T)(x)| = |\langle T, k^1(x-y)\,\varphi_Q(y)\rangle| \leq c^{-1}\, G(T).$$

Straightforward estimates yield

$$|\partial_y^s(k^1(x-y)\,\varphi_Q(y))| \leq C\, l(Q)^{-n}, \quad |s| = n-1,$$

which shows that $\partial_y^s(k^1(x-y)\,\varphi_Q(y))$ is a constant multiple of an atom, whence the claim.

We are then left with the case $x \in \frac{3}{2}Q$. Since $k^1 * T$ and φ_Q are bounded functions, we can write

$$|(k^1 * \varphi_Q T)(x)| \leq |(k^1 * \varphi_Q T)(x) - \varphi_Q(x)(k^1 * T)(x)| + \|\varphi_Q\|_\infty \|k^1 * T\|_\infty.$$

Let $\psi_Q \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be such that $\psi_Q \equiv 1$ in $2Q$, $\psi_Q \equiv 0$ in $(4Q)^c$ and $\|\partial^s \psi_Q\|_\infty \leq C_s\, l(Q)^{-|s|}$, for each multi-index s. Then one is tempted to write

$$|(k^1 * \varphi_Q T)(x) - \varphi_Q(x)(k^1 * T)(x)| \leq |\langle T, \psi_Q(y)(\varphi_Q(y) - \varphi_Q(x))k^1(x-y)\rangle| + \|\varphi_Q\|_\infty |\langle T, (1-\psi_Q(y))k^1(x-y)\rangle|.$$

The problem is that the first term in the right hand side above does not make any sense because T is acting on a function of y which is not necessarily differentiable at the point x. To overcome this difficulty one needs to resort to a standard regularization process. Take $\chi \in \mathcal{C}^\infty(B(0,1))$ such that $\int \chi(x)\,dx = 1$ and set $\chi_\varepsilon(x) = \varepsilon^{-n}\,\chi(x/\varepsilon)$. The plan is to estimate, uniformly on x and ϵ,

$$|(\chi_\varepsilon * k^1 * \varphi_Q T)(x) - \varphi_Q(x)(\chi_\varepsilon * k^1 * T)(x)|. \tag{36}$$

Clearly (36) tends, as ε tends to zero, to

$$|(k^1 * \varphi_Q T)(x) - \varphi_Q(x)(k^1 * T)(x)|,$$

for almost all $x \in \mathbb{R}^n$, which allows the transfer of uniform estimates. We now have

$$\begin{aligned}
|(\chi_\varepsilon * k^1 * \varphi_Q T)(x) &- \varphi_Q(x)(\chi_\varepsilon * k^1 * T)(x)| \\
&\le |\langle T, \psi_Q(y)(\varphi_Q(y) - \varphi_Q(x))(\chi_\varepsilon * k^1)(x-y)\rangle| \\
&\quad + \|\varphi_Q\|_\infty |\langle T, (1-\psi_Q(y))(\chi_\varepsilon * k^1)(x-y)\rangle| \\
&= A_1 + A_2,
\end{aligned}$$

where the last identity is the definition of A_1 and A_2. To deal with term A_1 set

$$k_\varepsilon^{1,x}(y) = (\chi_\varepsilon * k^1)(x-y).$$

We claim that, for an appropriate small dimensional constant c, the test function

$$f_Q(y) = c\, l(Q)\psi_Q(y)(\varphi_Q(y) - \varphi_Q(x))k_\varepsilon^{1,x}(y),$$

satisfies the normalization inequalities (15) in the definition of $G(T)$, with φ_Q replaced by f_Q and Q by $4Q$. If this is the case, then

$$A_1 \le c^{-1} l(Q)^{-1} |\langle T, f_Q\rangle| \le C\, G(T).$$

To prove the normalization inequalities (15) for the function f_Q it is enough, by the Sublemma, to show that the following holds

$$\|\partial^s f_Q\|_{L^1(4\,Q)} \le C l(Q) \tag{37}$$

$$\|R_i(\partial^s f_Q)\|_{L^1(8\,Q)} \le C l(Q), \tag{38}$$

for $1 \le i \le n$ and $|s| = n-1$.

For each $q > 1$ let p be its dual exponent. By Hölder's inequality we have

$$\|R_i(\partial^s f_Q)\|_{L^1(8Q)} \le C l(Q)^{\frac{n}{p}} \|R_i(\partial^s f_Q)\|_{L^q(\mathbb{R}^n)} \le C l(Q)^{\frac{n}{p}} \|\partial^s f_Q\|_{L^q(4Q)},$$

because the Riesz transforms preserve $L^q(\mathbb{R}^n)$. Therefore, (37) and (38) are a consequence of

$$\|\partial^s f_Q\|_{L^q(4Q)} \le C l(Q)^{\frac{n}{q} - |s|}, \;\; |s| = n-1. \tag{39}$$

To prove (39) we first notice that the regularized kernel $\chi_\varepsilon * k^1$ satisfies the inequalities

$$|(\chi_\varepsilon * \partial^s k^1)(x)| \le \frac{C}{|x|^{1+|s|}}, \quad x \in \mathbb{R}^n \setminus \{0\} \quad \text{and} \quad 0 \le |s| < n-1, \tag{40}$$

where C is a dimensional constant, which, in particular, is independent of ϵ. This can be proved by standard estimates which we omit. For $|s| = n-1$ the situation is a little bit more complicated. By (35) we have

$$(\chi_\varepsilon * \partial^s k^1)(x) = c\, \chi_\varepsilon(x) + (\chi_\varepsilon * S)(x),$$

where S is a smooth homogeneous convolution Calderón-Zygmund operator. As such, its kernel H satisfies the usual growth condition $|H(x)| \le C/|x|^n$. From this is not difficult to show that

$$|(\chi_\varepsilon * S)(x)| \le \frac{C}{|x|^n}, \quad x \in \mathbb{R}^n \setminus \{0\}, \tag{41}$$

for a dimensional constant C.

By Leibnitz formula, for $|s| = n-1$,

$$\begin{aligned}\partial^s \left(\psi_Q(\varphi_Q - \varphi_Q(x)) k_\varepsilon^{1,x}\right) &= \psi_Q\,(\varphi_Q - \varphi_Q(x)) \partial^s\, k_\varepsilon^{1,x} \\ &\quad + \sum_{|r|=1}^{n-1} c_{r,s}\, \partial^r (\psi_Q(\varphi_Q - \varphi_Q(x)))\, \partial^{s-r}\, k_\varepsilon^{1,x},\end{aligned} \tag{42}$$

and so

$$\begin{aligned}\|\partial^s f_Q\|_{L^q(4Q)} &\le C l(Q) \left(\int_{4Q} |\psi_Q(y)\,(\varphi_Q(y) - \varphi_Q(x))\, \partial^s k_\varepsilon^{1,x}(y)|^q\, dy\right)^{\frac{1}{q}} \\ &\quad + C l(Q) \sum_{|r|=1}^{n-1} \left(\int_{4Q} |\partial^r\, (\psi_Q(\varphi_Q - \varphi_Q(x))\ \partial^{s-r} k_\varepsilon^{1,x}(y)|^q\, dy\right)^{\frac{1}{q}} \\ &= A_{11} + A_{12}.\end{aligned}$$

Using (40) one obtains

$$\begin{aligned}A_{12} &\le C l(Q) \sum_{|r|=1}^{n-1} \frac{1}{l(Q)^{|r|}} \left(\int_{4Q} |(\partial^{s-r} k_\varepsilon^{1,x})(y)|^q\, dy\right)^{\frac{1}{q}} \\ &\le C l(Q)^{\frac{n}{q} - |s|}.\end{aligned}$$

To estimate A_{11} we resort to (41), which yields

$$
\begin{aligned}
A_{11} &= Cl(Q) \left(\int_{4Q} |\psi_Q(y)(\varphi_Q(y) - \varphi_Q(x)) \partial^s k_\varepsilon^{1,x}(y)|^q \, dy \right)^{\frac{1}{q}} \\
&\le Cl(Q) \|\partial \varphi_Q\|_\infty \left(\int_{4Q} \frac{dy}{|y-x|^{q(n-1)}} \, dy \right)^{\frac{1}{q}} \\
&\le Cl(Q)^{\frac{n}{q} - |s|}
\end{aligned}
$$

We now turn to A_2. By Lemma 6, there exists a point $x_0 \in Q$ such that $|(k^1 * \psi_Q T)(x_0)| \le C\, G(T)$. Then

$$
|(k^1 * (1-\psi_Q)T)(x_0)| \le C\, (\|k^1 * T\|_\infty + G(T)).
$$

The analogous inequality holds as well for the regularized potentials appearing in A_2, uniformly in ϵ, and therefore

$$
A_2 \le C\, |\langle T, (1-\psi_Q)(k_\varepsilon^{1,x} - k_\varepsilon^{1,x_0}) \rangle| + C\, (\|k^1 * T\|_\infty + G(T)).
$$

To estimate $|\langle T, (1-\psi_Q)(k_\varepsilon^{1,x} - k_\varepsilon^{1,x_0}) \rangle|$, we decompose $\mathbb{R}^n \setminus \{x\}$ into a union of rings

$$
N_j = \{ z \in \mathbb{R}^n : 2^j\, l(Q) \le |z - x| \le 2^{j+1}\, l(Q) \}, \quad j \in \mathbb{Z},
$$

and consider functions φ_j in $\mathcal{C}_0^\infty(\mathbb{R}^n)$, with support contained in

$$
N_j^* = \{ z \in \mathbb{R}^n : 2^{j-1}\, l(Q) \le |z - x| \le 2^{j+2}\, l(Q) \}, \quad j \in \mathbb{Z},
$$

such that $\|\partial^s \varphi_j\|_\infty \le C\, (2^j\, l(Q))^{-|s|}$, $|s| \ge 0$, and $\sum_j \varphi_j = 1$ on $\mathbb{R}^n \setminus \{x\}$. Since $x \in \frac{3}{2}Q$ the smallest ring N_j^* that intersects $(2Q)^c$ is N_{-3}^*. Therefore we have

$$
\begin{aligned}
|\langle T, (1-\psi_Q)(k_\varepsilon^{1,x} - k_\varepsilon^{1,x_0}) \rangle| &= \left| \left\langle T, \sum_{j \ge -3} \varphi_j (1-\psi_Q)(k_\varepsilon^{1,x} - k_\varepsilon^{1,x_0}) \right\rangle \right| \\
&\le \left| \left\langle T, \sum_{j \in I} \varphi_j (1-\psi_Q)(k_\varepsilon^{1,x} - k_\varepsilon^{1,x_0}) \right\rangle \right| \\
&\quad + \sum_{j \in J} |\langle T, \varphi_j (k_\varepsilon^{1,x} - k_\varepsilon^{1,x_0}) \rangle|,
\end{aligned}
$$

where I denotes the set of indices $j \ge -3$ such that the support of φ_j intersects $4Q$ and J the remaining indices, namely those $j \ge -3$ such that φ_j vanishes on $4Q$. Notice that the cardinality of I is bounded by a dimensional constant.

Set

$$
g = C\, l(Q) \sum_{j \in I} \varphi_j (1-\psi_Q)\, (k_\varepsilon^{1,x} - k_\varepsilon^{1,x_0}),
$$

and for $j \in J$

$$g_j = C\, 2^{2j}\, l(Q)\, \varphi_j\, (k_\varepsilon^{1,x} - k_\varepsilon^{1,x_0}).$$

We show now that the test functions g and g_j, $j \in J$, satisfy the normalization inequalities (15) in the definition of $G(T)$ for an appropriate choice of the (small) constant C . Once this is available, using the linear growth condition of T we obtain

$$\begin{aligned} |\langle T, (1-\psi_Q)(k_\varepsilon^{1,x} - k_\varepsilon^{1,x_0})\rangle| &\le C l(Q)^{-1} |\langle T, g\rangle| \\ &\quad + C \sum_{j\in J} (2^{2j} l(Q))^{-1} |\langle T, g_j\rangle| \\ &\le C\, G(T) + C \sum_{j\ge -3} 2^{-j}\, G(T) \le C\, G(T), \end{aligned}$$

which completes the proof of Lemma 5.

Checking the normalization inequalities for g and g_j is easy. First notice that the support of g is contained in a square $\lambda\, Q$ for some dilation factor λ depending only on n. On the other hand the support of g_j is conained in $2^{j+2}\, Q$. By the Sublemma, we have to show that for $|s| = n-1$, $1 \le i \le n$,

$$\|\partial^s g\|_{L^1(\lambda\, Q)} \le C l(Q), \quad \|R_i(\partial^s g)\|_{L^1(2\lambda\, Q)} \le C l(Q) \tag{43}$$

and for $1 \le j \le n$,

$$\|\partial^s g_j\|_{L^1(2^{j+2}\, Q)} \le C 2^j l(Q), \quad \|R_i(\partial^s g_j)\|_{L^1(2^{j+3}\, Q)} \le C 2^j l(Q). \tag{44}$$

As before, let $1 < q < \infty$ and call p the dual exponent to q. Apply Hölder's inequality and the fact that the Riesz transforms preserve $L^q(\mathbb{R}^n)$ to obtain

$$\|R_i(\partial^s g)\|_{L^1(2\lambda\, Q)} \le C l(Q)^{\frac{n}{p}} \|R_i(\partial^s g)\|_{L^q(\mathbb{R}^n)} \le C l(Q)^{\frac{n}{p}} \|\partial^s g\|_{L^q(\lambda\, Q)}$$

and

$$\|R_i(\partial^s g_j)\|_{L^1(2^{j+3} Q)} \le C \left(2^j l(Q)\right)^{\frac{n}{p}} \|R_i(\partial^s g_j)\|_{L^q(\mathbb{R}^n)} \le C \left(2^j l(Q)\right)^{\frac{n}{p}} \|\partial^s g_j\|_{L^q(2^{j+2}\, Q)}$$

hold. Therefore, (43) and (44) follow from

$$\|\partial^s g\|_{L^q(\lambda\, Q)} \le C\, l(Q)^{\frac{n}{q} - |s|} \tag{45}$$

and

$$\|\partial^s g_j\|_{L^q(2^{j+2} Q)} \le C \left(2^j l(Q)\right)^{\frac{n}{q} - |s|} \tag{46}$$

respectively.

To show (45) we take ∂^s in the definition of g, apply Leibnitz's formula and estimate in the supremum norm each term in the resulting sum . We get

$$\|\partial^s g\|_\infty \le C\, l(Q) \sum_{|r|=0}^{n-1} \frac{1}{l(Q)^{|r|}}\, \frac{1}{l(Q)^{1+|s|-|r|}} = C\, \frac{1}{l(Q)^{|s|}},$$

which yields (45) immediately.

For (46), applying a gradient estimate, we get

$$|\partial^{s-r} k_\varepsilon^{1,x}(y) - \partial^{s-r} k_\varepsilon^{1,x_0}(y)| \le C\, \frac{l(Q)}{(2^j\, l(Q))^{2+|s|-|r|}}, \quad y \in N_j^*, \quad j \in J.$$

Hence

$$\|\partial^s g_j\|_\infty \le C\, 2^{2j}\, l(Q) \sum_{|r|=0}^{n-1} \frac{1}{(2^j\, l(Q))^{|r|}}\, \frac{l(Q)}{(2^j\, l(Q))^{2+|s|-|r|}} = C\, \frac{1}{(2^j\, l(Q))^{|s|}},$$

which yields (46) readily. □

4.2 A continuity property for the capacity $\Gamma_{\hat{k}}$

In this section we prove a continuity property for the capacity $\Gamma_{\hat{k}}$, $1 \le k \le n$, which will be used in the proof of inequality (22). Although we state the result only for the capacities $\Gamma_{\hat{k}}$, $1 \le k \le n$, Lemma 7 below holds for the capacities κ_i, $1 \le i \le n$, defined in the Introduction, because the proof does not use any growth condition on distributions with bounded scalar Riesz potential.

Lemma 7. *Let $\{E_j\}_j$ be a decreasing sequence of compact sets, with intersection the compact set $E \subset \mathbb{R}^n$. Then, for $1 \le k \le n$,*

$$\Gamma_{\hat{k}}(E) = \lim_{j\to\infty} \Gamma_{\hat{k}}(E_j).$$

Proof. Since, by definition, the set function $\Gamma_{\hat{k}}$ in non-decreasing

$$\lim_{j\to\infty} \Gamma_{\hat{k}}(E_j) \ge \Gamma_{\hat{k}}(E),$$

and the limit clearly exists. For each $j \ge 1$, let T_j be a distribution such that the potentials $x_i/|x|^2 * T_j$ are in the unit ball of $L^\infty(\mathbb{R}^n)$, $i \ne k$, and

$$\Gamma_{\hat{k}}(E_j) - \frac{1}{j} < |\langle T_j, 1\rangle| \le \Gamma_{\hat{k}}(E_j).$$

We want to show that for each test function φ,

$$\langle T_j, \varphi\rangle \underset{j\to\infty}{\longrightarrow} \langle T, \varphi\rangle, \tag{47}$$

for some distribution T whose potentials $x_i/|x|^2 * T$ are in the unit ball of $L^\infty(\mathbb{R}^n)$ for $i \ne k$. If (47) holds and φ is a test function satisfying $\varphi \equiv 1$ in a neighbourhood of E, then

$$\lim_{j\to\infty} \Gamma_{\hat{k}}(E_j) = \lim_{j\to\infty} |\langle T_j, 1\rangle| = \lim_{j\to\infty} |\langle T_j, \varphi\rangle| = |\langle T, \varphi\rangle| \le \Gamma_{\hat{k}}(E).$$

To show (47), fix $i \neq k$ and assume, without loss of generality, that $i = 1$. Set $k^1(x) = x_1/|x|^2$ and $f_j = k^1 * T_j$. Write a point $x \in \mathbb{R}^n$ as $x = (x_1, x_2)$, with $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}^{n-1}$. Finally notice that $c\, k^1 = \partial_1 E$ where $E = log|x|$ and c is a constant. Moreover, for each test function φ one has

$$\varphi = c\Delta^{\frac{n}{2}} \varphi * E, \tag{48}$$

for some constant c. For $n = 2k$, identity (48) says that E is the fundamental solution of the k-Laplacian in $\mathbb{R}^n$, and for $n = 2k + 1$, (48) means that

$$\varphi = c\Delta^{k+1} \varphi * \frac{1}{|x|^{n-1}} * E. \tag{49}$$

We will only deal with the even case $n = 2k$, since, by using the reproduction formula (49), the arguments for the odd case turn to be very similar. Therefore, for each test function φ,

$$(T_j * \varphi)(x_1, x_2) = \int_{-\infty}^{x_1} \partial_1 (T_j * \varphi)(t, x_2)\, dt = c \int_{-\infty}^{x_1} \Delta^k (\varphi * f_j)(t, x_2)\, dt.$$

Setting $\overline{\varphi}(x) = \varphi(-x)$ we get

$$\langle T_j, \varphi \rangle = (T_j * \overline{\varphi})(0, 0) = c \int_{-\infty}^{0} \Delta^k (\overline{\varphi} * f_j)(t, 0)\, dt. \tag{50}$$

We remark, incidentally, that the above formula tells us how to recover a distribution from one of its scalar Riesz potentials.

Passing to a subsequence, we can assume that $f_j \longrightarrow f$ in the weak $*$ topology of $L^\infty(\mathbb{R}^n)$. But then $(f_j * \Delta^k \varphi)(x) \longrightarrow (f * \Delta^k \varphi)(x), \quad x \in \mathbb{R}^n$. This pointwise convergence is bounded because $|(f_j * \Delta^k \varphi)(x)| \leq \|\Delta^k \varphi\|_1 \|f_j\|_\infty \leq \|\Delta^k \varphi\|_1$. Hence the dominated convergence theorem yields

$$\lim_{j \to \infty} \langle T_j, \varphi \rangle = c \lim_{j \to \infty} \int_{-\infty}^{0} \Delta^k (\overline{\varphi} * f_j)(t, 0)\, dt = c \int_{-\infty}^{0} \Delta^k (\overline{\varphi} * f)(t, 0)\, dt.$$

Define the distribution T by

$$\langle T, \varphi \rangle = c \int_{-\infty}^{0} \Delta^k (\overline{\varphi} * f)(t, 0)\, dt.$$

Now we want to show that $f = k^1 * T$. For that we regularize f_j and T_j. Take $\chi \in \mathcal{C}_0^\infty(B(0,1))$ with $\int \chi(x)\, dx = 1$ and set $\chi_\varepsilon(x) = \varepsilon^{-n} \chi(x/\varepsilon)$. Then we have, as $j \to \infty$,

$$\left(\chi_\varepsilon * k^1 * T_j\right)(x) = \left(\chi_\varepsilon * f_j\right)(x) \longrightarrow \left(\chi_\varepsilon * f\right)(x), \quad x \in \mathbb{R}^n,$$

because f_j converges to f weak $*$ in $L^\infty(\mathbb{R}^n)$. On the other hand, since $\chi_\varepsilon * k_1 \in \mathcal{C}^\infty(\mathbb{R}^n)$ and T_j tends to T in the weak topology of distributions, with controlled supports, we have

$$\left(\chi_\varepsilon * k^1 * T_j\right)(x) \longrightarrow \left(\chi_\varepsilon * k^1 * T\right)(x), \quad x \in \mathbb{R}^n.$$

Hence

$$\chi_\varepsilon * k^1 * T = \chi_\varepsilon * f, \quad \varepsilon > 0,$$

and so, letting $\varepsilon \to 0$, $k^1 * T = f$. □

4.3 End of the proof of the inequality $\Gamma_{\hat{k}} \le C\,\Gamma_{\hat{k},\mathrm{op}}$

We claim that the inequality in the title of this subsection can be proved by adapting the scheme of the proof of Theorems 1.1 in [T2] and 7.1 in [T3]. As Lemma 7 shows, the capacities $\Gamma_{\hat{k}}$, $1 \le k \le n$, enjoy the exterior regularity property. This is also true for the capacities $\Gamma_{\hat{k},+}$, $1 \le k \le n$, defined by

$$\Gamma_{\hat{k},+}(E) = \sup\left\{\mu(E) : \mu \in L(E), \left\|\frac{x_j}{|x|^2} * \mu\right\|_\infty \le 1,\, 1 \le j \le n,\, j \ne k\right\},$$

just by the weak $\star$ compactness of the set of positive measures with total variation not exceeding 1. We can approximate a general compact set E by sets which are finite unions of cubes of the same side length in such a way that the capacities $\Gamma_{\hat{k}}$ and $\Gamma_{\hat{k},+}$ of the approximating sets are as close as we wish to those of E. As in (19), one has, using the Davie-Oksendal Lemma for several operators [MaPa, Lemma 4.2],

$$C^{-1}\,\Gamma_{\hat{k},\mathrm{op}}(E) \le \Gamma_{\hat{k},+}(E) \le C\,\Gamma_{\hat{k},\mathrm{op}}(E). \tag{51}$$

Thus we can assume, without loss of generality, that E is a finite union of cubes of the same size. This will allow to implement an induction argument on the size of certain (n-dimensional) rectangles. The first step involves rectangles of diameter comparable to the side length of the cubes whose union is E.

The starting point of the general inductive step in the proof of Tolsa's Theorem in [T2] (and [T3]) consists in the construction of a positive Radon measure μ supported on a compact set F which approximates E in an appropriate sense. The construction of F and μ gives readily that $\Gamma_{\hat{k}}(E) \le C\,\mu(F)$, and $\Gamma_{\hat{k},+}(F) \le C\,\Gamma_{\hat{k},+}(E)$, which tells us that F is not too small but also not too big. However, one cannot expect, in the context of [T2] and [T3], the Cauchy singular integral to be bounded on $L^2(\mu)$. In our case one cannot expect the operators R^j to be bounded on $L^2(\mu)$, for $1 \le j \le n$, $j \ne k$. Here R^j is the operator associated with the scalar Riesz kernel $(x_j - y_j)/|x-y|^2$.One has to carefully look for a compact subset G of F such that $\mu(F) \le C\,\mu(G)$, the restriction μ_G of μ to G has linear growth and the operators

R^j, $1 \le j \le n$, $j \neq k$, are bounded on $L^2(\mu_G)$ with dimensional constants. This completes the proof because then

$$\Gamma_{\hat{k}}(E) \le C\,\mu(F) \le C\,\mu(G) \le C\,\Gamma_{\hat{k},\mathrm{op}}(G) \le C\,\Gamma_{\hat{k},\mathrm{op}}(F)$$
$$\le C\,\Gamma_{\hat{k},+}(F) \le C\,\Gamma_{\hat{k},+}(E) \le C\,\Gamma_{\hat{k},\mathrm{op}}(E).$$

In [T2] and [T3] the set F is defined as the union of a special family of cubes $\{Q_i\}_{i=1}^N$ that cover the set E and approximate E at an appropriate intermediate scale. One then sets

$$F = \bigcup_{i=1}^{N} Q_i.$$

This part of the proof extends without any obstruction to our case because of the positivity properties of the symmetrization of the scalar Riesz kernels (see section 3). As in Lemma 7.2 in [T3], just by how the approximating set F is constructed, one gets $\Gamma_{\hat{k},+}(F) \le C\,\Gamma_{\hat{k},+}(E)$. By the definition of $\Gamma_{\hat{k}}(E)$ it follows that there exists a real distribution T_0 supported on E such that

1. $\Gamma_{\hat{k}}(E) \le 2|\langle T_0, 1\rangle|$.

2. T_0 has linear growth and $G(T_0) \le 1$.

3. $\|\frac{x_j}{|x|^2} * T_0\|_\infty \le 1, \quad 1 \le j \le n, \quad j \neq k.$

Consider now functions $\varphi_i \in \mathcal{C}_0^\infty(2Q_i)$, $0 \le \varphi_i \le 1$, $\|\partial^s \varphi_i\|_\infty \le C\,l(Q_i)^{-|s|}$, $0 \le |s| \le n-1$ and $\sum_{i=1}^N \varphi_i = 1$ on $\bigcup_i Q_i$. We define now simultaneously the measure μ and an auxiliary measure ν, which should be viewed as a model for T_0 adapted to the family of cubes $\{Q_i\}_{i=1}^N$. For each cube Q_i take a concentric segment Σ_i of length a small fixed fraction of $\Gamma_{\hat{k}}(E \cap Q_i)$ and set

$$\mu = \sum_{i=1}^{N} \mathcal{H}^1_{|\Sigma_i}$$

and

$$\nu = \sum_{i=1}^{N} \frac{\langle T_0, \varphi_i\rangle}{\mathcal{H}^1(\Sigma_i)} \mathcal{H}^1_{|\Sigma_i}.$$

We have $d\nu = b d\mu$, with $b = \dfrac{\langle \varphi_i, \nu_0\rangle}{\mathcal{H}^1(\Sigma_i)}$ on Σ_i. At this point we need to show that our function b is bounded, to apply later a suitable $T(b)$ Theorem. To estimate $\|b\|_\infty$ we use the localization inequalities

$$\left\| \frac{x_j}{|x|^2} * \varphi_i T_0 \right\|_\infty \le C, \quad 1 \le j \le n, \quad j \neq k, \quad 1 \le i \le N.$$

This was proved in Lemma 5 of Section 4.1. Since it is easily seen that $\varphi_i T_0$ has linear growth and $G(\varphi_i T_0) \leq C$, we obtain, by the definition of $\Gamma_{\hat{k}}$,

$$|\langle T_0, \varphi_i \rangle| \leq C\, \Gamma_{\hat{k}}(2Q_i \cap E), \quad \text{for } 1 \leq i \leq N. \tag{52}$$

It is now easy to see why $\Gamma_{\hat{k}}(E) \leq C\, \mu(F)$:

$$\begin{aligned} \Gamma_{\hat{k}}(E) &\leq 2\, |\langle T_0, 1 \rangle| = 2 \left| \sum_{i=1}^{N} \langle T_0, \varphi_i \rangle \right| \\ &\leq C \sum_{i=1}^{N} \Gamma_{\hat{k}}(2Q_i \cap E) = C\, \mu(F). \end{aligned} \tag{53}$$

We do not insist in summarizing the intricate details, which can be found in [T2] and [T3], of the definition of the set G and of the application of the $T(b)$ Theorem of [NTV2].

5 Counter-examples to the growth estimate

As we explained in the introduction, if T is a compactly supported distribution such that $x/|x|^2 * T$ is bounded, then T satisfies the linear growth condition (7) (see (10) and (13)). This is no longer true under the assumption that $n-1$ components of $x/|x|^2 * T$ are bounded, as the following examples show.

Proposition 8. *There exist a compactly supported real Radon measure μ in $\mathbb{R}^n$, such that for $1 \leq i \leq n-1$, $x_i/|x|^2 * \mu$ is in $L^\infty(\mathbb{R}^n)$ and $G(\mu) = \infty$.*

Proof. The idea of the proof is that there is no relation, in general, between the derivatives of a function with respect to different variables. The technical details of the proof differ according to the parity of the dimension, so we deal separately with even and odd dimensions. Indeed, we work in $\mathbb{R}^3$ and $\mathbb{R}^4$, the general case being a straightforward extension of these two.

1. *The odd case:*

   Set $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and let $h(x) = f(x_1) f(x_2) g(x_3)$, where f is the compactly supported infinitely differentiable function defined by

   $$f(t) = \begin{cases} 1 & \text{if } t \in [0,1] \\ 0 & \text{if } t \in [-1,2]^c \end{cases}. \tag{54}$$

   To define g, let ψ be an infinitely differentiable function supported on $[1/2, 1]$, increasing for $x \in [1/2, 3/4]$, decreasing for $x \in [3/4, 1]$ and such that $\psi(3/4) = 1$. Define g on $I_j = [2^{-j-1}, 2^{-j}]$, $j \geq 0$, by

   $$g(t) = \frac{\psi(2^j t)}{(j+1)^3}, \ t \in I_j. \tag{55}$$

Set $\mu = \partial_3^3 h = f(x_1)f(x_2)g^{3)}(x_3)$, and write $k^i(x) = x_i/|x|^2$, $1 \le i \le 3$, so that, for $i = 1, 2$,

$$(\mu * k^i)(x) = (\partial_3^3 h * k^i)(x) = (\partial_i \partial_3 h * \partial_3 k^3)(x).$$

We claim that $\|k^i * \mu\|_\infty \le C, \ \ 1 \le i \le 2$. Since for $m \ge 0$,

$$g^{m)}(t) = \frac{2^{mj}}{(j+1)^3}\psi^{m)}(2^j t), \ \ t \in I_j, \tag{56}$$

we have,

$$\begin{aligned}
|(\mu * k^i)(x)| &\le C \int_0^1 \int_{-1}^2 \int_{-1}^2 \frac{|f'(y_1)||f(y_2)||g'(y_3)|}{|y-x|^2} dy_1 dy_2 dy_3 \\
&\le C \sum_j \frac{2^j}{(j+1)^3} \int_{-1}^2 \int_{-1}^2 \int_{I_j} \frac{dy_3 dy_1 dy_2}{|y-x|^2} \\
&\le C \sum_j \frac{1}{(j+1)^2} \le C.
\end{aligned}$$

The next to the last inequality follows by decomposing the domain of integration into "annuli" $(|y_1 - x_1|^2 + |y_2 - x_2|^2)^{1/2} \simeq 2^k\, 2^{-j}, \ \ |y_3 - x_3| \simeq 2^{-j}, \ \ 0 \le k$. Hence $\|k^i * \mu\|_\infty \le C$ for $i = 1, 2$.

To see that the linear growth condition fails for the measure μ, take an interval $I_j^* \subset I_j$ such that, for some fixed small positive number δ, one has $l(I_j^*) \ge \delta\, l(I_j)$ and $g^{3)}(t) \ge \delta\, 2^{3j}/(1+j)^3, \ \ t \in I_j^*$. The existence of such δ and I_j^* follows readily from the definition of g on I_j. Take a non-negative function $\phi \in \mathcal{C}_0^\infty(I_j^*)$ with $\phi(t) = 1$ on $I_j^*/2$ (interval with the same center of I_j^* and half the length). Let Q_j be the cube $(I_j^*)^3$, $j \ge 0$ and set $\varphi_{Q_j}(x_1, x_2, x_3) = \phi(x_1)\phi(x_2)\phi(x_3)$. Then $\varphi_{Q_j} \in \mathcal{C}_0^\infty(Q_j)$ and φ_{Q_j}/C satisfies the normalization condition (15) for some absolute big constant C. Then, since $l(Q_j) = l(I_j^*) \approx l(I_j) = 2^{-j}$, by (56) for $m = 3$ we obtain,

$$\langle \mu, \varphi_{Q_j} \rangle \ = \left(\int_{I_j^*} \phi(t) dt \right)^2 \int_{I_j^*} \phi(t) g^{3)}(t) dt \approx l(Q_j)^2 \frac{2^{3j}}{(j+1)^3} l(Q_j) = \frac{2^j}{(j+1)^3} l(Q_j).$$

Thus

$$\frac{|\langle \mu, \varphi_{Q_j}\rangle|}{l(Q_j)} \underset{j\to\infty}{\longrightarrow} \infty,$$

which implies $G(\mu)=\infty$.

2. *The even case:*

For $x=(x_1,x_2,x_3,x_4)\in\mathbb{R}^4$ let $h(x)=f(x_1)f(x_2)f(x_3)g(x_4)$, where f is the function defined by (54) and g is defined by

$$g(t)=\psi(2^j t),\ t\in I_j,$$

that is, as in (55) except that the denominator $(j+1)^3$ is not needed in this case.

Define $\mu=\Delta^2 h$, the bilaplacian of h. Then

$$\begin{aligned}\mu &= g(x_4)\left(f^{4)}(x_1)f(x_2)f(x_3)+f(x_1)f^{4)}(x_2)f(x_3)+f(x_1)f(x_2)f^{4)}(x_3)\right)\\ &\quad + 2g(x_4)\left(f''(x_1)f''(x_2)f(x_3)+f''(x_1)f(x_2)f''(x_3)+f(x_1)f''(x_2)f''(x_3)\right)\\ &\quad + 2g''(x_4)\left(f''(x_1)f(x_2)f(x_3)+f(x_1)f''(x_2)f(x_3)+f(x_1)f(x_2)f''(x_3)\right)\\ &\quad + g^{4)}(x_4)f(x_1)f(x_2)f(x_3).\end{aligned}$$

Write $k^i(x)=x_i/|x|^2$, $1\le i\le 4$. Notice that $k^i(x)=c\,\partial_i E$, where E is the fundamental solution of the bilaplacian and c a constant. Then, for $1\le i\le 3$,

$$\|k^i{*}\mu\|_\infty=\|k^i{*}\Delta^2 h\|_\infty=\|c\,\partial_i(\Delta^2 h{*}E)\|_\infty=c\|\partial_i h\|_\infty=c\|f\|_\infty^2\|f'\|_\infty\|g\|_\infty\le C.$$

Although this is not necessary for the argument, notice that, by (56), we have

$$\|k^4{*}\mu\|_\infty=\|k^4{*}\Delta^2 h\|_\infty=\|c\,\partial_4(\Delta^2 h{*}E)\|_\infty=c\|\partial_4 h\|_\infty=c\|f\|_\infty^3\|g'\|_\infty=\infty.$$

Take an interval $I_j^*\subset I_j$ such that, for some fixed small positive number δ, one has $l(I_j^*)\ge\delta\,l(I_j)$ and $g^{4)}(t)\ge\delta\,2^{4j},\quad t\in I_j^*$. The existence of such δ and I_j^* follows readily from the definition of g on I_j. Take a non-negative function $\phi\in\mathcal{C}_0^\infty(I_j^*)$ with $\phi(t)=1$ on $I_j^*/2$ (interval with the same center of I_j^* and half the length). Let Q_j be the cube $(I_j^*)^4$, $j\ge 0$ and set $\varphi_{Q_j}(x_1,x_2,x_3,x_4)=\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)$. Then $\varphi_{Q_j}\in\mathcal{C}_0^\infty(Q_j)$ and φ_{Q_j}/C satisfies the normalization condition (15) for some absolute constant C. Then, since f'' and $f^{4)}$ are zero on I_j^* and $l(Q_j)=l(I_j^*)\approx l(I_j)=2^{-j}$, by (56) for $m=4$ we obtain,

$$\begin{aligned}\langle \mu,\varphi_{Q_j}\rangle &= \left(\int_{I_j^*}\phi(t)f(t)dt\right)^3\int_{I_j^*}\phi(t)g^{4)}(t)dt\\ &\approx l(Q_j)^3\,2^{4j}\,l(Q_j)\approx 2^j\,l(Q_j).\end{aligned}$$

Thus

$$\frac{|\langle \mu, \varphi_{Q_j} \rangle|}{l(Q_j)} \underset{j\to\infty}{\longrightarrow} \infty,$$

which implies $G(\mu) = \infty$.

□

On the plane, we do also have a counterexample in the setting of positive measures, based on a completely different idea.

Proposition 9. *There exists a positive Radon measure μ such that $x_1/|x|^2 * \mu$ is in $L^\infty(\mathbb{R}^2)$ and $G(\mu) = \infty$.*

Proof. Consider the function $f(t) = \log^+ \frac{1}{|t|}$, $t \in \mathbb{R}$. Then $f \in BMO(\mathbb{R}) \setminus L^\infty(\mathbb{R})$ and f is supported on the interval $[-1, 1]$. If $y > 0$, then

$$\begin{aligned} \left(\frac{i}{\pi z} * f\right)(x,y) &= \frac{1}{\pi}(k^2 * f)(x,y) + \frac{i}{\pi}(k^1 * f)(x,y) \\ &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-t)^2 + y^2} f(t)\,dt + \frac{i}{\pi}\int_{\mathbb{R}} \frac{x-t}{(x-t)^2+y^2} f(t)\,dt \\ &= (P_y f)(x) + i(Q_y f)(x), \end{aligned}$$

where $P_y f(x)$ and $Q_y f(x)$ are the Poisson transform and the conjugate Poisson transform of f respectively.

Therefore, if $Hf = \frac{1}{\pi}\text{p.v.}\frac{1}{x} * f$ is the Hilbert transform of f,

$$(k^1 * f dt)(x,y) = (Q_y f)(x) = P_y(Hf)(x).$$

We claim that

$$H(f) \in L^\infty(\mathbb{R}). \tag{57}$$

If (57) holds, then the positive measure $\mu = f(t)\,dt$ satisfies

$$|(k^1 * \mu)(x,y)| = |P_y(Hf)(x)| \le \|Hf\|_\infty, \quad x \in \mathbb{R}, \quad y > 0.$$

Since $(k^1 * \mu)(x,-y) = (k^1 * \mu)(x,y)$, we get $k^1 * \mu \in L^\infty(\mathbb{R}^2)$ and, on the other hand, μ has not linear growth, just because f is unbounded.

To show (57), we first observe that integrating by parts we have

$$\text{p.v.} \int_{-1}^{1} \log \frac{1}{|t|} \frac{dt}{x-t} = \lim_{\epsilon \to 0} \int_{1>|t|>\epsilon} \log|x-t| \,\frac{dt}{t}.$$

The function above is odd and so we can assume that x is positive. Making first the change of variables $\tau = -t$ and then $u = t/x$ we get

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{1>|t|>\epsilon} \log|x-t|\,\frac{dt}{t} &= -\lim_{\epsilon\to 0}\int_{1>|t|>\epsilon} \log|x+t|\,\frac{dt}{t}\\ &= \frac{1}{2}\int_{-1}^{1} \log\frac{|x-t|}{|x+t|}\,\frac{dt}{t}\\ &= \frac{1}{2}\int_{-\frac{1}{x}}^{\frac{1}{x}} \log\frac{|u-1|}{|u+1|}\,\frac{du}{u}.\end{aligned}$$

Hence

$$\left|\text{p.v.}\int_{-1}^{1}\log\frac{1}{|t|}\frac{dt}{x-t}\right| \le \frac{1}{2}\int_{-\infty}^{\infty}\left|\log\frac{|u-1|}{|u+1|}\,\frac{1}{u}\right|\,du,$$

which completes the proof because the last integral above is finite. □

It is worth mentioning that we do not know whether there exists a positive measure μ in $\mathbb{R}^n$, $n \ge 3$, with the $n-1$ potentials $\mu * x_i/|x|^2$, $1 \le i \le n-1$, in $L^\infty(\mathbb{R}^n)$, but not having linear growth.

6 Miscellaneous related results

As we have seen in the previous sections, the fact that the Cauchy kernel is complex is not as relevant as the fact that it is odd and has homogeneity -1. Indeed, in the plane, (6) shows that one recovers the theory of analytic capacity by replacing the Cauchy kernel $1/z$ by any of the real kernels $\mathrm{Re}(1/z)$ or $\mathrm{Im}(1/z)$. In $\mathbb{R}^n$, $n \ge 3$, the Theorem shows that an analogue of (6) holds in higher dimensions adding appropriate growth conditions on the admissible distributions.

A natural question is how one can extend this kind of results to the higher dimensional real variable setting in which the kernel $x/|x|^2$ is replaced by the vector valued Riesz kernel of homogeneity $-\alpha$

$$k_\alpha(x) = \frac{x}{|x|^{1+\alpha}}, \quad x \in \mathbb{R}^n, \quad 0 < \alpha < n,$$

and the capacity associated with this kernel is defined by (see [Pr1])

$$\Gamma_\alpha(E) = \sup\left\{ |\langle T, 1\rangle| : \mathrm{spt}(T) \subset E,\ \left\|\frac{x}{|x|^{1+\alpha}} * T\right\|_\infty \le 1\right\}.$$

The case $\alpha = n-1$, $n \ge 2$, is especially interesting, because it gives Lipschitz harmonic capacity (see (3)).

Unfortunately, as we show in subsections 6.1 and 6.2 below, the most obvious analogues of (6) and the Theorem fail in this setting.

6.1 Capacities associated with scalar α-Riesz potentials

Let T be a compactly supported distribution in $\mathbb{R}^n$ and $0 < \alpha < n$. As it was explained in the Introduction, the natural notion of distribution T of growth α should involve Hardy spaces. In our present case, one should replace the reproduction formula (11) by the following ones, depending on the nature of the parameter α:

- $\alpha \in \mathbb{Z}$. A standard Fourier transform computation shows that, for some constant c_n and each test function φ, one has

$$\varphi = c_n \sum_{j=1}^{n} \frac{x_j}{|x|^{1+\alpha}} * \partial_j(\Delta^{(n-1-\alpha)/2})\varphi \equiv c_n \frac{x}{|x|^{1+\alpha}} * \nabla(\Delta^{(n-\alpha-1)/2})\varphi.$$

- $\alpha \notin \mathbb{Z}$. A standard Fourier transform computation shows that, for some constant d_n and each test function φ, one has

$$\begin{aligned}\varphi &= d_n \sum_{j=1}^{n} \frac{x_j}{|x|^{1+\alpha}} * \frac{1}{|x|^{n-\{\alpha\}}} * \partial_j(\Delta^{(n-[\alpha]-1)/2})\varphi \\ &\equiv d_n \frac{x}{|x|^{1+\alpha}} * \frac{1}{|x|^{n-\{\alpha\}}} * \nabla(\Delta^{(n-[\alpha]-1)/2})\varphi,\end{aligned}$$

  where $\alpha = [\alpha] + \{\alpha\}$, with $[\alpha] \in \mathbb{Z}$ and $\{\alpha\} \in (0,1)$.

Now we are able to define the notion of a compactly supported distribution with growth α, $0 < \alpha < n$. We say that T has growth α provided

$$G_\alpha(T) = \sup_{\varphi_Q} \frac{|\langle T, \varphi_Q\rangle|}{l(Q)^\alpha} < \infty, \tag{58}$$

where the supremum is taken over all $\varphi_Q \in \mathcal{C}_0^\infty(Q)$ satisfying the following normalization inequalities :

1. For $\alpha \in \mathbb{Z}$, we require

$$\|\partial^s \varphi_Q\|_{H^1(\mathbb{R}^n)} \leq l(Q)^\alpha, \quad |s| = n - \alpha. \tag{59}$$

2. For $\alpha \notin \mathbb{Z}$, we require

$$\|\partial^s \varphi_Q * \frac{1}{|x|^{n-\{\alpha\}}}\|_{H^1(\mathbb{R}^n)} \leq l(Q)^\alpha, \quad |s| = n - [\alpha]. \tag{60}$$

For positive Radon measures μ in $\mathbb{R}^n$ the preceding notion of growth α is equivalent to the usual one. In subsection 6.5 complete details on this fact are provided.

For a compact set E in $\mathbb{R}^n$ we define $g_\alpha(E)$ as the set of all distributions T supported on E having growth α with constant $G_\alpha(T)$ at most 1.

For each coordinate k set

$$\Gamma_{\alpha,\hat{k}}(E) = \sup\{|\langle T, 1\rangle|\},$$

where the supremum is taken over those distributions $T \in g_\alpha(E)$, such that the j-th component of the α-Riesz potential $x_j/|x|^{1+\alpha} * T$ is in the unit closed ball of $L^\infty(\mathbb{R}^n)$, for $1 \le j \le n$, $j \ne k$.

The proof of Lemma 3.2 in [Pr1] tells us that if $k_\alpha * T$ is in the unit ball $L^\infty(\mathbb{R}^n, \mathbb{R}^n)$, then the distribution T has α-growth and $G_\alpha(T) \le C$. Hence $\Gamma_\alpha(E) \le C\,\Gamma_{\alpha,\hat{k}}(E)$. In this section we prove the following

Proposition 10. *Given $0 < \alpha < 1$, there exists a set $E \subset \mathbb{R}^n$ such that $\Gamma_\alpha(E) = 0$ and $\Gamma_{\alpha,\hat{k}}(E) > 0$.*

Therefore Γ_α and $\Gamma_{\alpha,\hat{k}}$ are not comparable and thus the direct analogue of the Theorem fails in this setting.

We proceed now to symmetrize the scalar α-Riesz kernels in order to get a better understanding of the capacities $\Gamma_{\alpha,\hat{k}}$ for $1 \le k \le n$ and $0 < \alpha < 1$.

For $0 < \alpha < n$ and $1 \le i \le n$ the quantity

$$\sum_\sigma \frac{x^i_{\sigma(2)} - x^i_{\sigma(1)}}{|x_{\sigma(2)} - x_{\sigma(1)}|^{1+\alpha}} \frac{x^i_{\sigma(3)} - x^i_{\sigma(1)}}{|x_{\sigma(3)} - x_{\sigma(1)}|^{1+\alpha}} \tag{61}$$

where the sum is taken over the permutations of the set $\{1, 2, 3\}$, is the analogue of the right hand side of (27) for the i-th coordinate of the Riesz kernel k_α. Notice that (61) is exactly

$$2\,p_{\alpha,i}(x_1, x_2, x_3),$$

where $p_{\alpha,i}(x_1, x_2, x_3)$ is defined as the sum in (61) only taken on the three permutations $(1, 2, 3), (2, 3, 1)$ and $(3, 1, 2)$.

We will now show that given three distinct points $x_1, x_2, x_3 \in \mathbb{R}^n$, for $1 \le i \le n$ and $0 < \alpha \le 1$, the quantity $p_{\alpha,i}(x_1, x_2, x_3)$ is non-negative. We will use this to study the L^2 boundedness of the scalar Riesz integral operator of homogeneity $-\alpha$.

The relationship between the quantity $p_{\alpha,i}(x, y, z)$, $0 < \alpha \le 1$, $1 \le i \le n$, and the L^2 estimates of the operator with kernel $k^i_\alpha = x_i/|x|^{1+\alpha}$ is as in (28). That is, if μ is a positive finite Radon measure in $\mathbb{R}^n$ with α-growth, $\varepsilon > 0$ and we set

$$R^i_{\alpha,\varepsilon}(\mu)(x) = \int_{|y-x|>\varepsilon} k^i_\alpha(y - x)\, d\mu(y),$$

then (see in [MeV] the argument for the Cauchy singular integral operator)

$$\left| \int |R^i_{\alpha,\varepsilon}(\mu)(x)|^2\, d\mu(x) - \frac{1}{3} p_{\alpha,i,\varepsilon}(\mu) \right| \le C\|\mu\|, \tag{62}$$

C being a positive constant depending only on n and α, and

$$p_{\alpha,\,i,\,\varepsilon}(\mu)=\iiint\limits_{S_\varepsilon} p_{\alpha,\,i}(x,y,z)\,d\mu(x)\,d\mu(y)\,d\mu(z),$$

with

$$S_\varepsilon=\{(x,y,z):|x-y|>\varepsilon,|x-z|>\varepsilon \text{ and } |y-z|>\varepsilon\}.$$

Lemma 11. *Let $0<\alpha<1$ and x_1, x_2, x_3 three different points in $\mathbb{R}^n$. For $1\le i\le n$ we have*

$$\frac{(2-2^\alpha)m^2}{L(x_1,x_2,x_3)^{2+2\alpha}}\le p_{\alpha,\,i}(x_1,x_2,x_3)\le\frac{3m^2}{L(x_1,x_2,x_3)^{2+2\alpha}},\tag{63}$$

where $m=\max(|x_2^i-x_1^i|,|x_3^i-x_2^i|,|x_3^i-x_1^i|)$ and $L(x_1,x_2,x_3)$ is the length of the largest side of the triangle determined by the three points x_1, x_2, x_3.

Moreover, $p_{\alpha,i}(x_1,x_2,x_3)=0$ if and only if the three points lie on a $(n-1)$-hypersurface perpendicular to the i axis, i.e. $x_1^i=x_2^i=x_3^i$.

Proof. Without loss of generality fix $i=1$. Write $a=x_2-x_1$ and $b=x_3-x_2$, then $a+b=x_3-x_1$. A simple computation yields

$$p_{\alpha,1}(x_1,x_2,x_3)=\frac{a_1^2|b|^{1+\alpha}+b_1^2|a|^{1+\alpha}+a_1b_1\left(|b|^{1+\alpha}+|a|^{1+\alpha}-|a+b|^{1+\alpha}\right)}{|a|^{1+\alpha}|b|^{1+\alpha}|a+b|^{1+\alpha}},\tag{64}$$

which makes the second inequality in (63) obvious. To prove the first inequality in (63), assume without loss of generality, that $1=|a|\le|b|\le|a+b|$. Then

$$p_{\alpha,1}(x_1,x_2,x_3)=\frac{1}{|b|^{1+\alpha}|a+b|^{1+\alpha}}\left(a_1^2|b|^{1+\alpha}+b_1^2+a_1b_1(1+|b|^{1+\alpha}-|a+b|^{1+\alpha})\right).$$

We distinguish now two cases,

- *Case* $a_1b_1\le 0$. Notice that since $|b|\le|a+b|$,

$$a_1b_1(1+|b|^{1+\alpha}-|a+b|^{1+\alpha})\ge a_1b_1.$$

Then, since $|b|\ge 1$,

$$\begin{aligned}p_{\alpha,1}(x_1,x_2,x_3)&=\frac{1}{|b|^{1+\alpha}|a+b|^{1+\alpha}}\left(a_1^2|b|^{1+\alpha}+b_1^2+a_1b_1(1+|b|^{1+\alpha}-|a+b|^{1+\alpha})\right)\\&\ge\frac{a_1^2|b|^{1+\alpha}+b_1^2+a_1b_1}{|b|^{1+\alpha}|a+b|^{1+\alpha}}\ge\frac{a_1^2+b_1^2+a_1b_1}{|b|^{1+\alpha}|a+b|^{1+\alpha}}\\&=\frac{1}{2}\frac{(a_1+b_1)^2+a_1^2+b_1^2}{|b|^{1+\alpha}|a+b|^{1+\alpha}}.\end{aligned}$$

- *Case* $a_1 b_1 > 0$. Then $\max\{a_1^2, b_1^2, (a_1+b_1)^2\} = (a_1+b_1)^2$. Write $t = |b| \geq 1$ and
$$f(t) = a_1^2 t^{1+\alpha} + b_1^2 + a_1 b_1 \left(1 + t^{1+\alpha} - (1+t)^{1+\alpha}\right).$$
By the triangle inequality,
$$p_{\alpha,1}(x_1,x_2,x_3) \geq \frac{f(t)}{|b|^{1+\alpha}|a+b|^{1+\alpha}} \geq \frac{\min_{t\geq 1} f(t)}{|b|^{1+\alpha}|a+b|^{1+\alpha}}.$$
Our function f has a minimum at the point $t^* = \left(\left(\frac{a_1}{b_1}+1\right)^{1/\alpha} - 1\right)^{-1}$.

  1. If $a_1/b_1 \geq 2^\alpha - 1$, then $t^* \leq 1$. Therefore
$$\begin{aligned} p_{\alpha,1}(x_1,x_2,x_3) &\geq \frac{f(1)}{|b|^{1+\alpha}|a+b|^{1+\alpha}} \\ &= \frac{a_1^2 + b_1^2 + 2a_1 b_1(1-2^\alpha)}{|b|^{1+\alpha}|a+b|^{1+\alpha}} \\ &= (2^\alpha - 1)\frac{(a_1-b_1)^2}{|b|^{1+\alpha}|a+b|^{1+\alpha}} + (2-2^\alpha)\frac{a_1^2+b_1^2}{|b|^{1+\alpha}|a+b|^{1+\alpha}} \\ &\geq \frac{2-2^\alpha}{2}\frac{(a_1+b_1)^2}{|b|^{1+\alpha}|a+b|^{1+\alpha}}. \end{aligned}$$
  2. If $a_1/b_1 < 2^\alpha - 1$, then $t^* > 1$. Hence,
$$p_{\alpha,1}(x_1,x_2,x_3) \geq \frac{f(t^*)}{|b|^{1+\alpha}|a+b|^{1+\alpha}}.$$
Since
$$f(t^*) = b_1^2\left(1+\frac{a_1}{b_1}\right)\left(1 - \frac{a_1}{\left((a_1+b_1)^{1/\alpha} - b_1^{1/\alpha}\right)^\alpha}\right),$$
then
$$\begin{aligned} f(t^*) &\geq b_1^2 \min_{a_1 < b_1(2^\alpha - 1)} \left(1 - \frac{a_1}{\left((a_1+b_1)^{1/\alpha} - b_1^{1/\alpha}\right)^\alpha}\right) \\ &= b_1^2(2-2^\alpha) \geq \frac{2-2^\alpha}{2^{2\alpha}}(a_1+b_1)^2, \end{aligned}$$
since the function
$$g(x) = 1 - \frac{x}{\left((x+b_1)^{1/\alpha} - b_1^{1/\alpha}\right)^\alpha}$$
is decreasing and $(a_1+b_1)^2 \leq (2^\alpha b_1)^2$.

Now, If $x_1^1 = x_2^1 = x_3^1$, then $a_1 = b_1 = 0$. Hence (64) gives us $p_{\alpha,1}(x_1, x_2, x_3) = 0$. On the other hand, if $p_{\alpha,1}(x_1, x_2, x_3) = 0$, inequality (63) gives us $\max((x_2^i - x_1^i)^2, (x_3^i - x_2^i)^2, (x_3^i - x_1^i)^2) = 0$, hence $a_1^2 = b_1^2 = (a_1 + b_1)^2 = 0$, which implies $x_1^1 = x_2^1 = x_3^1$. □

We are now ready to prove Proposition 10. Take a compact subset E of the x_1-axis with positive finite α-dimensional Hausdorff measure. Then by [Pr1, Theorem 1.1], $\Gamma_\alpha(E) = 0$. It remains to show that $\Gamma_{\alpha,\hat{1}}(E) > 0$. For this let μ be the α-dimensional Hausdorff measure restricted to E. Choosing appropriately E we can assume in addition that μ satisfies the Ahlfors regularity condition $\mu(B(x,r)) \simeq r^\alpha$, $0 < r < \operatorname{diam}(E)$. In particular, μ has growth α and is doubling. It is enough to show that the singular integral operator R_α^i associated with the scalar kernel $k_\alpha^i = x_i/|x|^{1+\alpha}$, $i \neq 1$, is bounded on $L^2(\mu)$. This reduction is possible because the Davie-Oksendal Lemma extends straightforwardly to several operators [MaPa, Lemma 4.2]. By Lemma 11 we have $p_{\alpha,i}(x_1, x_2, x_3) = 0$ for x_1, x_2 and x_3 in E and $i \neq 1$ and thus (62) yields

$$\int |R_{\alpha,\varepsilon}^i(\mu)(x)|^2 \, d\mu(x) \leq C \, \|\mu\|, \quad \epsilon > 0.$$

Replacing in the above inequality μ by $\chi_B \, \mu$ where B is any ball we get

$$\int_B |R_{\alpha,\varepsilon}^i(\chi_B \, \mu)(x)|^2 \, d\mu(x) \leq C \, \mu(B), \quad \epsilon > 0.$$

By the standard $T(1)$-Theorem of [DaJ] we conclude that R_α^i is bounded on $L^2(\mu)$. □

6.2 Lipschitz harmonic capacity is not comparable to the capacity associated with a scalar Riesz-potential

Inequality (6) says that in the plane, analytic capacity can be characterized in terms of either capacity κ_i, $i = 1, 2$. In particular this implies a weaker qualitative statement, namely, that if E is a compact set in the plane and there exists a non-zero distribution T supported on E with bounded potential $x_i/|x|^2 * T$, for $i = 1$ or $i = 2$, then there exists another non-zero distribution S supported on E with bounded potentials $x_i/|x|^2 * S$, $i = 1, 2$.

In $\mathbb{R}^n$ Lipschitz harmonic capacity is an excellent replacement for analytic capacity. Thus one may ask whether Lipschitz harmonic capacity can be described in terms of one of the capacities associated with a component of the kernel $x/|x|^n$ in which the growth condition $n-1$ has been required on the distributions involved. In a qualitative way we ask the following question. Assume that E is a compact set in $\mathbb{R}^n$ and that there exists a non-zero distribution T supported on E with growth $n-1$ and bounded potential $x_n/|x|^n * T$. Is it true that there exists another non-zero distribution S supported on E with bounded vector valued potential $x/|x|^n * T$? The answer is no for $n \geq 3$. We describe the example in $\mathbb{R}^3$.

Proposition 12. *There exists a compact set $E \subset \mathbb{R}^3$ which supports a non-zero distribution T with growth 2 and bounded scalar Riesz potential $x_3/|x|^3 * T$, but does not support any non-zero distribution S with bounded vector valued Riesz potential $x/|x|^3 * S$.*

Proof. Let $K \subset H = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0\}$ be the classical 1-dimensional planar Cantor set defined by taking the "corner quarters" at each generation. Then K has finite positive length but zero analytic capacity (see [Ga1], [Ga2] or [I]). In particular, K has zero Lipschitz harmonic capacity and by [MaPa] the same happens to $E = K \times [-1, 1]$. Thus E does not support any distribution S with bounded vector valued Riesz potential $x/|x|^3 * S$.

Let μ denote 2-dimensional Hausdorff measure restricted to $K \times \mathbb{R} \subset \mathbb{R}^3$ and let ν denote the restriction of μ to E. It is a simple matter to check that μ satisfies the growth condition

$$\mu(B(x, r)) \leq C\, r^2, \quad x \in K \times \mathbb{R}, \quad 0 < r.$$

Although the reverse inequality does not hold for large r, μ is a doubling measure. Indeed, $\mu(B(x, r))$ is comparable to r^2 for $0 < r \leq 1$ and to r for $1 \leq r$. Our goal is to show that the scalar Riesz singular integral operator R^3 with kernel $k^3(x) = x_3/|x|^3$ is bounded on $L^2(\nu)$. Once this is established the Davie-Oksendal lemma (see [Ch, Theorem 33] or [V3, Lemma 4.2]) provides a non-negative function $b \in L^\infty(\nu)$ such that $x_3/|x|^3 * b\nu$ is in $L^\infty(\mathbb{R}^3)$, which completes the proof.

The boundedness of R^3 on $L^2(\nu)$ follows directly from the boundedness of R^3 on $L^2(\mu)$. To show this we check that $R^3(1) = 0$ and then we apply the standard $T(1)$-Theorem for doubling measures (see [DaJ]). The computation of $R^3(1)$ is performed as follows. Set $K(x, \epsilon) = \{(y_1, y_2) \in K : |x_1 - y_1| > \epsilon \quad \text{and} \quad |x_2 - y_2| > \epsilon\}$, Then

$$\begin{aligned} R^3(1)(x) &= \lim_{\epsilon \to 0} \int_{|y-x|>\epsilon} \frac{x_3 - y_3}{|x-y|^3}\, d\mu(y) \\ &= \lim_{\epsilon \to 0} \int_{K(x,\epsilon)} \left(\int_{|y_3 - x_3|>\epsilon} \frac{x_3 - y_3}{|x-y|^3}\, dy_3 \right) dH^1(y_1, y_2) = 0, \end{aligned}$$

for each $x \in K \times \mathbb{R}$. □

Remarks

- Notice that in the above example one obtains that R^3 is bounded on $L^2(\nu)$, while the whole vector R is not bounded on $L^2(\nu)$. Therefore, the above example shows that corollary 3 does not hold if $n \geq 3$, namely, we cannot get $L^2(\nu)$ boundedness of the vector valued Riesz operator R_{n-1} associated with a Riesz kernel of homogeneity $-(n-1)$ from $L^2(\nu)$ boundedness of only one component R^i_{n-1}.

- It is an open question to decide whether, for $n \geq 3$, Lipschitz harmonic capacity is comparable to the capacities associated with $(n-1)$-components of the vector valued Riesz potential $x/|x|^n * T$.

6.3 Finiteness of the capacities κ_i

Indeed, we give a proof of a more general result, stating that for compact sets $E \subset \mathbb{R}^n$, $0 < \alpha < n$ and $1 \leq i \leq n$, the capacities

$$\kappa_{\alpha,\,i}(E) = \sup \left\{ |\langle T, 1 \rangle| : \operatorname{spt}(T) \subset E, \left\| \frac{x_i}{|x|^{1+\alpha}} * T \right\|_\infty \leq 1 \right\},$$

are finite.

Proposition 13. *For any cube $Q \subset \mathbb{R}^n$, $0 < \alpha < n$ and $1 \leq i \leq n$, we have*

$$\kappa_{\alpha,\,i}(Q) \leq C l(Q)^\alpha.$$

Proof. Without loss of generality assume $i = 1$. Assume also momentarily that the dimension n is odd, say $n = 2k+1$. Our argument uses a reproduction formula for test functions involving the kernel $k^i(y) = y_i/|y|^{1+\alpha}$, $1 \leq i \leq n$, [Pr1, Lemma 3.1]. For a test function g, the formula reads

$$g(x) = c_{n,\alpha} \sum_{j=1}^{n} \left(\Delta^k \partial_j g * \frac{1}{|y|^{n-\alpha}} * k^j \right)(x), \tag{65}$$

for some constant $c_{n,\alpha}$ depending only on the dimension n and on α. For $n = 2k$, there is an analogous reproduction formula that settles the even case [Pr1, Lemma 3.1].

Let T be a real distribution supported on Q such that $k^1 * T \in L^\infty(\mathbb{R}^n)$. Write the cube Q as $Q = I_1 \times Q'$, with I_1 being an interval in $\mathbb{R}$ and Q' an $n-1$ dimensional cube in $\mathbb{R}^{n-1}$, and let $\varphi_Q \in \mathcal{C}_0^\infty(2Q)$ be such that $\|\partial^s \varphi_Q\|_\infty \leq C_s l(Q)^{-|s|}$ and

$$\varphi_Q(x) = \varphi_1(x_1)\varphi_2(x_2, \ldots, x_n)$$

with $\varphi_1(x_1) = 1$ on I_1, $\varphi_1(x_1) = 0$ on $(2I_1)^c$ and $\int_{-\infty}^{\infty} \varphi_1 = 0$, and $\varphi_2 \geq 0$, $\varphi_2 \equiv 1$ on Q' and $\varphi_2 \equiv 0$ on $(2Q')^c$. Then, since our distribution T is supported on Q, using the reproduction formula (65),

$$\begin{aligned}
|\langle T, 1 \rangle| = |\langle T, \varphi_Q \rangle| &\leq C \sum_{j=1}^{n} \left| \left\langle T, \Delta^k \partial_j \varphi_Q * \frac{1}{|y|^{n-\alpha}} * k^j \right\rangle \right| \\
&= C \left| \left\langle k^1 * T, \Delta^k \partial_1 \varphi_Q * \frac{1}{|y|^{n-\alpha}} \right\rangle \right| + C \sum_{j=2}^{n} \left| \left\langle T, \Delta^k \partial_j \varphi_Q * \frac{1}{|y|^{n-\alpha}} * k^j \right\rangle \right| \\
&= A + B.
\end{aligned}$$

We first estimate the term A. We have

$$\int (k^1 * T)(x)\, \Delta^k \partial_1 \varphi_Q * \frac{1}{|y|^{n-\alpha}}(x)\, dx = \int_{3Q} (k^1 * T)(x)\, (\Delta^k \partial_1 \varphi_Q * \frac{1}{|y|^{n-\alpha}})(x)\, dx$$

$$+ \int_{\mathbb{R}^n \setminus 3Q} (k^1 * T)(x)\, (\varphi_Q * \Delta^k \partial_1 (\frac{1}{|y|^{n-\alpha}}))(x)\, dx.$$

Let Q_0 be the unit cube centered at 0. Dilating to bring the integrals on $3Q_0$ and $2Q_0$, and using $|\partial^s \varphi_Q| \le C_s l(Q)^{-|s|}$, we get

$$\begin{aligned} A &\le \|k^1 * T\|_\infty \left(\int_{3Q} \int_{2Q} \frac{|\Delta^k \partial_1 \varphi_Q(y)|}{|x-y|^{n-\alpha}}\, dy\, dx + \int_{\mathbb{R}^n \setminus 3Q} \int_{2Q} \frac{|\varphi_Q(y)|}{|x-y|^{2n-\alpha}}\, dy\, dx \right) \\ &\le C l(Q)^\alpha \left(\int_{3Q_0} \int_{2Q_0} \frac{dy\, dx}{|x-y|^{n-\alpha}} + \int_{\mathbb{R}^n \setminus 3Q_0} \int_{2Q_0} \frac{dy\, dx}{|x-y|^{2n-\alpha}} \right) \\ &\le C l(Q)^\alpha. \end{aligned}$$

We turn now to the estimate of B. The homogeneous differential operator Δ^k can be written as $\Delta^k = \sum_{|s|=2k} a_s\, \partial^s$, for certain constants a_s. Divide the set of multi-indexes s of length $2k$ into two classes I and J according to whether $s_1 \ge 1$ or $s_1 = 0$. In other words, $s \in I$ if ∂^s contains at least one partial derivative with respect to first variable. Thus $\Delta^k = \sum_{s\in I} a_s\, \partial^s + \sum_{s \in J} a_s\, \partial^s$, and so $B = B_1 + B_2$ where

$$B_1 = C \sum_{j=2}^n \left| \left\langle T, \sum_{s \in I} a_s\, \partial^s \partial_j \varphi_Q * \frac{1}{|y|^{n-\alpha}} * k^j \right\rangle \right|$$

and

$$B_2 = C \sum_{j=2}^n \left| \left\langle T, \sum_{s \in J} a_s\, \partial^s \partial_j \varphi_Q * \frac{1}{|y|^{n-\alpha}} * k^j \right\rangle \right|.$$

To estimate B_1 we bring in each term of the sum in $s \in I$ one derivative with respect to the first variable into the kernel k^j and use $\partial_1 k^j = \partial_j k^1$ to take back a derivative with respect to j into φ_Q. The effect of these moves is to replace k^j by k^1. Therefore

$$B_1 = C \sum_{j=2}^n \left| \left\langle k^1 * T, \sum_{|s|=2k} b_s\, \partial^s \partial_j \varphi_Q * \frac{1}{|y|^{n-\alpha}} \right\rangle \right|,$$

for some numbers b_s. This expression can be estimated as we did before with A.

To estimate B_2 we need to replace in some way the kernel k^j by k^1. We do this by showing that, for each j there exists a function $\psi_Q^j \in \mathcal{C}_0^\infty(2Q)$ satisfying

$$k^j * \varphi_Q = k^1 * \psi_Q^j, \quad 1 \le j \le n, \tag{66}$$

and $\|\partial^s \psi_Q^j\|_\infty \le C_s l(Q)^{-|s|}$. Before proving (66) we show how to estimate B_2.

By (66)

$$B_2 = C\sum_{j=2}^n \left|\left\langle T, \sum_{s\in J} a_s\, \partial^s \partial_j \varphi_Q * \frac{1}{|y|^{n-\alpha}} * k^j \right\rangle\right|$$

$$= C\sum_{j=2}^n \left|\left\langle T, \sum_{s\in J} a_s\, \partial^s \partial_j \psi_Q^j * \frac{1}{|y|^{n-\alpha}} * k^1 \right\rangle\right|$$

$$= C\sum_{j=2}^n \left|\left\langle k^1 * T, \sum_{s\in J} a_s\, \partial^s \partial_j \psi_Q^j * \frac{1}{|y|^{n-\alpha}} \right\rangle\right|,$$

which can be estimated as the term A.

We are left with proving (66). Taking Fourier transforms in (66) we obtain for some constant a,

$$a\, \hat\varphi_Q(\xi)\xi_j = \hat{\psi_Q^j}(\xi)\xi_1,$$

which becomes

$$a\, \partial_j \varphi_Q = \partial_1 \psi_Q^j.$$

Hence, for the non-trivial case $2 \le j \le n$,

$$\psi_Q^j(x) = a\int_{-\infty}^{x_1} \partial_j \varphi_Q(t, x_2, \dots, x_n)\, dt = a\, \partial_j \varphi_2(x_2, \dots, x_n) \int_{-\infty}^{x_1} \varphi_1(t)\, dt,$$

and the key remark is that the function above has support contained in $2Q$ because the integral of φ_1 on the real line vanishes. □

We conclude with the following corollary.

Corollary 14. *For any compact set $E \subset \mathbb{R}^n$, $0 < \alpha < n$ and $1 \le i \le n$, we have $\kappa_{\alpha, i}(E) \le C \operatorname{diam}(E)^\alpha$.*

When $n = 2$ and $\alpha = 1$, (6) implies that $\kappa_i(E) \le CM^1(E)$, $i = 1, 2$, where M stands for the one dimensional Hausdorff content. In general, we do not know whether in the preceding inequality the diameter of E can be replaced by the $\alpha-$dimensional Hausdorff content of E.

6.4 Localization and growth

The growth assumption on the distribution T in the localization lemma (Lemma 5) cannot be completely dispensed with. Indeed, if $x_i/|x|^2 \in L^\infty(\mathbb{R}^n)$ and one has the inequality

$$\left\| \frac{x_i}{|x|^2} * \varphi_Q T \right\|_\infty \le C \left\| \frac{x_i}{|x|^2} * T \right\|_\infty, \tag{67}$$

for all $\varphi_Q \in \mathcal{C}_0^\infty(Q)$ satisfying the normalization condition (15), then necessarily T has linear growth. This can be shown by an argument very close to that of the previous subsection. We only deal with the details of the case $n = 2$. The case of even dimensions is very similar, while the case of odd dimensions needs some additional care. We also assume $i = 1$.

Let Q be square and φ_Q a function in $\mathcal{C}_0^\infty(Q)$ satisfying the normalization condition (15). Set $Q = I_1 \times I_2$ and $\psi(x_1, x_2) = \psi_1(x_1)\psi(x_2)$, where, for $j = 1, 2$, $\psi_j \in \mathcal{C}_0^\infty(I_j)$, $\psi_j = 1$ on I_j, $\int_{-\infty}^{\infty} \psi(x_1)\, dx_1 = 0$ and $\|d^k\psi_j/(dx_j)^k\|_\infty \le C\, l(I_j)^{-k}$, $0 \le k \le 2$. We then have

$$\langle T, \varphi_Q\rangle = \langle \varphi_Q\, T, 1\rangle = \langle \varphi_Q\, T, \psi\rangle.$$

We want now to find a function χ such that $\psi = k^1 * \chi$, where $k^1 = x_1/|x|^2$. Taking the Fourier transform we get $\hat{\psi}(\xi) = a(\xi_1/|\xi|^2)\, \hat{\chi}(\xi)$ for some constant a. Hence $\partial_1\chi = b\, \Delta\psi$, for some other constant b. Thus

$$\begin{aligned}\chi &= b \int_{-\infty}^{x_1} \Delta\psi(t, x_2)\, dt \\ &= b\left(\partial_1\psi_1(x_1)\, \psi_2(x_2) + \left(\int_{-\infty}^{x_1} \psi_1(t)\, dt\right) \partial_2^2\psi_2(x_2)\right).\end{aligned}$$

Notice that χ is supported on Q and $\|\chi\|_\infty \le C\, l(Q)^{-1}$. Therefore

$$|\langle T, \varphi_Q\rangle| = |\langle k^1 * \varphi_Q T, \chi\rangle| \le C\, \|k^1 * \varphi_Q T\|_\infty\, \|\chi\|_{L^1(Q)} \le C\, l(Q). \qquad \square$$

6.5 The growth condition for positive measures

We start by showing that the usual linear growth condition for a positive Radon measure is equivalent to the linear growth condition for distributions as defined in (14). Later on we treat also the case of the α-growth condition for $0 < \alpha < n$.

Given a positive Radon measure μ set

$$L(\mu) = \sup_Q \frac{\mu(Q)}{l(Q)},$$

where the supremum is taken over all cubes Q with sides parallel to the coordinate axis.

If $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, then by an inequality of Mazya [Mz, 1.2.2, p. 24]

$$|\langle \mu, \varphi\rangle| = |\int \varphi\, d\mu| \le \int |\varphi|\, d\mu \le C\, L(\mu) \int |\nabla^{n-1}\varphi(x)|\, dx,$$

where $\nabla^{n-1}\varphi$ denotes, as usual , the vector of all derivatives $\partial^s\varphi$ of order $|s| = n-1$. Thus

$$G(\mu) \le C\, L(\mu).$$

The reverse inequality is immediate. Indeed, given a cube Q let φ_Q be a function in $\mathcal{C}_0^\infty(2Q)$ such that $1 \le \varphi_Q$ on Q and $\|\partial^s \varphi_Q\|_\infty \le C_s\, l(Q)^{-|s|}$, $|s| \ge 0$. Then

$$\mu(Q) \le \int \varphi_Q\, d\mu = |\langle \mu, \varphi_Q \rangle| \le C\, G(\mu)\, l(Q)$$

because $C_s^{-1}\, l(Q)^{-1}\, \partial^s \varphi$ is an atom for $|s| = n-1$, and so $\|\partial^s \varphi\|_{H^1(\mathbb{R}^n)} \le C\, l(Q), |s| = n-1$.

We proceed now to treat the case of a general α-growth condition, $0 < \alpha < n$. Set

$$L_\alpha(\mu) = \sup_Q \frac{\mu(Q)}{l(Q)^\alpha},$$

where the supremum is taken over all cubes Q with sides parallel to the coordinate axis. We consider first the inequality $L_\alpha(\mu) \le C\, G_\alpha(\mu)$. The definition of G_α is in (58). Given a cube Q let φ_Q be a function in $\mathcal{C}_0^\infty(2Q)$ such that $1 \le \varphi_Q$ on Q and $\|\partial^s \varphi_Q\|_\infty \le C_s\, l(Q)^{-|s|}$, $|s| \ge 0$. We claim that $c\, \varphi_Q$ satisfies the normalization inequalities (59) or (60) for a sufficiently small positive constant c. If this is the case, then

$$\mu(Q) \le \int \varphi_Q\, d\mu = |\langle \mu, \varphi_Q \rangle| \le c^{-1}\, G_\alpha(\mu)\, l(Q).$$

We treat first the case of integer α. Clearly $\|\partial^s \varphi_Q\|_{L^1} \le C l(Q)^\alpha$, $|s| = n - \alpha$. By Hölder's inequality and the fact that Riesz transforms preserve $L^q(\mathbb{R}^n)$, $1 < q < \infty$,

$$\|R_j(\partial^s \varphi_Q)\|_{L^1(4Q)} \le C l(Q)^{\frac{n}{p}} \|R_j(\partial^s \varphi_Q)\|_{L^q(\mathbb{R}^n)} \le C l(Q)^{\frac{n}{p}} \|\partial^s \varphi_Q\|_{L^q(\mathbb{R}^n)} \le C l(Q)^\alpha.$$

Then, by the Sublemma in subsection 4.1, the function φ_Q satisfies the normalization inequalities (59).

If $\alpha \notin \mathbb{Z}$, write $\alpha = [\alpha] + \{\alpha\}$, with $[\alpha] \in \mathbb{Z}$ and $0 < \{\alpha\} < 1$. For the claim we have to show that for $|s| = n - [\alpha]$, and $1 \le j \le n$,

$$\|\partial^s \varphi_Q * \frac{1}{|x|^{n-\{\alpha\}}}\|_{L^1(\mathbb{R}^n)} \le l(Q)^\alpha \tag{68}$$

$$\|R_j(\partial^s \varphi_Q * \frac{1}{|x|^{n-\{\alpha\}}})\|_{L^1(\mathbb{R}^n)} \le l(Q)^\alpha. \tag{69}$$

Inequality (68) is proven as follows. By Fubini,

$$\int_{4Q} |(\partial^s \varphi_Q * \frac{1}{|x|^{n-\{\alpha\}}})(x)| dx \le C l(Q)^\alpha.$$

As in the Sublemma, integrating by parts to take one derivative from $\partial^s \varphi_Q$ to the kernel $1/|x|^{n-\{\alpha\}}$ we obtain

$$\int_{(4Q)^c} |(\partial^s \varphi_Q * \frac{1}{|x|^{n-\{\alpha\}}})(x)| dx \le C l(Q)^\alpha,$$

which proves (68).

The prove of inequality (69) we use that, for some constant $c = c(n, \alpha)$,

$$R_j(\partial^s \varphi_Q * \frac{1}{|x|^{n-\{\alpha\}}}) = c\, \partial^s \varphi_Q * \frac{x_j}{|x|^{n+1-\{\alpha\}}}.$$

This can be easily checked by taking the Fourier transform. Now the argument described above to prove (68) applies with small changes to prove (69).

For the reverse inequality, namely $G_\alpha(\mu) \leq C\, L_\alpha(\mu)$, it is convenient to distinguish two cases.

- α *is integer.* The argument is exactly as in the case $\alpha = 1$. If $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, then by an inequality of Mazya [Mz, 1.2.2, p. 24]

$$|\langle \mu, \varphi \rangle| = |\int \varphi\, d\mu| \leq \int |\varphi|\, d\mu \leq C\, L_\alpha(\mu) \int |\nabla^{n-[\alpha]} \varphi(x)|\, dx.$$

  Thus

$$G_\alpha(\mu) \leq C\, L_\alpha(\mu).$$

- α *is not integer.* If $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, then by another inequality of Mazya [Mz, 3.4.1, p. 134]

$$|\langle \mu, \varphi \rangle| = |\int \varphi\, d\mu| \leq \int |\varphi|\, d\mu \leq C\, L_\alpha(\mu) \int |\nabla^{n-[\alpha]} \varphi(x) * \frac{1}{|x|^{n-\{\alpha\}}}|\, dx.$$

  Thus

$$G_\alpha(\mu) \leq C\, L_\alpha(\mu).$$

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