Estimates for the maximal singular integral in terms of the singular integral: the case of even kernels

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Abstract

Let T be a smooth homogeneous Calderón-Zygmund singular integral operator in $\mathbb{R}^n$. In this paper we study the problem of controlling the maximal singular integral $T^\star f$ by the singular integral Tf. The most basic form of control one may consider is the estimate of the $L^2(\mathbb{R}^n)$ norm of $T^\star f$ by a constant times the $L^2(\mathbb{R}^n)$ norm of Tf. We show that if T is an even higher order Riesz transform, then one has the stronger pointwise inequality $T^\star f(x) \leq C\, M(Tf)(x)$, where C is a constant and M is the Hardy-Littlewood maximal operator. We prove that the L^2 estimate of $T^\star$ by T is equivalent, for even smooth homogeneous Calderón-Zygmund operators, to the pointwise inequality between $T^\star$ and $M(T)$. Our main result characterizes the L^2 and pointwise inequalities in terms of an algebraic condition expressed in terms of the kernel $\frac{\Omega(x)}{|x|^n}$ of T, where Ω is an even homogeneous function of degree 0, of class $C^\infty(S^{n-1})$ and with zero integral on the unit sphere S^{n-1}. Let $\Omega = \sum P_j$ the expansion of Ω in spherical harmonics P_j of degree j. Let A stand for the algebra generated by the identity and the smooth homogeneous Calderón-Zygmund operators. Then our characterizing condition states that T is of the form $R \circ U$, where U is an invertible operator in A and R is a higher order Riesz transform associated with a homogeneous harmonic polynomial P which divides each P_j in the ring of polynomials in n variables with real coefficients.

1 Introduction

Let T be a smooth homogeneous Calderón-Zygmund singular integral operator on $\mathbb{R}^n$ with kernel

$$K(x) = \frac{\Omega(x)}{|x|^n}, \quad x \in \mathbb{R}^n \setminus \{0\}\,, \tag{1}$$

where Ω is a (real valued) homogeneous function of degree 0 whose restriction to the unit sphere S^{n-1} is of class $C^\infty(S^{n-1})$ and satisfies the cancellation property

$$\int_{|x|=1} \Omega(x)\, d\sigma(x) = 0\,,$$

σ being the normalized surface measure on S^{n-1}. Recall that Tf is the principal value convolution operator

$$Tf(x) = P.V. \int f(x-y)\, K(y)\, dy \equiv \lim_{\epsilon \to 0} T^{\epsilon} f(x)\,, \tag{2}$$

where T^{ϵ} is the truncation at level ϵ defined by

$$T^{\epsilon} f(x) = \int_{|y-x|>\epsilon} f(x-y) K(y)\, dy\,.$$

As we know, the limit in (2) exists for almost all x for f in $L^p(\mathbb{R}^n)$, $1 \le p < \infty$.

Let $T^{\star}$ be the maximal singular integral

$$T^{\star} f(x) = \sup_{\epsilon>0} |T^{\epsilon} f(x)|, \quad x \in \mathbb{R}^n\,.$$

In this paper we consider the problem of controlling $T^{\star}f$ by Tf. The most basic form of control one may think of is the L^2 estimate

$$\|T^{\star} f\|_2 \le C \|Tf\|_2, \quad f \in L^2(\mathbb{R}^n)\,. \tag{3}$$

Another way of saying that $T^{\star}f$ is dominated by Tf, apparently much stronger, is provided by the pointwise inequality

$$T^{\star} f(x) \le C\, M(Tf)(x), \quad x \in \mathbb{R}^n\,, \tag{4}$$

where M denotes the Hardy-Littlewood maximal operator. Notice that (4) may be viewed as an improved version of classical Cotlar's inequality

$$T^{\star} f(x) \le C\left(M(Tf)(x) + Mf(x)\right), \quad x \in \mathbb{R}^n\,,$$

because the term involving Mf is missing in the right hand side of (4).

We prove that if T is an even higher order Riesz transform, then (4) holds. Recall that T is a higher order Riesz transform if its kernel is given by a function Ω of the form

$$\Omega(x) = \frac{P(x)}{|x|^d}, \quad x \in \mathbb{R}^n \setminus \{0\}\,,$$

with P a homogeneous harmonic polynomial of degree $d \ge 1$. If $P(x) = x_j$, then one obtains the j-th Riesz transform R_j. If the homogeneous polynomial P is not required to be harmonic, but has still zero integral on the unit sphere, then we call T a polynomial operator.

Thus, if $T = R$ is an even higher order Riesz transform, one has the weak L^1 type inequality

$$\|R^{\star} f\|_{1,\infty} \le C \|Rf\|_1\,, \tag{5}$$

which combined with the classical weak L^1 type estimate

$$\|R^{\star} f\|_{1,\infty} \le C \|f\|_1\,,$$

yields the sharp inequality

$$\|R^\star f\|_{1,\infty} \leq C \min\{\|f\|_1, \|Rf\|_1\}\,.$$

In [MV2] one proved (5) for the Beurling transform in the plane and also that (5) fails for the Riesz transforms R_j. Therefore the assumption that the operator is even is crucial.

The question of estimating $T^\star f$ by Tf was first raised in [MV2]. The problem originated in an attempt to gain a better understanding of how one can obtain a. e. existence of principal values of truncated singular integrals from L^2 boundedness, for underlying measures more general than the Lebesgue measure in $\mathbb{R}^n$. This in turn is motivated by a problem of David and Semmes [DS], which consists in deriving uniform rectifiability of a d-dimensional Ahlfors regular subset of $\mathbb{R}^n$ from the L^2 boundedness of the Riesz kernel of homogeneity $-d$ with respect to d-dimensional Hausdorff measure on the set . For more details on that see the last section of [MaV].

Our main result states that for even operators inequalities (3) and (4) are equivalent to an algebraic condition involving the expansion of Ω in spherical harmonics. This condition may be very easily checked in practice and so, in particular, we can produce extremely simple examples of even polynomial operators for which (3) and (4) fail. For these operators no control of $T^\star f$ by Tf seems to be known. To state our main result we need to introduce a piece of notation.

Recall that Ω has an expansion in spherical harmonics, that is,

$$\Omega(x) = \sum_{j=1}^{\infty} P_j(x), \quad x \in S^{n-1}\,, \tag{6}$$

where P_j is a homogeneous harmonic polynomial of degree j. If Ω is even, then only the P_j of even degree j may be non-zero.

An important role in this paper will be played by the algebra A consisting of the bounded operators on $L^2(\mathbb{R}^n)$ of the form

$$\lambda I + S\,,$$

where λ is a real number and S a smooth homogeneous Calderón-Zygmund operator.

Our main result reads as follows.

Theorem. *Let T be an even smooth homogeneous Calderón-Zygmund operator with kernel* (1) *and assume that Ω has the expansion* (6)*. Then the following are equivalent.*

(i)

$$T^\star f(x) \leq C\, M(Tf)(x), \quad x \in \mathbb{R}^n\,.$$

(ii)

$$\|T^\star f\|_2 \leq C \|Tf\|_2, \quad f \in L^2(\mathbb{R}^n)\,.$$

(iii) The operator T can be factorized as $T = R \circ U$, where U is an invertible operator in the algebra A and R is a higher order Riesz transform associated with a harmonic homogeneous polynomial P which divides each P_j in the ring of polynomials in n variables with real coefficients.

Two remarks are in order.

Remark 1. Observe that condition (iii) is algebraic in nature. This is one of the reasons that makes the proof difficult. Condition (iii) can be reformulated in a more concrete fashion as follows. Assume that the expansion of Ω in spherical harmonics is

$$\Omega(x) = \sum_{j=j_0}^{\infty} P_{2j}(x), \quad P_{2j_0} \neq 0 .$$

Then (iii) is equivalent to the following

(iv) For each j there exists a homogeneous polynomial Q_{2j-2j_0} of degree $2j - 2j_0$ such that $P_{2j} = P_{2j_0} Q_{2j-2j_0}$ and $\sum_{j=j_0}^{\infty} \gamma_{2j} Q_{2j-2j_0}(\xi) \neq 0, \quad \xi \in S^{n-1}$.

Here for a positive integer j we have set

$$\gamma_j = i^{-j} \pi^{\frac{n}{2}} \frac{\Gamma(\frac{j}{2})}{\Gamma(\frac{n+j}{2})} . \tag{7}$$

The quantities γ_j appear in the computation of the Fourier multiplier of the higher order Riesz transform R with kernel given by a homogeneous harmonic polynomial P of degree j. One has (see [St, p. 73])

$$\widehat{Rf}(\xi) = \gamma_j \frac{P(\xi)}{|\xi|^j} \hat{f}(\xi), \quad f \in L^2(\mathbb{R}^n) .$$

As we will show later, the series $\sum_{j=j_o}^{\infty} \gamma_{2j} Q_{2j-2j_0}(x)$ is convergent in $C^\infty(S^{n-1})$. If U is defined as the operator in the algebra A whose Fourier multiplier is $\gamma_{2j_0}^{-1}$ times the sum of the preceding series, then $T = R \circ U$ for the higher order Riesz transform R given by the polynomial P_{2j_0}. This shows that (iii) follows from (iv).

To show that (iii) implies (iv) we prove that $P = \lambda P_{2j_0}$ for some real number $\lambda \neq 0$. Since P divides P_{2j_0} by assumption, we only need to show that the degree d of P must be $2j_0$. Now, let $\mu(\xi)$ denote the Fourier multiplier of U, so that μ is a smooth function with no zeros on the sphere. The Fourier multiplier of T is

$$\sum_{j=j_0}^{\infty} \gamma_{2j} P_{2j}(\xi) = \gamma_d P(\xi) \mu(\xi), \quad \xi \in S^{n-1} .$$

If d is less than $2j_0$, then P is orthogonal to all P_{2j} [St, p. 69] and so

$$\int_{|\xi|=1} P(\xi)^2 \mu(\xi) \, d\xi = 0 ,$$

which yields $P(\xi) = 0$, $|\xi| = 1$, a contradiction.

Remark 2. Condition (iii) is rather easy to check in practice. For instance, take $n = 2$ and consider the polynomial of fourth degree

$$P(x, y) = xy + x^4 + y^4 - 6\,x^2y^2\,.$$

The polynomial operator associated to P does not satisfy (i) nor (ii), because the definition of P above is also the spherical harmonics expansion of P and xy clearly does not divide $x^4 + y^4 - 6\,x^2y^2$. Section 7 contains other examples of polynomial operators that do not satisfy (i) nor (ii).

On the other hand, the polynomial operator associated with

$$P(x, y) = xy + x^3y - y^3x\,,$$

does satisfy (i) and (ii), but this is not the case for the operator determined by

$$P(x, y) = xy + 2(x^3y - y^3x)\,,$$

although xy obviously divides $x^3y - y^3x$. See section 8 for the details.

Thus the condition on Ω so that T satisfies (i) or (ii) is rather subtle.

Having clarified the statement of the Theorem and some of its implications, we say now a few words on the proofs and the organization of the paper.

We devote sections 2, 3 and 4 to the proof of "(iii) implies (i)", which we call the sufficient condition. In section 2 we prove that the even higher order Riesz transforms satisfy (i). Section 3 is devoted to the proof of the sufficient condition for polynomial operators. The argument is an extension of that used in the previous section. The drawback is that we loose control on the dependence of the constants on the degree of the polynomial. The main difficulty we have to overcome in section 4 to complete the proof of the sufficient condition in the general case, is to find a second approach to the polynomial case which gives some estimates with constants independent of the degree of the polynomial. This allows us to use a compactness argument to finish the proof. It is an intriguing fact that the approach in section 3 cannot be dispensed with, because it provides certain properties which are vital for the final argument and do not follow otherwise.

In sections 5 and 6 we prove the necessary condition, that is, "(ii) implies (iii)". In section 5 we deal with the polynomial case. Analysing the inequality (ii) via Plancherel at the frequency side we obtain various inclusion relations among zero sets of certain polynomials. This requires a considerable combinatorial effort for reasons that will become clear later on. We found in Maple a formidable ally in formulating the right identities which were needed, which we proved rigorously afterwards. In a second step we solve the division problem which leads us to (iii) by a recurrent argument with some algebraic geometry ingredients, the Hilbert's Nullstellensatz in particular. The question of independence on the degree of the polynomial appears again, this time related to the coefficients of certain expansions. We deal with this problem in section 6. Section 7 is devoted to the proof of the intricate combinatorial lemmas used in the previous sections. In section 8 we discuss some examples and we ask a couple of questions that we have not been able to answer.

Our methods are a combination of classical Fourier analysis techniques and Calderón-Zygmund theory with potential theoretic ideas coming from our previous work [MO], [MPV], [MV1], [MNOV], [Ve1] and [Ve2].

As it was discovered in [MV2], there is remarkable difference between the odd and even cases for the problem we consider. To keep at a reasonable size the length of this article we decided to only deal here with the even case, which is more difficult, because one needs special L^∞ estimates for singular integrals, which hold only in the even case. The results for the odd case will be published elsewhere [MOPV].

The L^∞ estimates mentioned above are not obvious even for the simplest even homogeneous Calderón-Zygmund operator, the Beurling transform, which plays an important role in planar quasiconformal mapping theory. An application of our estimates to planar quasiconformal mappings is given in [MOV].

2 Even higher order Riesz transforms

In this section we prove that if T is an even higher order Riesz transform, then

$$T^\star f(x) \le C\, M(Tf)(x), \quad x \in \mathbb{R}^n\,. \tag{8}$$

Let B be the open ball of center 0 and radius 1, ∂B its boundary and $\overline{B}$ its closure. In proving (8) we will encounter the following situation. We are given a function φ defined by different formulae in B and $\mathbb{R}^n \setminus \overline{B}$, which is differentiable up to order N on $B \cup (\mathbb{R}^n \setminus \overline{B})$ and whose derivatives up to order $N-1$ extend continuously up to ∂B. The question is to compare the distributional derivatives of order N with the expressions one gets on B and $\mathbb{R}^n \setminus \overline{B}$ by taking ordinary derivatives. The next simple lemma is a sample of what we need.

Lemma 1. *Let φ be a continuously differentiable function on $B \cup (\mathbb{R}^n \setminus \overline{B})$ which extends continuously to ∂B. Then we have the identity*

$$\partial_j \varphi = \partial_j \varphi(x) \chi_B(x) + \partial_j \varphi(x) \chi_{\mathbb{R}^n \setminus \overline{B}}(x)\,,$$

where the left hand side is the j-th distributional derivative of φ.

Proof. Let ψ be a test function. Then

$$\langle \partial_j \varphi, \psi \rangle = -\int \varphi\, \partial_j \psi = -\int_B \varphi\, \partial_j \psi - \int_{\mathbb{R}^n \setminus \overline{B}} \varphi\, \partial_j \psi\,.$$

Now apply Green-Stokes' theorem to the domains B and $\mathbb{R}^n \setminus \overline{B}$ to move the derivatives from ψ to φ. The boundary terms cancel precisely because of the continuity of φ on ∂B, and we get

$$\langle \partial_j \varphi, \psi \rangle = \int (\chi_B\, \partial_j \varphi + \chi_{\mathbb{R}^n \setminus \overline{B}}\, \partial_j \varphi) \psi\, dx\,.$$

$\square$

We need an analog of the previous statement for second order derivatives and radial functions, which is the case we take up in the next corollary.

Corollary 2. *Assume that φ is a radial function of the form*

$$\varphi(x) = \varphi_1(|x|)\chi_B(x) + \varphi_2(|x|)\chi_{\mathbb{R}^n\setminus\overline{B}}(x)\,,$$

where φ_1 is continuously differentiable on $[0,1)$ and φ_2 on $(1,\infty)$. Let L be a second order differential operator with constant coefficients. Then the distribution $L\varphi$ satisfies

$$L\varphi = L\varphi(x)\chi_B(x) + L\varphi(x)\chi_{\mathbb{R}^n\setminus\overline{B}}(x)\,,$$

provided φ_1, φ_1', φ_2 and φ_2' extend continuously to the point 1 and the two conditions

$$\varphi_1(1) = \varphi_2(1), \quad \varphi_1'(1) = \varphi_2'(1)\,,$$

are satisfied.

Proof. The proof reduces to applying Lemma 1 twice. Before the second application one should remark that the hypothesis $\varphi_1'(1) = \varphi_2'(1)$ gives the continuity of all first order partial derivatives of φ. $\square$

We proceed now to describe in detail the main argument for the proof of (8). By translating and dilating one reduces the proof of (8) to

$$|T^1 f(0)| \le C\, M(Tf)(0)\,, \tag{9}$$

where

$$T^1 f(0) = \int_{|y|>1} f(y)K(y)\,dy$$

is the truncated integral at level 1. Recall that the kernel of our singular integral is

$$K(x) = \frac{\Omega(x)}{|x|^n} = \frac{P(x)}{|x|^{n+d}}\,,$$

where P is an even homogeneous harmonic polynomial of degree $d \ge 2$. The idea is to obtain an identity of the form

$$K(x)\chi_{\mathbb{R}^n\setminus\overline{B}}(x) = T(b)(x)\,, \tag{10}$$

for some measurable bounded function b supported on B. Once (10) is at our disposition we get, for f in some $L^p(\mathbb{R}^n)$, $1 \le p < \infty$,

$$\begin{aligned} T^1 f(0) &= \int \chi_{\mathbb{R}^n\setminus\overline{B}}(y)\, K(y)\, f(y)\,dy \\ &= \int T(b)(y)\, f(y)\,dy \\ &= \int_B b(y)\, Tf(y)\,dy\,, \end{aligned}$$

and so (9) follows with $C = V_n \, \|b\|_\infty$, V_n being the volume of the unit ball of $\mathbb{R}^n$.

Let us turn our attention to the proof of (10). Set $d = 2N$ and let E be the standard fundamental solution of the N-th power $\triangle^N$ of the Laplacean. Consider the function

$$\varphi(x) = E(x)\, \chi_{\mathbb{R}^n \setminus \overline{B}}(x) + (A_0 + A_1\,|x|^2 + \cdots + A_{d-1}\,|x|^{2d-2})\, \chi_B(x)\,, \tag{11}$$

where the constants $A_0, A_1, \dots, A_{d-1}$ are chosen as follows. Since $\varphi(x)$ is radial, the same is true of $\triangle^j \varphi$ for each positive integer j. Thus, in order to apply the preceding corollary N times one needs $2N = d$ conditions, which (uniquely) determine $A_0, A_1, \dots, A_{d-1}$. Therefore, for some constants $\alpha_1, \alpha_2, \dots, \alpha_{N-1}$,

$$\triangle^N \varphi = (\alpha_0 + \alpha_1 |x|^2 + \cdots + \alpha_{N-1} |x|^{2(N-1)}) \chi_B(x) = b(x)\,, \tag{12}$$

where the last identity is a definition of b. Since

$$\varphi = E \star \triangle^N \varphi\,,$$

taking derivatives of both sides we obtain

$$P(\partial)\, \varphi = P(\partial)\, E \star \triangle^N \varphi\,. \tag{13}$$

To compute $P(\partial)E$ we take the Fourier transform

$$\widehat{P(\partial)E}(\xi) = P(i\xi)\, \hat{E}(\xi) = \frac{P(\xi)}{|\xi|^d}\,.$$

On the other hand, as it is well known ([St, p. 73],

$$\widehat{P.V.\frac{P(x)}{|x|^{n+d}}}\,(\xi) = \gamma_d \frac{P(\xi)}{|\xi|^d}\,.$$

See (7) for the precise value of γ_d, which is not important now. We conclude that, for some constant c_d depending on d,

$$P(\partial)E = c_d\, P.V.\frac{P(x)}{|x|^{n+d}}\,.$$

Thus

$$P(\partial)\varphi = c_d\, P.V.\frac{P(x)}{|x|^{n+d}} \star \triangle^N \varphi = c_d\, T(b)\,.$$

The only thing left is the computation of $P(\partial)\, \varphi$. We have, by Corollary 2,

$$P(\partial)\, \varphi = c_d\, K(x)\, \chi_{\mathbb{R}^n \setminus \overline{B}} + P(\partial)(A_0 + A_1\,|x|^2 + \cdots + A_{d-1}\,|x|^{2d-2})(x)\, \chi_B(x)\,,$$

and so, to complete the proof of (10), we only have to show that

$$P(\partial)(|x|^{2j}) = 0, \quad 1 \le j \le d-1\,. \tag{14}$$

Notice that the degree of P may be much smaller than the degree of $|x|^{2j}$ and so the previous identity is not obvious. Taking the Fourier transform we obtain

$$\widehat{P(\partial)(|x|^{2j})} = c_j\, P(\xi)\, \triangle^j\, \delta\,,$$

where δ is the Dirac delta at the origin and c_j a constant depending on j. Let ψ be a test function. Then, since P is harmonic,

$$\begin{aligned}\langle P(\xi)\, \triangle^j\, \delta, \varphi\rangle &= \langle \triangle^j\, \delta,\, P(\xi)\, \varphi(\xi)\rangle \\ &= \langle \triangle^{j-1}\, \delta,\, 2\, \nabla P(\xi) \cdot \nabla\varphi(\xi) + P(\xi)\, \triangle\varphi(\xi)\rangle\,.\end{aligned}$$

Iterating the previous computation we obtain that

$$\langle P(\xi)\, \triangle^j\, \delta,\, \varphi\rangle = \langle \delta, D(\xi)\rangle = D(0)\,,$$

where D is a linear combination of products of the form $\partial^\alpha\, \varphi(\xi)\, \partial^\beta\, P(\xi)$, with multi-indeces β of length $|\beta| \le j \le d-1$. Therefore $\partial^\beta\, P(\xi)$ is a homogeneous polynomial of degree at least $d-j \ge 1$, and so $\partial^\beta\, P(0) = 0$. This yields $D(0) = 0$ and completes the proof of (14) and, thus, of (10).

In fact (14) follows immediately from an identity of Lyons and Zumbrun [LZ] which will be discussed in the next section, but we prefer to present here the above independent natural argument for the reader's convenience.

3 Proof of the sufficient condition: the polynomial case

In this section we assume that T is an even polynomial operator. This amounts to say that for some even integer $2N$, $N \ge 1$, the function $|x|^{2N}\, \Omega(x)$ is a homogeneous polynomial of degree $2N$. Such a polynomial may be written as [St, p. 69]

$$|x|^{2N}\, \Omega(x) = P_2(x)|x|^{2N-2} + \cdots + P_{2j}(x)|x|^{2N-2j} + \cdots + P_{2N}(x)\,,$$

where P_{2j} is a homogeneous harmonic polynomial of degree $2j$, $1 \le j \le N$. In other words, the expansion of $\Omega(x)$ in spherical harmonics is

$$\Omega(x) = P_2(x) + P_4(x) + \cdots + P_{2N}(x), \quad |x| = 1\,.$$

As in the previous section, we want to obtain an expression for the kernel $K(x)$ off the unit ball B. For this we need the differential operator $Q(\partial)$ defined by the polynomial

$$Q(x) = \gamma_2\, P_2(x)|x|^{2N-2} + \cdots + \gamma_{2j}\, P_{2j}(x)|x|^{2N-2j} + \cdots + \gamma_{2N}\, P_{2N}(x)\,.$$

If E is the standard fundamental solution of Δ^N, then

$$Q(\partial)E = P.V.\, K(x)\,,$$

which may be easily verified by taking the Fourier E transform of both sides.

Take now the function φ of the previous section. We have $\varphi = E \star \triangle^N \varphi$ and thus

$$Q(\partial)\varphi = Q(\partial)E \star \triangle^N \varphi = P.V.\, K(x) \star b = T(b)\,,$$

where b is defined as $\triangle^N \varphi$. On the other hand, by Corollary 2

$$Q(\partial)\,\varphi = K(x)\,\chi_{\mathbb{R}^n \setminus \overline{B}} + Q(\partial)(A_0 + A_1\,|x|^2 + \cdots + A_{2N-1}\,|x|^{4N-2})(x)\,\chi_B(x)\,. \quad (15)$$

Contrary to what happened in the previous section, the term

$$S(x) := -Q(\partial)(A_0 + A_1\,|x|^2 + \cdots + A_{2N-1}\,|x|^{4N-2})(x)$$

does not necessarily vanish, the reason being that now Q does not need to be harmonic.

Our goal is to find a function $\beta \in L^\infty(\mathbb{R}^n)$, satisfying the decay estimate

$$|\beta(x)| \le \frac{C}{|x|^{n+1}}, \quad |x| \ge 2\,, \quad (16)$$

and

$$S(x)\chi_B(x) = T(\beta)(x)\,. \quad (17)$$

Once this is achieved the proof of (i) is just a variation of the argument presented in section 2, which we now explain. By (15), the definition of $S(x)$ and (17), we get

$$K(x)\chi_{\mathbb{R}^n \setminus \overline{B}}(x) = T(b)(x) + T(\beta)(x)\,. \quad (18)$$

Set $\gamma = b + \beta$. We show (9) by arguing as follows. For f in any $L^p(\mathbb{R}^n)$, $1 \le p < \infty$, we have

$$\begin{aligned} T^1 f(0) &= \int \chi_{\mathbb{R}^n \setminus \overline{B}}(y)\, K(y)\, f(y)\, dy \\ &= \int T(\gamma)(y)\, f(y)\, dy \\ &= \int \gamma(y)\, T f(y)\, dy \\ &= \int_{2B} \gamma(y)\, T f(y)\, dy + \int_{\mathbb{R}^n \setminus 2B} \gamma(y)\, T f(y)\, dy \end{aligned}$$

and thus, by the decay inequality (16) with β replaced by γ,

$$\begin{aligned} |T^1 f(0)| &\le C \left(\|\gamma\|_\infty \frac{1}{|2B|} \int_{2B} |T f(y)|\, dy + \int_{\mathbb{R}^n \setminus 2B} \frac{|T f(y)|}{|y|^{n+1}}\, dy \right) \\ &\le C\, M(T f)(0)\,. \end{aligned}$$

To construct β satisfying (16) and (17) we resort to our hypothesis, condition (iii) in the Theorem, which says that $T = R \circ U$, where R is a higher order Riesz transform, U is an invertible operator in the algebra A and the polynomial P which determines R divides P_{2j}, $1 \leq j \leq N$, in the ring of polynomials in n variables with real coefficients. The construction of β is performed in two steps.

The first step consists in proving that there exists a function β_1 in $L^\infty(B)$, satisfying a Lipschitz condition of order 1 on B, $\int \beta_1(x)\,dx = 0$ and such that

$$S(x)\chi_B(x) = R(\beta_1)(x)\,. \tag{19}$$

It will become clear later on how the Lipschitz condition on β_1 is used. To prove (19) we need an explicit formula for $S(x)$ and for that we will make use of the following formula of Lyons and Zumbrun [LZ].

Lemma 3. *Let L be a homogeneous polynomial of degree l and let f be a smooth function of one variable. Then*

$$L(\partial)f(r) = \sum_{\nu\geq 0} \frac{1}{2^\nu\,\nu!}\,\Delta^\nu L(x)\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{l-\nu} f(r), \quad r = |x|\,.$$

An immediate consequence of Lemma 3 is

Lemma 4. *Let P_{2j} a homogeneous harmonic polynomial of degree $2j$ and let k be a non-negative integer. Then*

$$P_{2j}(\partial)(|x|^{2k}) = 2^{2j}\frac{k!}{(k-2j)!}\,P_{2j}(x)\,|x|^{2(k-2j)} \quad \text{if } 2j \leq k\,,$$

and

$$P_{2j}(\partial)(|x|^{2k}) = 0, \quad \text{if } 2j > k\,.$$

On the other hand, a routine computation gives

$$\triangle^j(|x|^{2k}) = 4^j\frac{j!\,k!}{(k-j)!}\binom{\frac{n}{2}+k-1}{j}|x|^{2(k-j)}, \quad k \geq j\,, \tag{20}$$

and

$$\triangle^j(|x|^{2k}) = 0, \quad k < j\,. \tag{21}$$

By Lemma 4, (20) and (21) we get that for some constants c_{jk} one has, in view of the definitions of $Q(x)$ and $S(x)$,

$$S(x) = \sum_{j=1}^{N-1}\sum_{k=j}^{N-1} c_{jk}\,P_{2j}(x)\,|x|^{2(k-j)}\,. \tag{22}$$

Therefore it suffices to prove (19) with $S(x)$ replaced by $P_{2j}(x)\,|x|^{2k}$, for $1 \le j \le N$ and each non-negative integer k. The idea is to look for an appropriate function ψ such that

$$P(\partial)\psi(x) = P_{2j}(x)\,|x|^{2k}\,\chi_B(x)\,. \tag{23}$$

Indeed, if (23) holds and $2d$ is the degree of P, then

$$\psi = E \star \Delta^d \psi\,,$$

provided E is the fundamental solution of $\triangle^d$ and ψ is good enough. Hence

$$P(\partial)\psi = P(\partial)E \star \Delta^d\psi = c\,P.V.\,\frac{P(x)}{|x|^{n+2d}} \star \Delta^d\psi = R(\beta_1)\,,$$

if $\beta_1 = c\,\Delta^d\psi$. The conclusion is that we have to solve (23) in such a way that $\Delta^d\psi$ is supported on B, is a Lipschitz function on B and has zero integral.

Taking Fourier transforms in (23) we get

$$(-1)^d P(\xi)\,\widehat{\psi}(\xi) = (-1)^{j+k}\,P_{2j}(\partial)\,\triangle^k\left(\widehat{\chi_B}(\xi)\right)\,. \tag{24}$$

Recall that for $m = n/2$ one has [Gr, A-10]

$$\widehat{\chi_B}(\xi) = \frac{J_m(\xi)}{|\xi|^m}, \quad \xi \in \mathbb{R}^n\,,$$

where J_m is the Bessel function of order m. Set

$$G_\lambda(\xi) = \frac{J_\lambda(\xi)}{|\xi|^\lambda}, \quad \xi \in \mathbb{R}^n, \quad \lambda > 0\,.$$

In computing the right hand side of (24) we apply Lemma 3 to $L(x) = P_{2j}(x)\,|x|^{2k}$ and $f(r) = G_m(r)$ and we get

$$P(\xi)\,\widehat{\psi}(\xi) = (-1)^{j+k+d}\sum_{\nu\ge 0}\frac{(-1)^\nu}{2^\nu\,\nu!}\,\triangle^\nu\left(P_{2j}(\xi)\,|\xi|^{2k}\right)\,G_{m+2j+2k-\nu}(\xi)\,,$$

owing to the well known formula, e.g. [Gr, A-6],

$$\frac{1}{r}\,\frac{d}{dr}\,G_\lambda(r) = -G_{\lambda+1}(r)\,, \quad r > 0, \quad \lambda > 0\,.$$

Since $P_{2j}(\xi)$ is homogeneous of degree $2j$, $\nabla P_{2j}(\xi)\cdot\xi = 2j\,P_{2j}(\xi)$, and hence one may readily show by an inductive argument that

$$\triangle^\nu\left(P_{2j}(\xi)\,|\xi|^{2k}\right) = a_{jk\nu}\,P_{2j}(\xi)\,|\xi|^{2(k-\nu)}\,,$$

for some constants $a_{jk\nu}$. Thus, for some other constants $a_{jk\nu}$, we get

$$P(\xi)\,\widehat{\psi}(\xi) = \sum_{\nu\ge 0} a_{jk\nu}\,P_{2j}(\xi)\,|\xi|^{2(k-\nu)}\,G_{m+2j+2k-\nu}(\xi)\,. \tag{25}$$

By hypothesis P divides P_{2j} in the ring of polynomials in n variables and so

$$P_{2j}(\xi) = P(\xi)\, Q_{2j-2d}(\xi)\,,$$

for some homogeneous polynomial Q_{2j-2d} of degree $2j - 2d$. Cancelling out the factor $P(\xi)$ in (25) we conclude that

$$\widehat{\psi}(\xi) = Q_{2j-2d}(\xi) \sum_{\nu=0}^{k} a_{jk\nu}\, |\xi|^{2(k-\nu)}\, G_{m+2j+2k-\nu}(\xi)\,.$$

Since [Gr, A-10]

$$((1 - \widehat{|x|^2)^\lambda}\, \chi_B(x))(\xi) = c_\lambda\, G_{m+\lambda}(\xi)\,,$$

we finally obtain

$$\psi(x) = Q_{2j-2d}(\partial) \sum_{\nu=0}^{k} a_{jk\nu}\, \triangle^{k-\nu} \left((1 - |x|^2)^{2j+2k-\nu}\, \chi_B(x)\right)\,.$$

Observe that ψ restricted to B is a polynomial which vanishes on ∂B up to order $2d$ and ψ is zero off B. Therefore $\triangle^d \psi$ is supported on B and its restriction to B is a polynomial with zero integral. This completes the first step of the construction of β.

The second step proceeds as follows. Since by hypothesis $T = R \circ U$, with U invertible in the algebra A, we have

$$R(\beta_1) = T(U^{-1}\beta_1)\,.$$

Setting

$$\beta = U^{-1}\beta_1\,, \tag{26}$$

we are only left with the task of showing that

$$\beta \in L^\infty(\mathbb{R}^n) \tag{27}$$

and that, for some positive constant C,

$$|\beta(x)| \le \frac{C}{|x|^{n+1}}, \quad |x| \ge 2\,. \tag{28}$$

Since $U^{-1} \in A$, for some real number λ and some smooth homogeneous Calderón-Zygmund operator V,

$$U^{-1} = \lambda\, I + V\,.$$

Thus

$$\beta = \lambda\, \beta_1 + V(\beta_1)\,.$$

Now β_1 is supported on B and has zero integral on B and this is enough to insure the decay estimate (28). Indeed, let $L(x)$ be the kernel of V and assume that $|x| \geq 2$. Then

$$\begin{aligned} V(\beta_1)(x) &= \int L(x-y)\,\beta_1(y)\,dy \\ &= \int \left(L(x-y) - L(x)\right)\,\beta_1(y)\,dy\,, \end{aligned} \tag{29}$$

and so

$$\begin{aligned} |V(\beta_1)(x)| &\leq \int |\left(L(x-y) - L(x)\right)|\,|\beta_1(y)|\,dy, \\ &\leq C \int \frac{|y|}{|x|^{n+1}}\,|\beta_1(y)|\,dy, \\ &= \frac{C}{|x|^{n+1}}\,. \end{aligned} \tag{30}$$

The boundedness of β is a more delicate issue. It follows immediately from the next lemma applied to the operator V and the function β_1. Is precisely here where we use the fact that β_1 satisfies a Lipschitz condition.

The constant of the kernel $K(x) = \Omega(x)/|x|^n$ of the smooth homogeneous Calderón-Zygmund operator T is

$$\|T\|_{CZ} \equiv \|K\|_{CZ} = \|\Omega\|_\infty + \||x|\,\nabla\Omega(x)\|_\infty\,. \tag{31}$$

We adopt the standard notation for the minimal Lipschitz constant of a Lipschitz function f on B, namely

$$\|f\|_{\mathrm{Lip}(1,B)} = \sup\left\{\frac{|f(x)-f(y)|}{|x-y|} : x, y \in B,\, x \neq y\right\} < \infty\,.$$

Lemma 5. *Let T be the homogeneous singular integral operator with kernel $K(x) = \frac{\Omega(x)}{|x|^n}$, where Ω is an even homogeneous function of degree 0, continuously differentiable and with zero integral on the unit sphere. Then*

$$\|T(f\,\chi_B)\|_{L^\infty(\mathbb{R}^n)} \leq C\,\|K\|_{CZ}\left(\|f\|_{L^\infty(B)} + \|f\|_{\mathrm{Lip}(1,B)}\right)\,,$$

where C is a positive constant which depends only on n.

Proof. We start by examining the behaviour of $T(f\,\chi_B)$ on the unit sphere. We claim that

$$|T_\epsilon(f\,\chi_B)(a)| \leq C\,\|K\|_{CZ}\left(\|f\|_{L^\infty(B)} + \|f\|_{\mathrm{Lip}(1,B)}\right)\,, \quad |a| = 1, \quad \epsilon > 0\,.$$

Indeed, if one follows in detail the proof of the claim, which we discuss below, one will realize that the principal value integral $T(f\,\chi_B)(a)$ exists for all a in the sphere and satisfies the desired estimate.

We have

$$
\begin{aligned}
T_\epsilon(f\,\chi_B)(a) &= \int_{\epsilon<|x-a|<1/2} \chi_B(x)\,f(x)\,K(a-x)\,dx + \int_{1/2<|x-a|} \cdots \\
&= I_\epsilon + II\,.
\end{aligned}
$$

Clearly,

$$
|II| \le \int_{1/2<|x-a|} \chi_B(x)\,|f(x)|\,\frac{|\Omega(x-a)|}{|x-a|^n}\,dx \le 2^n\,|B|\,\|\Omega\|_\infty\,\|f\|_{L^\infty(B)}\,.
$$

To deal with the term I_ϵ we write

$$
\begin{aligned}
I_\epsilon &= \int_{\epsilon<|x-a|<1/2} \chi_B(x)\,(f(x)-f(a))\,K(a-x)\,dx \\
&\quad + f(a)\int_{\epsilon<|x-a|<1/2} \chi_B(x)\,K(a-x)\,dx \\
&= III_\epsilon + f(a)\,IV_\epsilon\,,
\end{aligned}
$$

and we remark that III_ϵ can easily be estimated as follows

$$
|III_\epsilon| \le \|f\|_{\mathrm{Lip}(1,B)} \int_B |x-a||K(a-x)|\,dx \le C\,\|\Omega\|_\infty\,\|f\|_{\mathrm{Lip}(1,B)}\,.
$$

Taking care of IV_ϵ is not so easy. Take spherical coordinates centered at the point a, $x = a + r\,\omega$ with $0 \le r$ and $|\omega| = 1$. Then

$$
IV_\epsilon = \int_\epsilon^{1/2} \left(\int_{A(r)} \Omega(\omega)\,d\sigma(\omega) \right) \frac{dr}{r}\,, \tag{32}
$$

where

$$
A(r) = \{\omega : |\omega| = 1 \text{ and } |a + r\omega| < 1\}\,.
$$

Let H be the tangent hiperplane to $S = \{\omega : |\omega| = 1\}$ at the point a. Call V the half space with boundary H containing the origin. Clearly $A(r) \subset S \cap V$. Since Ω is even,

$$
0 = \int_S \Omega(\omega)\,d\sigma(\omega) = 2\int_{S\cap V} \Omega(\omega)\,d\sigma(\omega)\,.
$$

Thus

$$
\int_{A(r)} \Omega(\omega)\,d\sigma(\omega) = -\int_{(S\cap V)\setminus A(r)} \Omega(\omega)\,d\sigma(\omega)\,,
$$

and so

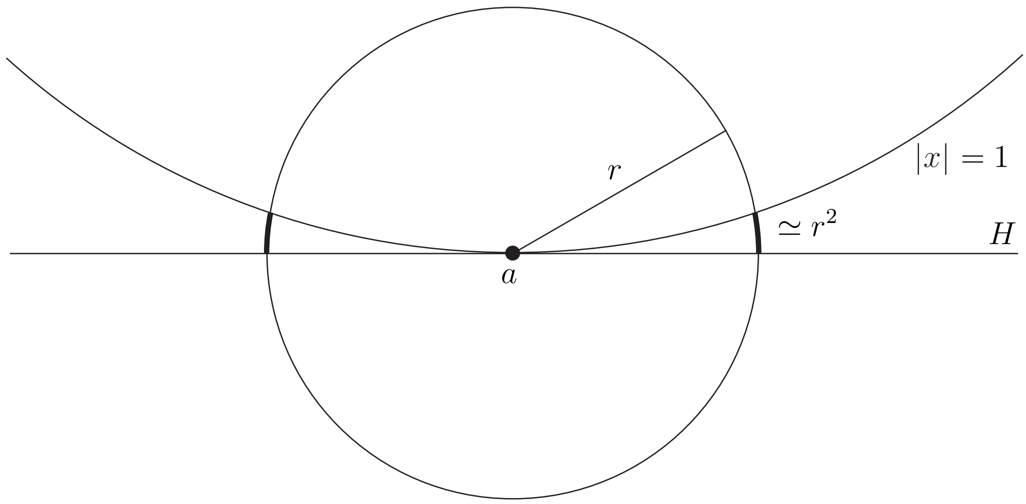
$$
\left| \int_{A(r)} \Omega(\omega)\,d\sigma(\omega) \right| \le \|\Omega\|_\infty\,\sigma((S\cap V)\setminus A(r))\,.
$$

Since H is tangent to S at the point a, we obtain

$$\sigma((S \cap V) \setminus A(r)) \leq C\, r\,,$$

which yields, by (32),

$$|IV_\epsilon| \leq C\, \|\Omega\|_\infty\,.$$



Assume now that $|a| < 1$. Proceeding as before we estimate in the same way the terms II and III_ϵ, so that we are again left with IV_ϵ. Let ϵ_0 stand for the distance from a to the boundary of B. In estimating IV_ϵ we can assume, without loss of generality, that $\epsilon_0 \leq 1/4$. .Set $a_0 = a/|a|$,

$$A = \{x \in B : \epsilon_0 < |x - a| < 1/2\}$$

and

$$A_0 = \{x \in B : \epsilon_0 < |x - a_0| < 1/2\}\,.$$

We compare IV_ϵ to the expression we get replacing a by a_0 and ϵ by ϵ_0 in the definition of IV_ϵ. For $\epsilon \leq \epsilon_0$ we have

$$\int_{\epsilon < |x-a| < 1/2} \chi_B(x)\, K(a-x)\, dx = \int_{\epsilon_0 < |x-a| < 1/2} \chi_B(x)\, K(a-x)\, dx$$

and then

$$\left|\int_{\epsilon<|x-a|<1/2} \chi_B(x)\, K(a-x)\, dx - \int_{\epsilon_0<|x-a_0|<1/2} \chi_B(x)\, K(a_0-x)\, dx\right|$$

$$\begin{aligned}
&= \left|\int_A K(a-x)\, dx - \int_{A_0} K(a_0-x)\, dx\right| \\
&\le \int_{A\cap A_0} |K(a-x) - K(a_0-x)|\, dx \\
&\quad + \left|\int_{A\setminus A_0} \chi_B(x)\, K(a-x)\, dx\right| + \left|\int_{A_0\setminus A} \chi_B(x)\, K(a_0-x)\, dx\right| \\
&= J_1 + J_2 + J_3\,.
\end{aligned}$$

If $x \in A \cap A_0$, then

$$|K(a-x) - K(a_0-x)| \le C\, \|K\|_{CZ}\, \frac{|a-a_0|}{|x-a|^{n+1}}\,.$$

Hence

$$J_1 \le C\, \|K\|_{CZ}\, |a-a_0| \int_{|x-a|>\epsilon_0} \frac{dx}{|x-a|^{n+1}} \le C\, \|K\|_{CZ}\,.$$

To estimate J_2 observe that

$$A \setminus A_0 = (A \cap B(a_0, \epsilon_0)) \cup (A \cap (\mathbb{R}^n \setminus B(a_0, 1/2)))\ .$$

Now, it is obvious that if $|x-a_0| \ge 1/2$, then $|x-a| \ge 1/4$, and so

$$J_2 \le \|\Omega\|_\infty \left(\int_{|x-a_0|<\epsilon_0} \frac{dx}{\epsilon_0^n} + \int_B 4^n\, dx\right) \le C\, \|\Omega\|_\infty\,.$$

A similar argument does the job for J_3.

The case $|a| > 1$ is treated in a completely analogous way. □

The construction of β is then completed and the Theorem is proved for polynomial operators.

We remark that a variant of Lemma 5 holds, with the same proof, replacing B by $\mathbb{R}^n \setminus \overline{B}$. To control the term II we have to assume, in addition to the hypothesis of Lemma 5, that f satisfies a decay inequality of the type

$$|f(x)| \le \frac{\|f\|_{L^\infty(\mathbb{R}^n\setminus\overline{B})}}{|x|^\eta}, \quad |x| \ge 1\,.$$

Then we conclude that

$$\|T(f\, \chi_{\mathbb{R}^n\setminus\overline{B}})\|_{L^\infty(\mathbb{R}^n)} \le C\, \|K\|_{CZ} \left(\|f\|_{L^\infty(\mathbb{R}^n\setminus\overline{B})} + \|f\|_{\mathrm{Lip}(1,\mathbb{R}^n\setminus\overline{B})}\right)\,,$$

where C depends on n and η. We will use later on this variant of Lemma 5 with $f(x) = K(x)$ on $\mathbb{R}^n \setminus \overline{B}$, so that $\eta = n$ and the constant C will depend only on n.

We mention another straightforward extension of Lemma 5 that will not be used in this paper. The function f may be assumed to be in $\mathrm{Lip}(\alpha, B)$ $0 < \alpha \leq 1$, and the unit ball may be replaced by a domain with boundary of class $C^{1+\epsilon}$.

After the paper was completed we learned from Stephen Semmes that Lemma 5 is known in dimension 2 [Ch, p.52] or [BM, p.348] and that was used to prove global regularity of vortex patches for incompressible perfect fluids.

4 Proof of the sufficient condition: the general case

We start this section by clarifying several facts about the convergence of the series (6). Let us then assume that Ω is a function in $C^\infty(S^{n-1})$ with zero integral. Then Ω has an expansion (6) in spherical harmonics. For each positive integer r, one has the identity [St, p. 70]

$$\sum_{j\geq 1} (j(j+n-2))^r \; \|P_j\|_2^2 = (-1)^r \int_{S^{n-1}} \triangle_S^r \Omega \, \Omega \, d\sigma \,, \tag{33}$$

where Δ_S stands for the spherical Laplacean. Then

$$\sum_{j\geq 1} (j(j+n-2))^r \; \|P_j\|_2^2 \leq \|\triangle_S^r \Omega\|_2 \, \|\Omega\|_2 \,,$$

where the L^2 norm is taken with respect to $d\sigma$. Thus, by Schwarz's inequality, for each positive integer M

$$\sum_{j\geq 1} j^M \, \|P_j\|_2 < \infty \,. \tag{34}$$

We want to see that we also have

$$\sum_{j\geq 1} j^M \, \|P_j\|_\infty < \infty \,, \tag{35}$$

where the supremum norm is taken on S^{n-1}. This follows immediately from the next lemma, whose proof was indicated to us by Fulvio Ricci.

Lemma 6. *For all homogeneous polynomials of degree q*

$$\|Q\|_\infty \leq C \, q^{\frac{n-1}{2}} \, \|Q\|_2 \,,$$

where C is a positive constant which depends only on n.

Proof. Take an orthonormal base $Q_1, \dots, Q_d$, $d = d_q$, of the subspace of $L^2(d\sigma)$ consisting of the restrictions to S^{n-1} of all homogeneous polynomials of degree q. Consider the function

$$S(x) = \sum_{j=1}^{d} Q_j(x)^2, \quad x \in S^{n-1}.$$

We claim that S is rotation invariant, and, hence, constant. Since $d\sigma$ is a probability measure this constant must be $\sum_{j=1}^{d} \int Q_j(x)^2 \, d\sigma(x) = d$. Now let Q be a homogeneous polynomial of degree q and set $Q = \sum_{j=1}^{d} \lambda_j Q_j$. Then

$$|Q(x)| \le \left(\sum_{j=1}^{d} \lambda_j^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{d} Q_j(x)^2 \right)^{\frac{1}{2}} = \|Q\|_2 \, d^{\frac{1}{2}}, \quad x \in S^{n-1},$$

which proves the lemma because $d = \binom{n+q-1}{q} \simeq q^{n-1}$ ([SW, p. 139]).

To show the claim take a rotation ρ. Then we have

$$Q_j(\varrho(x)) = \sum_{j=1}^{d} a_{jk} \, Q_k(x),$$

for some matrix (a_{jk}) which is orthogonal, because the polynomials $Q_j(\rho(x))$ form also an orthonormal basis due to the rotation invariance of σ. Hence

$$\sum_{j=1}^{d} Q_j(\varrho(x))^2 = \sum_{j=1}^{d} Q_j(x)^2, \quad x \in S^{n-1}.$$

□

Let us return now to the context of the Theorem. Thus T is an even smooth homogeneous Calderón-Zygmund operator with kernel $K(x) = \Omega(x)/|x|^n$, and the expansion of Ω in spherical harmonics is

$$\Omega(x) = \sum_{j \ge 1}^{\infty} P_{2j}(x). \tag{36}$$

By hypothesis there is a homogeneous harmonic polynomial P of degree $2d$ which divides each P_{2j}. In other words, $P_{2j} = P \, Q_{2j-2d}$, where Q_{2j-2d} is a homogeneous polynomial of degree $2j - 2d$. We want to show that the series $\sum_j Q_{2j-2d}(x)$ is convergent in $C^\infty(S^{n-1})$, that is, that for each positive integer M

$$\sum_{j \ge d} j^M \, \|Q_{2j-2d}\|_\infty < \infty. \tag{37}$$

The next lemma states that when one divides two homogeneous polynomials, then the supremum norm (on S^{n-1}) of the quotient is controlled by the supremum norm of the dividend.

Lemma 7. *Let P be a homogeneous polynomial non identically zero. Then there exists a positive ϵ and a positive constant $C = C(n, P)$ such that*

$$\|Q\|_\infty \leq C\, q^{2(n-1)/\epsilon}\, \|P\, Q\|_\infty\,,$$

for each homogeneous polynomial Q of degree q.

Proof. Assume that we can prove that for some positive ϵ

$$\int_{|x|=1} \frac{1}{|P(x)|^\epsilon}\, d\sigma(x) < \infty\,. \tag{38}$$

Then, by Lemma 6 and Schwarz's inequality,

$$\begin{aligned}
\|Q\|_\infty \leq\ & C\, q^{(n-1)/2}\, \|Q\|_2 \\
\leq\ & C\, q^{(n-1)/2} \left(\int_{|x|=1} \frac{1}{|P(x)|^\epsilon}\, d\sigma(x)\right)^{1/4} \left(\int_{|x|=1} |P(x)|^\epsilon\, |Q|^4\, d\sigma(x)\right)^{1/4} \\
\leq\ & C\, q^{(n-1)/2}\, \|P\, Q\|_\infty^{\epsilon/4} \left(\int_{|x|=1} |Q|^{4-\epsilon}\, d\sigma(x)\right)^{1/4} \\
\leq\ & C\, q^{(n-1)/2}\, \|P\, Q\|_\infty^{\epsilon/4}\, \|Q\|_\infty^{1-\epsilon/4}\,,
\end{aligned}$$

which completes the proof of the lemma.

Let us prove (38). Let d be the degree of P. By a well-known result of Ricci and Stein [RS], $|P(x)|$ is a weight in the class A^∞. Indeed, if $\epsilon\, d < 1$, then

$$\int_{|x|<1} \frac{1}{|P(x)|^\epsilon}\, dx \leq C(\epsilon, d) \left(\int_{|x|<1} |P(x)|\, dx\right)^{-\epsilon} < \infty\,.$$

Since P is an homogeneous polynomial, (38) follows by changing to spherical coordinates. □

Now (37) may be proved readily from Lemma 7 and (35). Indeed, setting $M_0 = 2(n-1)/\epsilon$, we have

$$\|Q_{2j-2d}\|_\infty \leq\ C(n, P)\, (2j)^{M_0}\, \|P_{2j}\|_\infty\,,$$

and

$$\sum_{j \geq d} j^M\, \|Q_{2j-2d}\|_\infty \leq\ C(n, P) \sum_{j \geq 1} (2j)^{M+M_0}\, \|P_{2j}\|_\infty < \infty\,.$$

The scheme for the proof of the sufficient condition in the general case is as follows. Taking a large partial sum of the series (36) we pass to a polynomial operator T_N (associated to a polynomial of degree $2N$), which still satisfies the hypothesis (*iii*) of the Theorem. Then we may apply the construction of section 3 to T_N and get functions b_N and β_N. Unfortunately what was done in section 3 does not give any uniform estimate in N, which is precisely what we need to try a compactness argument. The rest of the section is devoted to get the appropriate uniform estimates and to describe the final compactness argument.

By hypothesis, $T = R \circ U$, where R is the higher order Riesz transform associated to the harmonic polynomial P of degree $2d$ that divides all P_{2j}, and U is invertible in the algebra A. The Fourier multiplier of T is

$$\sum_{j=d}^{\infty} \gamma_{2j} \frac{P_{2j}(\xi)}{|\xi|^{2j}} = \gamma_{2d} \frac{P(\xi)}{|\xi|^{2d}} \sum_{j \geq d} \frac{\gamma_{2j}}{\gamma_{2d}} \frac{Q_{2j-2d}(\xi)}{|\xi|^{2j-2d}}, \quad \xi \in \mathbb{R}^n \setminus \{0\}.$$

Therefore the Fourier multiplier of U is

$$\mu(\xi) = \gamma_{2d}^{-1} \sum_{j \geq d} \gamma_{2j} \frac{Q_{2j-2d}(\xi)}{|\xi|^{2j-2d}}, \tag{39}$$

and the series is convergent in $C^\infty(S^{n-1})$ because $\gamma_{2j} \simeq (2j)^{-n/2}$ [SW, p. 226]. Set, for $N \geq d$,

$$\mu_N(\xi) = \gamma_{2d}^{-1} \sum_{j=d}^{N} \gamma_{2j} \frac{Q_{2j-2d}(\xi)}{|\xi|^{2j-2d}}, \quad \xi \in \mathbb{R}^n \setminus \{0\}. \tag{40}$$

If

$$K_N(x) = \sum_{j=d}^{N} \frac{P_{2j}(x)}{|x|^{2j+n}}, \quad x \in \mathbb{R}^n \setminus \{0\},$$

and T_N is the polynomial operator with kernel K_N, then $T_N = R \circ U_N$, where U_N is the operator in the algebra A with Fourier multiplier $\mu_N(\xi)$. From now on N is assumed to be big enough so that $\mu_N(\xi)$ does not vanish on S^{n-1}. In fact, we will need later on the inequality

$$|\partial^\alpha \mu_N^{-1}(\xi)| \leq C, \quad |\xi| = 1, \quad 0 \leq |\alpha| \leq 2(n+3), \tag{41}$$

which may be taken for granted owing to the convergence in $C^\infty(S^{n-1})$ of the series (39). In (41) C is a positive constant depending only on the dimension n and μ.

Notice that T_N satisfies condition (iii) in the Theorem (with T replaced by T_N), because $\mu_N(\xi) \neq 0$, $|\xi| = 1$, and so we can apply the results of section 3. In particular,

$$K_N(x)\chi_{\mathbb{R}^n \setminus \overline{B}}(x) = T_N(b_N)(x) + T_N(\beta_N)(x),$$

where b_N and β_N are respectively the functions b and β defined in (18). It is important to remark that b_N does not depend on T. As (12) shows, the function b_N depends on N only through the fundamental solution of the operator $\triangle^N$. The uniform estimate we need on b_N is given by part (i) of the next lemma. The polynomial estimates in N of (ii) and (iii) are also basic for the compactness argument we are looking for.

Lemma 8. *There exist a constant C depending only on n such that*

(i)

$$|\widehat{b_N}(\xi)| \le C, \quad \xi \in \mathbb{R}^n\,,$$

(ii)

$$\|b_N\|_{L^\infty(B)} \le C\,(2N)^{2n+2}\,,$$

and

(iii)

$$\|\nabla b_N\|_{L^\infty(B)} \le C\,(2N)^{2n+4}\,.$$

Proof. We first prove (i). Let $h_1, \dots, h_d$ be an orthonormal basis of the subspace of $L^2(d\sigma)$ consisting of all homogeneous harmonic polynomials of degree $2N$. As in the proof of Lemma 6 we have $h_1^2 + \cdots + h_d^2 = d$, on S^{n-1}. Set

$$H_j(x) = \frac{1}{\gamma_{2N}\sqrt{d}}\, h_j(x), \quad x \in \mathbb{R}^n\,,$$

and let S_j be the higher order Riesz transform with kernel $K_j(x) = H_j(x)/|x|^{2N+n}$. The Fourier multiplier of S_j^2 is

$$\frac{1}{d}\,\frac{h_j(\xi)^2}{|\xi|^{4N}}, \quad 0 \ne \xi \in \mathbb{R}^n\,,$$

and thus

$$\sum_{j=1}^{d} S_j^2 = I\,.$$

By (10), we get

$$K_j(x)\,\chi_{\mathbb{R}^n \setminus \overline{B}}(x) = S_j(b_N)(x), \quad x \in \mathbb{R}^n, \quad 1 \le j \le d\,,$$

and so

$$b_N = \sum_{j=1}^{d} S_j\left(K_j(x)\,\chi_{\mathbb{R}^n \setminus \overline{B}}(x)\right)\,. \tag{42}$$

We now appeal to a lemma of Calderón and Zygmund ([CZ]; see [LS] for a simpler proof), which can be stated as follows.

Lemma (Calderón and Zygmund)**.** *If K is the kernel of a higher order Riesz transform, then, for some constant C depending only on n,*

$$|(\widehat{K(x)\,\chi_{\mathbb{R}^n \setminus \overline{B}}}(x))(\xi)| \le C\,|(\widehat{P.V.\,K}(x))(\xi)|, \quad \xi \in \mathbb{R}^n \setminus \{0\}\,.$$

By (42) and the preceding lemma, we get

$$\begin{aligned}|\widehat{b_N}(\xi)| &\leq \sum_{j=1}^{d} |\widehat{P.V.\ K_j}(x)(\xi)|\,|(\widehat{K_j(x)\,\chi_{\mathbb{R}^n\setminus\overline{B}}}(x))(\xi)| \\ &\leq C \sum_{j=1}^{d} |\widehat{P.V.\ K_j}(x)(\xi)|^2 \\ &= C\,.\end{aligned}$$

We now turn to the proof of (ii) in Lemma 8. In view of the expression (42) for b_N, we apply Lemma 5 to the operators S_j and the functions $K_j(x)$, which satisfy a Lipschitz condition on $\mathbb{R}^n \setminus \overline{B}$. We obtain

$$\|b_N\|_\infty \leq C\,d \max_{1\leq j\leq d} \|K_j\|_{CZ}\,(\|K_j\|_{L^\infty(\mathbb{R}^n\setminus\overline{B})} + \|K_j\|_{\mathrm{Lip}(1,\mathbb{R}^n\setminus\overline{B})})\,. \tag{43}$$

As it is well known, $d \simeq (2N)^{n-2}$ [SW, p. 140]. On the other hand

$$\|K_j\|_{CZ} \leq \|H_j\|_\infty + \|\nabla H_j\|_\infty\,,$$

where the supremum norms are taken on S^{n-1}. Clearly

$$\|H_j\|_\infty = \frac{1}{\gamma_{2N}}\|\frac{h_j}{\sqrt{d}}\|_\infty \leq \frac{1}{\gamma_{2N}} \simeq (2N)^{n/2}\,.$$

For the estimate of the gradient of H_j we use the inequality [St, p. 276]

$$\|\nabla H_j\|_\infty \leq C\,(2N)^{n/2+1}\,\|H_j\|_2\,, \tag{44}$$

where the L^2 norm is taken with respect to $d\sigma$. Since the h_j are an orthonormal system,

$$\|H_j\|_2 = \frac{1}{\sqrt{d}\,\gamma_{2N}} \simeq \frac{(2N)^{n/2}}{(2N)^{(n-2)/2}} \simeq 2N\,.$$

Gathering the above inequalities we get

$$\|K_j\|_{CZ} \leq C\,(2N)^{n/2+2}\,.$$

On the other hand, a straightforward computation yields

$$\|K_j\|_{L^\infty(\mathbb{R}^n\setminus\overline{B})} + \|K_j\|_{\mathrm{Lip}(1,\mathbb{R}^n\setminus\overline{B})} \leq C\,N\|H_j\|_\infty + \|\nabla H_j\|_\infty \leq C\,(2N)^{n/2+2}\,,$$

and therefore

$$\|b_N\|_{L^\infty(B)} \leq C\,(2N)^{n-2}\,(2N)^{n/2+2}\,(2N)^{n/2+2} = C\,(2N)^{2n+2}\,.$$

We are only left with the proof of (iii) in Lemma 8. Recalling the definition of b in (12) we see that b_N has the form

$$b_N(x) = \alpha_0 + \alpha_1\,|x|^2 + \cdots + \alpha_{N-1}\,|x|^{2N-2}, \quad |x| < 1\,,$$

for some real coefficients α_j, $0 \le j \le N-1$. Define the polynomial $p(t)$ of the real variable t as

$$p(t) = \alpha_0 + \alpha_1 t^2 + \cdots + \alpha_{N-1} t^{2N-2},$$

so that $b_N(x) = p(|x|)$, $|x| < 1$. By part (ii) of the lemma

$$\sup_{0 \le t \le 1} |p(t)| \le C\,(2N)^{2n+2},$$

and thus, appealing to Markov's inequality [Lo, p. 40],

$$\sup_{0 \le t \le 1} |p'(t)| \le (2N-2)^2 \sup_{0 \le t \le 1} |p(t)| \le C\,(2N)^{2n+4}.$$

Now (iii) follows from the obvious identity $\frac{\partial b_N}{\partial x_j} = p'(|x|)\,\frac{\partial |x|}{\partial x_j}$, which gives $|\nabla b_N(x)| \le p'(|x|)$, $|x| < 1$. $\square$

Our goal is now to show that under condition *(iii)* of the Theorem we can find a function γ in $L^\infty(\mathbb{R}^n)$ such that

$$K(x)\chi_{\mathbb{R}^n \setminus \overline{B}}(x) = T(\gamma)(x), \quad x \in \mathbb{R}^n. \tag{45}$$

If T is a polynomial operator this was proven in the preceding section for a γ of the form $b + \beta$ (see (18)). The approach we take up now has the advantage that when applied to T_N gives a uniform bound on $\gamma_N = b_N + \beta_N$.

Since Ω has the expansion (36) in spherical harmonics, we have

$$\begin{aligned} K(x)\chi_{\mathbb{R}^n \setminus \overline{B}}(x) &= \sum_{j \ge 1} \frac{P_{2j}(x)}{|x|^{2j+n}}\,\chi_{\mathbb{R}^n \setminus \overline{B}}(x) \\ &= \sum_{j \ge 1} T_j(b_j)(x), \end{aligned}$$

where T_j is the higher order Riesz transform with kernel $P_{2j}(x)/|x|^{2j+n}$ and b_j is the function constructed in section 2 (see (10) and (12)). The Fourier multiplier of T_j is

$$\gamma_{2j}\,\frac{P_{2j}(\xi)}{|\xi|^{2j}} = \gamma_{2d}\,\frac{P(\xi)}{|\xi|^{2d}}\,\frac{\gamma_{2j}}{\gamma_{2d}}\,\frac{Q_{2j-2d}(\xi)}{|\xi|^{2j-2d}}, \quad \xi \in \mathbb{R}^n \setminus \{0\}.$$

Let S_j be the operator whose Fourier multiplier is

$$\frac{\gamma_{2j}}{\gamma_{2d}}\,\frac{Q_{2j-2d}(\xi)}{|\xi|^{2j-2d}}, \quad \xi \in \mathbb{R}^n \setminus \{0\}, \tag{46}$$

so that $T_j = R \circ S_j$. Then

$$\begin{aligned} K(x)\chi_{\mathbb{R}^n \setminus \overline{B}}(x) &= \sum_{j \ge d} (R \circ S_j)(b_j) \\ &= \sum_{j \ge d} T\left((U^{-1} \circ S_j)(b_j)\right) \\ &= T\left(\sum_{j \ge d} (U^{-1} \circ S_j)(b_j)\right). \end{aligned}$$

The latest identity is justified by the absolute convergence of the series $\sum_{j\geq d}(U^{-1}\circ S_j)(b_j)$ in $L^2(\mathbb{R}^n)$, which follows from the estimate

$$\begin{aligned}\sum_{j\geq d}\|(U^{-1}\circ S_j)(b_j)\|_2 &\leq C\sum_{j\geq d}\|Q_{2j-2d}\|_\infty\,\|b_j\|_{L^2(\mathbb{R}^n)}\\ &\leq C\sum_{j\geq d}\|Q_{2j-2d}\|_\infty\,\|b_j\|_{L^\infty(B)}\\ &\leq C\sum_{j\geq d}(2N)^{2n+2}\,\|Q_{2j-2d}\|_\infty\ <\infty\,.\end{aligned}$$

We claim now that the series $\sum_{j\geq d}(U^{-1}\circ S_j)(b_j)$ converges uniformly on $\mathbb{R}^n$ to a function γ, which will prove (45) . Observe that the operator $U^{-1}\circ S_j\in A$ is not necessarily a Calderón-Zygmund operator because the integral on the sphere of its multiplier does not need to vanish. However it can be written as $U^{-1}\circ S_j=c_jI+V_j$, where

$$c_j=\frac{\gamma_{2j}}{\gamma_{2d}}\int_{S^{n-1}}\mu(\xi)^{-1}\,Q_{2j-2d}(\xi)\,d\sigma(\xi)$$

and V_j is the Calderón-Zygmund operator with multiplier

$$\mu(\xi)^{-1}\frac{\gamma_{2j}}{\gamma_{2d}}\,\frac{Q_{2j-2d}(\xi)}{|\xi|^{2j-2d}}-c_j\,. \tag{47}$$

Now

$$\sum_{j\geq d}(U^{-1}\circ S_j)(b_j)=\sum_{j\geq d}c_j\,b_j+\sum_{j\geq d}V_j(b_j)$$

and the first series offers no difficulties because, by Lemma 8 (ii) and (37)

$$\sum_{j\geq d}|c_j|\,\|b_j\|_{L^\infty(B)}\leq C\sum_{j\geq d}(2j)^{-n/2}(2j)^{2n+2}\|Q_{2j-2d}\|_\infty<\infty\,.$$

The second series is more difficult to treat. By Lemma 5 and Lemma 8 (ii) and (iii),

$$\begin{aligned}\|V_j(b_j)\|_{L^\infty(\mathbb{R}^n)} &\leq C\,\|V_j\|_{CZ}\left(\|b_j\|_{L^\infty(B)}+\|\nabla b_j\|_{L^\infty(B)}\right)\\ &\leq C\,(2j)^{2n+4}\,\|V_j\|_{CZ}\,.\end{aligned}$$

Estimating the Calderón-Zygmund constant of the kernel of the operator V_j is not an easy task, because we do not have an explicit expression for the kernel. We do know, however, the multiplier (47) of V_j. We need a way of estimating the constant of the kernel in terms of the multiplier and this is what the next lemma supplies.

Lemma 9. *Let V be a smooth homogeneous Calderón-Zygmund operator with Fourier multiplier m. Then for some constant C depending only on n,*

$$\|V\|_{CZ}\leq C\,\|\triangle_S^{n+3}m\|_2^{1/2}\,\|m\|_2^{1/2}\,,$$

where $\triangle_S$ is the spherical Laplacean and the L^2 norm is taken with respect to $d\sigma$.

Proof. Let $\omega(x)/|x|^n$ be the kernel of V, so that ω is a homogeneous function of degree zero, of class $C^\infty(S^{n-1})$ and with zero integral on the sphere. Consider the expansion of ω in spherical harmonics $\omega(x) = \sum_{j\geq 1} p_j(x)$, $|x| = 1$, so that the kernel of V is $\sum_{j\geq 1} p_j(x)/|x|^{j+n}$, $x \in \mathbb{R}^n \setminus \{0\}$ and its Fourier multiplier is $m(\xi) = \sum_{j\geq 1} \gamma_j \, p_j(\xi)$, $|\xi| = 1$.. By the definition (31) of the constant of the kernel of a Calderón-Zygmund operator we have

$$\|V\|_{CZ} \leq C \sum_{j\geq 1} \left(j \, \|p_j\|_\infty + \|\nabla p_j\|_\infty\right) \,,$$

where the supremum is taken on S^{n-1}. By (44) with H_j replaced by p_j, and Lemma 6

$$\begin{aligned}\|V\|_{CZ} &\leq C \sum_{j\geq 1} \left(j^{1+(n-1)/2} \, \|p_j\|_2 + j^{n/2+1} \, \|p_j\|_2\right) \\ &\leq C \sum_{j\geq 1} \left(j^{n/2+2} \, \|p_j\|_2\right) \,.\end{aligned}$$

Since $\gamma_j \simeq j^{-n/2}$, the above sum can be estimated, using Schwarz's inequality and (33) with Ω replaced by m and P_j by $\gamma_j \, p_j$, by

$$\begin{aligned}\sum_{j\geq 1} j^{n+2} \, \|\gamma_j \, p_j\|_2 &\leq C \left(\sum_{j\geq 1} j^{2n+6} \, \|\gamma_j \, p_j\|_2^2\right)^{1/2} \\ &\leq C \left(\sum_{j\geq 1} (j(j+n-2))^{n+3} \, \|\gamma_j \, p_j\|_2^2\right)^{1/2} \\ &= C \left((-1)^n \int_{S^{n-1}} \triangle_S^{n+3} m \, m \, d\sigma\right)^{1/2} \\ &\leq C \, \|\triangle_S^{n+3} m\|_2^{1/2} \, \|m\|_2^{1/2} \,.\end{aligned}$$

□

Since the multiplier of V_j is given by (47) and μ^{-1} is in $C^\infty(S^{n-1})$, Lemma 9 reduces the estimate of $\|V_j\|_{CZ}$ to the estimate of the $L^2(d\sigma)$ norm of $\nabla^k Q_{2j-2d}$, for $0 \leq k \leq 2(n+3)$. Let us consider first the case $k = 1$.

Since $P_{2j} = P \; Q_{2j-2d}$, we have

$$\nabla P_{2j} = \nabla P \, Q_{2j-2d} + P \, \nabla Q_{2j-2d} \,,$$

and so, by Lemma 7 and (44) with H_j replaced by P_{2j}, there is a large positive integer $M = M(n, P)$ such that

$$\begin{aligned}
\|\nabla Q_{2j-2d}\|_\infty &\le C\, j^M\, \|P\,\nabla Q_{2j-2d}\|_\infty \\
&\le C\, j^M\, \left(\|\nabla P_{2j}\|_\infty + \|Q_{2j-2d}\|_\infty\right) \\
&\le C\, j^M\, \left(C\, j^{n/2+1}\, \|P_{2j}\|_2 + C\, j^M\, \|P_{2j}\|_\infty\right) \\
&\le C\, j^M\, \|P_{2j}\|_2\,,
\end{aligned}$$

where in the latest inequality M has been increased without changing the notation.

By induction we get, for some large integer $M = M(n, P)$,

$$\|\nabla^k Q_{2j-2d}\|_\infty \le C\, j^M\, \|P_{2j}\|_2, \quad 0 \le k \le 2(n+3)\,.$$

Therefore, the estimate we finally obtain for the constant of the kernel of V_j is

$$\|V_j\|_{CZ} \le C\, j^M\, \|P_{2j}\|_2\,,$$

and thus

$$\|V_j(b_j)\|_{L^\infty(\mathbb{R}^n)} \le C\, j^M\, \|P_{2j}\|_2\,,$$

where again $M = M(n, P)$ is a positive integer. Hence the series $\sum_{j\ge d}(U^{-1}\circ S_j)(b_j)$ converges uniformly on $\mathbb{R}^n$ and the proof of (45) is complete.

We are now ready for the discussion of the final compactness argument that will complete the proof of the sufficient condition. The reader is invited to review the definitions of the operators T_N (with kernel K_N) and U_N given in this section just before Lemma 8. We know from section 3 (see (18)) that

$$K_N(x)\chi_{\mathbb{R}^n\setminus\overline{B}}(x) = T_N(b_N)(x) + T_N(\beta_N)(x)\,. \tag{48}$$

On the other hand, by the construction of the function γ we have just described, we also have

$$K_N(x)\chi_{\mathbb{R}^n\setminus\overline{B}}(x) = T_N(\gamma_N)(x), \quad \gamma_N = \sum_{j\ge d}^{N}(U_N^{-1}\circ S_j)(b_j)\,. \tag{49}$$

Notice that (41) guaranties that the estimate of the supremum norm of γ on the whole of $\mathbb{R}^n$ is applicable to the operator T_N, and thus we get an estimate for $\|\gamma_N\|_{L^\infty(\mathbb{R}^n)}$ which is uniform in N. Since T_N is injective, (48) and (49) imply

$$b_N + \beta_N = \gamma_N \tag{50}$$

and, in particular, we conclude that the functions $b_N + \beta_N$ are uniformly bounded in $L^\infty(\mathbb{R}^n)$, a fact that cannot be derived from the work done in section 3. It is worth mentioning that numerical computations indicate that b_N, and thus β_N, are not uniformly bounded. On the other hand, section 3 tells us that γ_N satisfies the decay estimate (16) with β replaced by γ_N, which we cannot infer from the preceding construction of γ. The advantages of both approaches will be combined now to get both the boundedness and decay property for γ.

In view of (49) and the expressions of the multipliers of U_N and S_j (see (46)),

$$\widehat{\gamma_N}(\xi) = \sum_{j=d}^{N} \frac{1}{\mu_N(\xi)} \frac{\gamma_{2j}}{\gamma_{2d}} \frac{Q_{2j-2d}(\xi)}{|\xi|^{2j-2d}} \widehat{b_j}(\xi)\,,$$

which yields, by Lemma 8 and (37) for $M = 0$,

$$\begin{aligned}\|\widehat{\gamma_N}\|_{L^\infty(\mathbb{R}^n)} &\leq C \sum_{j=d}^{N} \|Q_{2j-2d}\|_\infty \\ &\leq C \sum_{j=d}^{\infty} \|Q_{2j-2d}\|_\infty \\ &\leq C\,,\end{aligned} \tag{51}$$

where C does not depend on N. Recall that, from (26) in section 3, we have

$$\beta_N = U_N^{-1}(\beta_{1,N})\,,$$

with $\beta_{1,N}$ a bounded function supported on B satisfying $\int \beta_{1,N}(x)\,dx = 0$. Since

$$\widehat{\beta_{1,N}} = \mu_N\, \widehat{\beta_N} = \mu_N\, (\widehat{\gamma_N} - \widehat{b_N})\,,$$

we have, again by Lemma 8,

$$\|\widehat{\beta_{1,N}}\|_{L^\infty(\mathbb{R}^n)} \leq C\,.$$

Therefore, passing to a subsequence, we may assume that, as N goes to ∞,

$$\widehat{b_N} \longrightarrow a_0 \qquad \text{and} \qquad \widehat{\beta_{1,N}} \longrightarrow a_1\,,$$

weak $\star$ in $L^\infty(\mathbb{R}^n)$. Hence

$$b_N \longrightarrow \Phi_0 = \mathcal{F}^{-1}a_0 \qquad \text{and} \qquad \beta_{1,N} \longrightarrow \Phi_1 = \mathcal{F}^{-1}a_1\,,$$

in the weak $\star$ topology of tempered distributions, $\mathcal{F}^{-1}$ being the inverse Fourier transform. In particular, Φ_0 and Φ_1 are distributions supported on $\overline{B}$ and

$$\langle \Phi_1, 1\rangle = \lim_{N\to\infty} \int \beta_{1,N}(x)\,dx = 0\,. \tag{52}$$

We would like now to understand the convergence properties of the sequence of the β_N's . Since

$$\widehat{\beta_N}(\xi) = \mu_N^{-1}(\xi)\, \widehat{\beta_{1,N}}(\xi)\,,$$

and we have pointwise bounded convergence of $\mu_N^{-1}(\xi)$ towards $\mu^{-1}(\xi)$ on $\mathbb{R}^n \setminus \{0\}$, we get that $\widehat{\beta_N} \to \mu^{-1}\, a_1$, in the weak $\star$ topology of $L^\infty(\mathbb{R}^n)$. Thus $\beta_N \to U^{-1}(\Phi_1)$ in the weak $\star$ topology of tempered distributions. Letting $N \to \infty$ in (50) we obtain

$$\Phi_0 + U^{-1}(\Phi_1) = \gamma\,.$$

We come now to the last key point of the proof, namely, that one has decay estimate

$$|\gamma(x)| \leq \frac{C}{|x|^{n+1}}, \quad |x| \geq 2\,. \tag{53}$$

Since Φ_0 and Φ_1 are supported on $\overline{B}$ and $U^{-1}(\Phi_1) = \lambda\, \Phi_1 + V(\Phi_1)$, where λ is a real number and V a smooth homogeneous Calderón-Zygmund operator, it is enough to show that $V(\Phi_1)$ has the appropriate behavior off the ball $B(0,2)$. Let L be the kernel of V. Regularizing Φ_1 one checks that, for a fixed x satisfying $|x| \geq 2$,

$$\begin{aligned} V(\Phi_1)(x) &= \langle \Phi_1, L(x-y) \rangle \\ &= \langle \Phi_1, L(x-y) - L(x) \rangle\,, \end{aligned} \tag{54}$$

where the latest identity follows from (52). Since Φ_1 is a distribution supported on $\overline{B}$ there exists a positive integer ν and a constant C such that

$$|\langle \Phi_1, \varphi \rangle| \leq C \sup_{|\alpha| \leq \nu} \sup_{|y| \leq 3/2} |\partial^\alpha \varphi(y)|\,, \tag{55}$$

for each infinitely differentiable function φ on $\mathbb{R}^n$. The kernel L satisfies

$$|\frac{\partial^\alpha}{\partial y^\alpha}\,(L(x-y) - L(x))\,| \leq \frac{C_\alpha}{|x|^{n+1+|\alpha|}}, \quad |y| \leq 3/2\,,$$

and hence by (54) and (55)

$$|V(\Phi_1)(x)| \leq \frac{C}{|x|^{n+1}}, \quad |x| \geq 2\,,$$

which proves (53) and then completes the proof of the sufficient condition in the general case.

5 Proof of the necessary condition: the polynomial case

We assume in this section that T is a polynomial operator with kernel

$$K(x) = \frac{\Omega(x)}{|x|^n} = \frac{P_2(x)}{|x|^{2+n}} + \frac{P_4(x)}{|x|^{4+n}} + \cdots + \frac{P_{2N}(x)}{|x|^{2N+n}}, \quad x \neq 0\,,$$

where P_{2j} is a homogeneous harmonic polynomials of degree $2j$. Let Q be the homogeneous polynomial of degree $2N$ defined by

$$Q(x) = \gamma_2 \, P_2(x)|x|^{2N-2} + \cdots + \gamma_{2j} \, P_{2j}(x)|x|^{2N-2j} + \cdots + \gamma_{2N} \, P_{2N}(x)\,.$$

Then

$$\widehat{P.V.K}(\xi) = \frac{Q(\xi)}{|\xi|^{2N}}, \quad \xi \neq 0\,.$$

Our assumption is now the L^2 estimate between $T^\star$ and T (see (ii) in the statement of the Theorem). Since the truncated operator T^1 at level 1 is obviously dominated by $T^\star$, we have

$$\int (T^1 f)^2(x)\,dx \leq \int (T^\star f)^2(x)\,dx \leq C \int (Tf)^2(x)\,dx\,.$$

The kernel of T^1 is (see (15))

$$K(x)\,\chi_{\mathbb{R}^n \setminus \overline{B}}(x) = T(b)(x) + S(x)\,\chi_B(x)\,, \tag{56}$$

where b is given in equation (12) and

$$-S(x) = Q(\partial)(A_0 + A_1\,|x|^2 + \cdots + A_{2N-1}\,|x|^{4N-2})(x), \quad x \in \mathbb{R}^n\,.$$

The reader may consult the beginning of section 3 to review the context of the definition of S. In view of (56) we have, for each $f \in L^2(\mathbb{R}^n)$,

$$\begin{aligned} \|S\,\chi_B \star f\|_2 &\leq C\,\|T^1 f\|_2 + \|b \star Tf\|_2 \\ &\leq C\,(\|Tf\|_2 + \|\widehat{b}\|_\infty \|Tf\|_2) \\ &= C\,\|Tf\|_2\,. \end{aligned}$$

By Plancherel, the above L^2 inequality translates into a pointwise inequality between the Fourier multipliers, namely,

$$|\widehat{S\,\chi_B}(\xi)| \leq C\,|\widehat{P.V.K}(\xi)| = C\,\frac{|Q(\xi)|}{|\xi|^{2N}}\,. \tag{57}$$

Our next goal is to show that (57) provides interesting relations between the zero sets of Q and the P_{2j}. For each function f on $\mathbb{R}^n$ set $Z(f) = \{x \in \mathbb{R}^n : f(x) = 0\}$.

Lemma (Zero Sets Lemma)**.**

$$Z(Q) \subset Z(P_{2j}), \quad 1 \leq j \leq N\,.$$

Proof. We know that S has an expression of the form (see (22))

$$S(x) = \sum_{l=N+1}^{2N-1} \sum_{j=1}^{l-N} c_{lj}\, P_{2j}(x)\, |x|^{2(l-N-j)}\,. \tag{58}$$

Since $\widehat{\chi_B} = G_m$, $m = n/2$, Lemma 3 yields

$$\begin{aligned} \widehat{S\,\chi_B}(\xi) &= S(\imath\,\partial)\,\widehat{\chi_B}(\xi) \\ &= \sum_{l=N+1}^{2N-1} \sum_{j=1}^{l-N} c_{lj}\,(-1)^{l-N}\; P_{2j}(\partial)\;\triangle^{l-N-j}\,G_m(\xi) \\ &= \sum_{l=N+1}^{2N-1} \sum_{j=1}^{l-N} \sum_{k=0}^{l-N-j} c_{l,\,j,\,k}\, P_{2j}(\xi)\,|\xi|^{2(l-N-j-k)} G_{m+2l-2N-k}(\xi)\,. \end{aligned} \tag{59}$$

The function $G_p(\xi)$ is, for each $p \geq 0$, a radial function which is the restriction to the real positive axis of an entire function [Gr, A-8]. Set $\xi = r\,\xi_0$, $|\xi_0| = 1$, $r \geq 0$. Then

$$\widehat{S\chi_B}(r\xi_0) = \sum_{p=1}^{\infty} a_{2p}(\xi_0)\, r^{2p}\,, \tag{60}$$

and the power series has infinite radius of convergence for each ξ_0. Assume now that $Q(\xi_0) = 0$. Then, by (57), $\widehat{S\chi_B}(r\xi_0) = 0$ for each $r \geq 0$, and hence $a_{2p}(\xi_0) = 0$, for each $p \geq 1$. For $p = 1$ one has $a_2(\xi_0) = P_2(\xi_0)\, C_2$, where

$$C_2 = \sum_{l=N+1}^{2N-1} c_{l,\,1,\,l-N-1}\, G_{m+l-N+1}(0)\,.$$

It will be shown later that $C_2 \neq 0$, and then we get $P_2(\xi_0) = 0$. Let us make the inductive hypothesis that $P_2(\xi_0) = \cdots = P_{2(j-1)}(\xi_0) = 0$. Then we obtain, if $j \leq N-1$, $a_{2j}(\xi_0) = P_{2j}(\xi_0)\, C_{2j}$, where

$$C_{2j} = \sum_{l=N+j}^{2N-1} c_{l,\,j,\,l-N-j}\, G_{m+l-N+j}(0)\,. \tag{61}$$

Since we will show that $C_{2j} \neq 0$, $P_{2j}(\xi_0) = 0$, $1 \leq j \leq N-1$. We have

$$0 = Q(\xi_0) = \sum_{j=1}^{N} \gamma_{2j}\, P_{2j}(\xi_0)\,,$$

and so we also get $P_{2N}(\xi_0) = 0$. Therefore the zero sets Lemma is completely proved provided we have at our disposition the following.

Lemma 10.

$$C_{2j} = \frac{-\pi^{\frac{n}{2}}}{V_n\, 2^{\frac{n}{2}}\, \Gamma(\frac{n}{2}+1)}\, \frac{(-1)^j}{j\, 4^j\, \Gamma(2j+\frac{n}{2})}, \quad 1 \le j \le N-1\,.$$

The proof of Lemma 10 is lengthy and rather complicated from the computational point of view, and so we postpone it to section 7. $\square$

Notice that, although the constants C_{2j} are non-zero, they become rapidly small as the index j increases.

The reason why Lemma 10 is involved is that one has to trace back the exact values of the constants C_{2j} from the very beginning of our proof of (56). This forces us to take into account the exact values of various other constants. For instance, those which appear in the expression of the fundamental solution of $\triangle^N$ and the constants $A_0, A_1, \dots, A_{2N-1}$ in formula (11). Finally, we need to prove some new identities involving a triple sum of combinatorial numbers, in the spirit of those that can be found in the book of R. Graham D. Knuth and O. Patashnik [GKP].

The following is elementary folklore, but is proved here for the reader's sake.

Lemma (Dimension Lemma)**.** *If f is a real valued continuous function on $\mathbb{R}^n$ which changes sign, then $H^{n-1}(Z(f)) > 0$, H^{n-1} being Hausdorff measure in dimension $n-1$. In particular, the Hausdorff dimension of $Z(f)$ is at least $n-1$.*

Proof. Assume, without loss of generality, that $f(0) > 0$ and $f(p) < 0$, where $p = (0, \dots, 0, 1)$. For $\epsilon > 0$ small enough we have $f(x) > 0$ if $|x| < \epsilon$ and $f(x) < 0$ if $|x-p| < \epsilon$. Set $B = \{x \in \mathbb{R}^n : x_n = 0 \quad \text{and} \quad |x| < \epsilon\}$. Bolzano's theorem tells us that, for each $x \in B$, f vanishes at some point of the segment $(x_1, \dots, x_{n-1}, t)$, $0 \le t \le 1$. Hence the orthogonal projection of the set $Z(f)$ onto the hyperplane $\{x : x_n = 0\}$ contains B and so $H^{n-1}(Z(f)) \ge H^{n-1}(B) > 0$. $\square$

We turn now our attention to an algebraic lemma which plays a key role in obtaining the necessary condition we are looking for.

Lemma (Division Lemma)**.** *Let F and G be polynomials in $\mathbb{R}[x_1, \dots, x_n]$. Assume that G is irreducible and that $H^{n-1}(Z(F) \cap Z(G)) > 0$. Then there exists a polynomial H in $\mathbb{R}[x_1, \dots, x_n]$ such that $F = G\,H$.*

Proof. Denote by $V(P)$ the complex hyper-surface $\{z \in \mathbb{C}^n : P(z) = 0\}$ of a polynomial P. By hypothesis $V(F) \cap V(G)$ is not empty. If $V(G)$ is not contained in $V(F)$ then the complex dimension of $V(G) \cap V(F)$ is not greater than $n-2$ [K, 3.2 p. 131]. Since the real dimension of a variety is less than or equal to the complex dimension, we conclude that $Z(G) \cap Z(F)$ has real dimension not greater than $n-2$, which contradicts the fact that it has positive $n-1$-dimensional Hausdorff measure. Thus $V(G) \subset V(F)$, and therefore $F = G\,H$ for some polynomial H in $\mathbb{C}[x_1, \dots, x_n]$. Since F and G have real coefficients, the same happens to H. $\square$

We proceed to the proof of the necessary condition.

Let j_0 be the first positive index such that P_{2j_0} does not vanish identically. We want to show that P_{2j_0} divides P_{2j} for $j_0 \leq j \leq N$.

Since $\mathbb{R}[x_1, \ldots, x_n]$ is a unique factorization domain we can express P_{2j_0} as a product of irreducible factors, say R_k, $1 \leq k \leq M$, which are also homogeneous. Clearly $Z(P_{2j_0}) = \cup_k Z(R_k)$ and so

$$Z(Q) = \bigcup_k (Z(Q) \cap Z(R_k)) \,.$$

Since the integral of Q on the sphere is 0, Q changes sign and thus by the Dimension Lemma there is at least a k such that $H^{n-1}(Z(Q) \cap Z(R_k)) > 0$. Change notation if necessary so that $k = 1$. Then R_1 divides Q, by the Division Lemma. We may also apply the Division Lemma to R_1 and P_{2j} for each $j_0 \leq j \leq N$, because $Z(Q) \cap Z(R_1) \subset Z(P_{2j}) \cap Z(R_1)$ by the Zero Sets Lemma. Hence R_1 also divides P_{2j}, for $j_0 \leq j \leq N$. Set

$$Q = R_1 \, Q_1 \qquad \text{and} \qquad P_{2j} = R_1 \, P_{2j,1}, \quad j_0 \leq j \leq N \,,$$

for certain homogeneous polynomials Q_1 and $P_{2j,1}$.

If $M = 1$ we are done. Otherwise our intention is to repeat as many times as we can the above division process. With this in mind we use (60) to rewrite inequality (57) in the form

$$\left| \sum_{p=1}^{\infty} a_{2p}(\xi_0) \, r^{2p} \right| \leq C \, |Q(\xi_0)|, \quad 0 < r \,. \tag{62}$$

The definition of the coefficients a_{2p} and (59) show that there exist real numbers $\mu_j(p)$ such that

$$a_{2p}(\xi_0) = \sum_{j=j_0}^{N-1} \mu_j(p) \, P_{2j}(\xi_0)$$

and so

$$\begin{aligned} a_{2p}(\xi_0) &= R_1(\xi_0) \sum_{j=j_0}^{N-1} \mu_j(p) \, P_{2j,1}(\xi_0) \\ &= R_1(\xi_0) \, a_{2p,1}(\xi_0) \,, \end{aligned} \tag{63}$$

where the last identity provides the definition of the numbers $a_{2p,1}(\xi_0)$. We can simplify the common factor $R_1(\xi_0)$ in (62) and get

$$\left| \sum_{p=1}^{\infty} a_{2p,1}(\xi_0) \, r^{2p} \right| \leq C \, |Q_1(\xi_0)|, \quad 0 < r \,. \tag{64}$$

Equipped with (64), we are ready to begin the second step in the division process. If $Q_1(\xi_0) = 0$, then $a_{2p,1}(\xi_0) = 0$, for each $p \geq 1$. Hence, as in the proof of the Zero Sets Lemma,

$$Z(Q_1) \subset Z(P_{2j,1}), \quad j_0 \leq j \leq N\,.$$

To apply the Division Lemma we need to ascertain that the zero set of Q_1 is big enough and for that it suffices to show, by the Dimension Lemma, that Q_1 changes sign. As we are assuming that M is greater than 1, the degree of R_1 is less than $2j_0$. Considering the expansions of R_1 and Q in spherical harmonics, we see that they are orthogonal in $L^2(d\sigma)$ [St, p. 69]. Hence

$$0 = \int R_1(\xi)\, Q(\xi)\, d\sigma(\xi) = \int R_1^2(\xi)\, Q_1(\xi)\, d\sigma(\xi)\,,$$

which tells us that Q_1 changes sign.

Since $P_{2j_0,1} = \prod_{k=2}^{M} R_k$, we conclude that one of the R_k, say R_2, divides the $P_{2j,1}$, $j_0 \leq j \leq N$. An inductive argument gives that P_{2j_0} divides the P_{2j}, $j_0 \leq j \leq N$. At the k-th step one should observe that $Q = \prod_{l=1}^{k} R_l\, Q_k$ and

$$0 = \int \prod_{l=1}^{k} R_l(\xi)\, Q(\xi)\, d\sigma(\xi) = \int \prod_{l=1}^{k} R_l^2(\xi)\, Q_k(\xi)\, d\sigma(\xi)\,,$$

so that Q_k changes sign. It is also important to remark that we have

$$a_{2p,k}(\xi_0) = \sum_{j=j_0}^{N-1} \mu_j(p)\, P_{2j,k}(\xi_0), \quad 1 \leq k \leq M\,.$$

Thus at the M-th step we get for $p = j_0$

$$a_{2j_0,M}(\xi_0) = C_{2j_0} \neq 0\,. \tag{65}$$

We have $P_{2j} = P_{2j_0}\, Q_{2j-2j_0}$ for some homogeneous polynomials Q_{2j-2j_0} of degree $2j - 2j_0$ and so

$$\begin{aligned} Q(\xi) &= \sum_{j=j_0}^{N} \gamma_{2j}\, P_{2j}(\xi) |\xi|^{2N-2j} \\ &= P_{2j_0}(\xi) \sum_{j=j_0}^{N} \gamma_{2j}\, Q_{2j-2j_0}(\xi) |\xi|^{2N-2j}\,. \end{aligned}$$

By (62) and the definition of the coefficients $a_{2p,M}(\xi_0)$, for $|\xi_0| = 1$ and $0 < r$, we get

$$\left| \sum_{p=j_0}^{\infty} a_{2p,M}(\xi_0)\, r^{2p} \right| \leq C \left| \sum_{j=j_0}^{N} \gamma_{2j}\, Q_{2j-2j_0}(\xi_0) \right|, \quad 0 < r\,.$$

Taking into account (65) we conclude that

$$\sum_{j=j_0}^{N} \gamma_{2j}\, Q_{2j-2j_0}(\xi_0) \neq 0, \quad |\xi_0| = 1\,,$$

which completes the proof of the necessary condition in the polynomial case.

6 Proof of the necessary condition: the general case

In this section the kernel of our operator has the general form $K(x) = \Omega(x)/|x|^n$ with Ω a homogeneous function of degree 0, with vanishing integral on the sphere and of class $C^\infty(S^{n-1})$. Then $\Omega(x) = \sum_{j\geq 1}^{\infty} P_{2j}(x)/|x|^{2j}$ with P_{2j} a homogeneous harmonic polynomial of degree $2j$. The strategy consists in passing to the polynomial case by looking at a partial sum of the series above. Set, for each $N \geq 1$, $K_N(x) = \Omega_N(x)/|x|^n$, where $\Omega_N(x) = \sum_{j=1}^{N} P_{2j}(x)/|x|^{2j}$, and let T_N be the operator with kernel K_N. The difficulty is now that there is no obvious way of obtaining the inequality

$$\|T_N^\star f\|_2 \leq C\|T_N f\|_2, \quad f \in L^2(\mathbb{R}^n)\,, \tag{66}$$

from our hypothesis, namely,

$$\|T^\star f\|_2 \leq C\|Tf\|_2, \quad f \in L^2(\mathbb{R}^n)\,.$$

Instead we try to get (66) with $\|T_N f\|_2$ replaced by $\|Tf\|_2$ in the right hand side plus an additional term which becomes small as N tends to ∞. We start as follows.

$$\begin{aligned}\|T_N^1 f\|_2 &\leq \|T^1 f\|_2 + \|T^1 f - T_N^1 f\|_2 \\ &\leq C\,\|Tf\|_2 + \|\sum_{j>N} \frac{P_{2j}(x)}{|x|^{2j+n}}\, \chi_{\mathbb{R}^n\setminus\overline{B}} * f\|_2\,.\end{aligned}$$

By (10) there exists a bounded function b_j supported on B such that

$$\frac{P_{2j}(x)}{|x|^{2j+n}}\, \chi_{\mathbb{R}^n\setminus\overline{B}}(x) = P.V.\frac{P_{2j}(x)}{|x|^{2j+n}} * b_j\,.$$

By Lemma 8 (i) $\|\widehat{b_j}\|_{L^\infty(\mathbb{R}^n)}$ is bounded uniformly in j and then an application of Plancherel yields

$$\begin{aligned}\left\|\sum_{j>N} \frac{P_{2j}(x)}{|x|^{2j+n}}\, \chi_{\mathbb{R}^n\setminus\overline{B}} * f\right\|_2 &= \left\|\sum_{j>N} P.V.\frac{P_{2j}(x)}{|x|^{2j+n}} * b_j * f\right\|_2 \\ &\leq C\left(\sum_{j>N} \|P_{2j}\|_\infty\right)\|f\|_2\,,\end{aligned}$$

where the supremum norm is taken on the sphere. By (56) applied to T_N

$$T_N^1 f = K_N\, \chi_{\mathbb{R}^n \setminus \overline{B}} * f = K_N * b_N * f + S_N\, \chi_B * f\,.$$

Hence, for each f in $L^2(\mathbb{R}^n)$,

$$\begin{aligned}
\|S_N \chi_B * f\|_2 &\le \|T_N^1 f\|_2 + \|K_N * f * b_N\|_2 \\
&\le C\|Tf\|_2 + C\left(\sum_{j>N} \|P_{2j}\|_\infty\right) \|f\|_2 + \|K_N * f * b_N\|_2 \\
&\le C\|Tf\|_2 + \|Tf * b_N\|_2 + C\left(\sum_{j>N} \|P_{2j}\|_\infty\right) \|f\|_2 \\
&\le C\|Tf\|_2 + C\left(\sum_{j>N} \|P_{2j}\|_\infty\right) \|f\|_2\,,
\end{aligned}$$

where in the latest inequality Lemma 8 (i) was used. The above L^2 inequality translates, via Plancherel, into the pointwise estimate

$$|\widehat{S_N \chi_B}(\xi)| \le C|\widehat{P.V.K}(\xi)| + C\left(\sum_{j>N} \|P_{2j}\|_\infty\right), \quad \xi \neq 0\,. \tag{67}$$

The idea is now to take limits, as N goes to ∞, in the preceding inequality. The remainder of the convergent series will disappear and we will get a useful analog of (57). The first task is to clarify how the left hand side converges.

Set $\xi = r\,\xi_0$, with $|\xi_0| = 1$ and $r > 0$. Rewrite (60) with S replaced by S_N and a_{2p} by a_{2p}^N:

$$\widehat{S_N \chi_B}(r\xi_0) = \sum_{p=1}^{\infty} a_{2p}^N(\xi_0)\, r^{2p}\,.$$

It is a remarkable key fact that for a fixed p the sequence of the a_{2p}^N stabilizes for N large. This fact depends on a laborious computation of various constants and will be proved in section 7 in the following form.

Lemma 11. *If $p+1 \le N$, then $a_{2p}^N = a_{2p}^{p+1}$.*

If $p \ge 1$ and $p+1 \le N$ we set $a_{2p} = a_{2p}^N$. We need an estimate for the a_{2p}^N, which will be proved as well in section 7.

Lemma 12. *We have, for a constant C depending only on n,*

$$|a_{2p}| \le \frac{C}{(p-1)!\, 4^p} \sum_{j=1}^{p} \|P_{2j}\|_\infty, \quad 1 \le p \le N-1\,, \tag{68}$$

and

$$|a_{2p}^N| \le \frac{C}{4^p} \binom{\frac{n}{2}+N-1}{N-1} \sum_{j=1}^{N-1} \|P_{2j}\|_\infty, \quad 1 < N \le p\,. \tag{69}$$

Let us prove that for each ξ_0 in the sphere the sequence $S_N\,\chi_B(r\,\xi_0)$ converges uniformly on $0 \le r \le 1$. For $1 \le N \le M$

$$\begin{aligned}\left|\widehat{S_N\,\chi_B}(r\xi_0) - \widehat{S_M\,\chi_B}(r\xi_0)\right| &\le \sum_{p\ge N} |a_{2p}^N| r^{2p} + \sum_{p=N}^{M-1} |a_{2p}| r^{2p} + \sum_{p\ge M} |a_{2p}^M| r^{2p} \\ &\le C\left(\frac{1}{4^N}\binom{\frac{n}{2}+N-1}{N-1} + \sum_{p\ge N}\frac{1}{(p-1)!\,4^p}\right)\sum_{j=1}^{\infty}\|P_{2j}\|_\infty\,,\end{aligned}$$

which clearly tends to 0 as N goes to ∞. Letting N go to ∞ in (67) we get

$$\left|\sum_{p=1}^{\infty} a_{2p}(\xi_0)\,r^{2p}\right| \le C\,|\widehat{P.V.K}(\xi_0)|, \quad 0 \le r \le 1, \quad |\xi_0| = 1\,. \tag{70}$$

At this point we may repeat almost verbatim the proof we presented in the previous section, because the coefficients a_{2p}^N stabilize. This allows us to argue as in the polynomial case. The only difference lays in the fact that now we are dealing with infinite sums. However, no convergence problems will really arise because of (35).

7 Proof of the combinatorial Lemmata

This section will be devoted to prove Lemmas 10, 11 and 12 stated and used in the preceding sections.

For the proof of Lemma 10 (see section 5) we need to carefully trace back the path that led us to the constants C_{2j}. To begin with we need a formula for the coefficients A_l in (11) and for that it is essential to have the expression for the fundamental solution $E_N = E_N^n$ of $\triangle^N$ in $\mathbb{R}^n$. One has [ACL]

$$E_N(x) = \frac{1}{|x|^{n-2N}}(\alpha(n,N) + \beta(n,N)\log|x|^2)\,,$$

where α and β are constants that depend on n and N. To write in close form α and β we consider different cases. Let ω_n be the surface measure of S^{n-1}.

Case 1: n is odd. Then

$$\alpha(n,N) = \frac{\Gamma(2-\frac{n}{2})}{4^{N-1}\,(N-1)!\,\Gamma(N+1-\frac{n}{2})\,(2-n)\,\omega_n}$$

and

$$\beta(n,N) = 0\,.$$

Case 2: n is even, $n \neq 2$ and $N \leq \frac{n}{2} - 1$. Then

$$\alpha(n,N) = \frac{(-1)^{N-1}\,(\frac{n}{2} - N - 1)!}{4^{N-1}(N-1)!\,(\frac{n}{2}-2)!\,(2-n)\,\omega_n}$$

and

$$\beta(n,N) = 0\,.$$

Case 3: n is even, $n \neq 2$ and $N \geq \frac{n}{2}$. Then

$$\beta(n,N) = \frac{1}{(-1)^{\frac{n}{2}+1}(N-\frac{n}{2})!\,4^{N-1}\,(N-1)!\,(\frac{n}{2}-2)!\,(2-n)\,\omega_n}$$

and

$$\alpha(n,N) = 2\,\beta(n,N)\,S_{N-\frac{n}{2}}\,,$$

where $S_0 = 0$ and

$$S_L = \sum_{k=1}^{L} \frac{1}{2k} + \sum_{k=\frac{n}{2}}^{L+\frac{n}{2}-1} \frac{1}{2k}, \quad 1 \leq L\,.$$

Case 4: $n = 2$.

$$\beta(2,N) = \frac{1}{2}\,\frac{1}{4^{N-1}(N-1)!^2\,\omega_2}$$

and

$$\alpha(2,N) = 2\,\beta(2,N)\,S_{N-1}\,.$$

Recall that the constants $A_0, A_1, \ldots, A_{2N-1}$ are chosen so that the function (see (11))

$$\varphi(x) = E(x)\,\chi_{\mathbb{R}^n\setminus\overline{B}}(x) + (A_0 + A_1\,|x|^2 + \cdots + A_{2N-1}\,|x|^{4N-2})\,\chi_B(x)\,, \tag{71}$$

and all its partial derivatives of order not greater than $2N-1$ extend continuously up to ∂B.

Lemma 13. *For $L = N+1, \ldots, 2N-1$ we have*

$$A_L = \frac{(-1)^{N+L}\binom{N+\frac{n}{2}-1}{N-1}}{V_n\,4^N\,(L+\frac{n}{2}-N)\,(2N-L-1)!\,L!}\,,$$

where V_n is the volume of the unit ball.

Proof. Set $t = |x|^2$, so that

$$E_N^n(x) \equiv E(t) = t^{N-\frac{n}{2}}(\alpha + \beta \log(t)) \,. \tag{72}$$

Let $P(t)$ be the polynomial $\sum_{L=0}^{2N-1} A_L t^L$. By Corollary 2 in section 2 we need that

$$P^{k)}(1) = E^{k)}(1), \quad 0 \le k \le 2N-1 \,.$$

By Taylor's expansion we have that $P(t) = \sum_{i=0}^{2N-1} \frac{E^{i)}(1)}{i!}(t-1)^i$, and hence, by the binomial formula applied to $(t-1)^i$,

$$A_L = \sum_{i=L}^{2N-1} \frac{E^{i)}(1)}{i!}(-1)^{i-L}\binom{i}{L}, \quad 0 \le L \le 2N-1 \,.$$

Now we want to compute $E^{i)}(1)$. Clearly

$$\left(\frac{d}{dt}\right)^i (t^{N-\frac{n}{2}}) = \left(N - \frac{n}{2}\right)\cdots\left(N - \frac{n}{2} - i + 1\right) t^{N-\frac{n}{2}-i}.$$

Notice that the logarithmic term in (72) only appears when the dimension n is even. In this case, for each $i \ge N+1$

$$\left(\frac{d}{dt}\right)^i (t^{N-\frac{n}{2}}\log t) = \left(N - \frac{n}{2}\right)!(-1)^{i-N+\frac{n}{2}-1}\left(i - N + \frac{n}{2} - 1\right)!\, t^{-i+N-\frac{n}{2}} \,.$$

Hence, for $i \ge N+1$, we obtain

$$\begin{aligned} E^{i)}(1) = \alpha(n,N)\left(N - \frac{n}{2}\right)\cdots\left(n - \frac{n}{2} - i + 1\right) \\ + \beta(n,N)\left(N - \frac{n}{2}\right)!(-1)^{i-N+\frac{n}{2}-1}\left(i - N + \frac{n}{2} - 1\right)! \end{aligned}$$

Consequently,

$$\begin{aligned} A_L = (-1)^L\alpha(n,N)\sum_{i=L}^{2N-1}\left(N - \frac{n}{2}\right)\cdots\left(N - \frac{n}{2} - i + 1\right)\frac{(-1)^i}{i!}\binom{i}{L} \\ + (-1)^{L-N+\frac{n}{2}-1}\beta(n,N)\left(N - \frac{n}{2}\right)!\sum_{i=L}^{2N-1}\left(i - N + \frac{n}{2} - 1\right)!\frac{\binom{i}{L}}{i!} \,. \end{aligned} \tag{73}$$

Let's remark that for the cases $n = 2$ or n even and $N \ge \frac{n}{2}$ the first term in (73) is zero, while for the cases n odd or n even and $N \le \frac{n}{2} - 1$ the second term is zero because $\beta(n,N) = 0$. This explains why we compute below the two terms separately.

For the first term we show that

$$\sum_{i=L}^{2N-1}\left(N - \frac{n}{2}\right)\cdots\left(N - \frac{n}{2} - i + 1\right)\frac{(-1)^i}{i!}\binom{i}{L} = (-1)^L\binom{N - \frac{n}{2}}{L}\binom{\frac{n}{2} + N - 1}{2N - 1 - L} \,. \tag{74}$$

Indeed, the left hand side of (74) is, setting $k = i - L$,

$$\frac{1}{L!}\sum_{k=0}^{2N-1-L}\Big(N-\frac{n}{2}\Big)\cdots\Big(N-\frac{n}{2}-L-k+1\Big)\frac{(-1)^{L+k}}{k!}$$

$$= (-1)^L\binom{N-\frac{n}{2}}{L}\sum_{k=0}^{2N-1-L}\binom{\frac{n}{2}+L-N+k-1}{k}$$

$$= (-1)^L\binom{N-\frac{n}{2}}{L}\binom{N+\frac{n}{2}-1}{2N-1-L},$$

where the last identity comes from ([GKP, (5.9), p. 159]).

To compute the second term we first show that

$$\sum_{i=L}^{2N-1}\Big(i-N+\frac{n}{2}-1\Big)!\frac{1}{i!}\binom{i}{L} = \frac{(L-N+\frac{n}{2}-1)!}{L!}\binom{N+\frac{n}{2}-1}{2N-1-L}. \tag{75}$$

As before, setting $k = i - L$ and applying [GKP, (5.9), p. 159], we see that the left hand side of (75) is

$$\frac{1}{L!}\sum_{k=0}^{2N-1-L}\Big(L+k-N+\frac{n}{2}-1\Big)!\frac{1}{k!}$$

$$= \Big(L-N+\frac{n}{2}-1\Big)!\sum_{k=0}^{2N-1-L}\binom{\frac{n}{2}+L-N+k-1}{k}$$

$$= \frac{(L-N+\frac{n}{2}-1)!}{L!}\binom{N+\frac{n}{2}-1}{2N-1-L}.$$

We are now ready to complete the proof of the lemma distinguishing 4 cases.

Case 1: n odd.

Since $\beta(n,N)=0$, replacing in (73) $\alpha(n,N)$ by its value and using (74) we get, by elementary arithmetics and the identity $n\,V_n=\omega_n$,

$$\begin{aligned}
A_L &= \frac{(-1)^L\Gamma(2-\frac{n}{2})}{4^{N-1}(N-1)!\Gamma(N+1-\frac{n}{2})(2-n)\omega_n}(-1)^L\binom{N-\frac{n}{2}}{L}\binom{\frac{n}{2}+N-1}{2N-1-L}\\
&= \frac{(-1)^{N+L}(N+\frac{n}{2}-1)\cdots(\frac{n}{2}+1)}{4^N\,V_n\,L!\,(2N-1-L)!\,(L+\frac{n}{2}-N)(N-1)!}\\
&= \frac{(-1)^{N+L}\binom{N+\frac{n}{2}-1}{N-1}}{4^N\,V_n\,L!\,(2N-1-L)!\,(L+\frac{n}{2}-N)}.
\end{aligned}$$

Case 2: n even, $n\neq 2$ and $N\leq \frac{n}{2}-1$.

As in case 1 $\beta(n,N)=0$, and we proceed similarly using (74) to obtain

$$\begin{aligned}
A_L &= \frac{(-1)^L(-1)^{N-1}(\frac{n}{2}-N-1)!}{4^N(N-1)!(\frac{n}{2}-2)!(2-n)\omega_n}(-1)^L\binom{N-\frac{n}{2}}{L}\binom{N+\frac{n}{2}-1}{2N-1-L}\\
&= \frac{(-1)^{N+1}\binom{N+\frac{n}{2}-1}{N-1}\frac{n}{2}!(\frac{n}{2}-N-1)!\{(N-\frac{n}{2})\cdots(N-\frac{n}{2}-L+1)\}}{(2N-1-L)!\,L!\,4^{N-1}\,(2-n)\,\omega_n\,(\frac{n}{2}-2)!\,(\frac{n}{2}-N+L)!}\\
&= \frac{(-1)^N\binom{N+\frac{n}{2}-1}{N-1}}{4^N\,V_n\,L!\,(2N-1-L)!}\,\frac{(-1)L(\frac{n}{2}-N+L-1)!}{(\frac{n}{2}-N+L)!}\\
&= \frac{(-1)^{N+L}\binom{N+\frac{n}{2}-1}{N-1}}{4^N\,V_n\,L!\,(2N-1-L)!\,(L+\frac{n}{2}-N)}\,.
\end{aligned}$$

Case 3: n even, $n\neq 2$ and $N\geq\frac{n}{2}$.

Replacing in (73) $\alpha(n,N)$ and $\beta(n,N)$ by their values and using (75) we get, by elementary arithmetics and the identity $n\,V_n=\omega_n$,

$$\begin{aligned}
A_L &= \beta(n,N)(N-\frac{n}{2})!(-1)^{N+\frac{n}{2}-1+L}\sum_{i=L}^{2N-1}(i-N+\frac{n}{2}-1)!\frac{1}{i!}\binom{i}{L}\\
&= \frac{(-1)^{N+L}(L-N+\frac{n}{2}-1)!}{L!4^{N-1}(\frac{n}{2}-2)!(N-1)!(2-n)\omega_n}\binom{N+\frac{n}{2}-1}{2N-1-L}\\
&= \frac{(-1)^{L+N}\frac{n}{2}(\frac{n}{2}-1)}{4^{N-1}(2-n)\omega_n L!(2N-1-L)!(\frac{n}{2}-N+L)}\binom{N+\frac{n}{2}-1}{N-1}\\
&= \frac{(-1)^{L+N}\binom{N+\frac{n}{2}-1}{N-1}}{4^N L! V_n(2N-1-L)!(\frac{n}{2}-N+L)}\,.
\end{aligned}$$

Case 4: $n=2$.

Proceeding as in case 3 and we obtain

$$\begin{aligned}
A_L &= \beta(n,N)(N-\frac{n}{2})!(-1)^{N+\frac{n}{2}-1+L}\sum_{i=L}^{2N-1}(i-N+\frac{n}{2}-1)!\frac{1}{i!}\binom{i}{L}\\
&= \frac{(-1)^{N+L}N!}{2\omega_2 4^{N-1}L!(N-1)!(2N-1-L)!(L+1-N)}\\
&= \frac{(-1)^{L+N}\binom{N}{N-1}}{V_2 4^N L!(2n-1-L)!(L+1-N)}\,.
\end{aligned}$$

$\square$

Proof of Lemma 10. Recall that (see (61))

$$C_{2j} = \sum_{l=N+j}^{2N-1} c_{l,\,j,\,l-N-j}\, G_{\frac{n}{2}+l-N+j}(0)\,.$$

Thus, we have to compute the constants $c_{l,j,k}$ appearing in the expression (59) for $\widehat{S\,\chi_B}(\xi)$. For that we need the constants $c_{l,j}$ appearing in the formula (22) for $S(x)$. We start by computing $P_{2j}(\partial)\Delta^{N-j}(|x|^{2l})$. Using (20) and Lemma 4 one gets

$$P_{2j}(\partial)\Delta^{N-j}(|x|^{2l}) = \frac{4^N l!(N-j)!}{(l-N-j)!}\binom{l-1+\frac{n}{2}}{N-j} P_{2j}(x)|x|^{2(l-N-j)}$$

if $l-N-j \geq 0$ (and $=0$ if $l-N-j<0$).

As in (25) (Section 3), we want to compute $P_{2j}(\partial)\Delta^{l-N-j}G_{\frac{n}{2}}(\xi)$ by using Lemma 3 applied to $f(r) = G_{\frac{n}{2}}(r)$ and the homogeneous polynomial $L(x) = P_{2j}(x)\,|x|^{2(l-N-j)}$. We obtain

$$\begin{aligned} P_{2j}(\partial)\Delta^{l-N-j}G_{\frac{n}{2}}(\xi) &= (-1)^{2(l-N)}\sum_{k\geq 0}\frac{(-1)^k}{2^k k!}\Delta^k(P_{2j}(x)|x|^{2(l-N-j)})\, G_{\frac{n}{2}+2(l-N)-k}(\xi) \\ &= \sum_{k=0}^{l-N-j}\frac{(-1)^k}{2^k k!}4^k\frac{(l-N-j)!}{(l-N-j-k)!}k!\binom{\frac{n}{2}+j+l-N-1}{k} \\ &\quad\times P_{2j}(\xi)|\xi|^{2(l-N-j-k)}\, G_{\frac{n}{2}+2(l-N)-k}(\xi)\,. \end{aligned}$$

In view of the definitions of $Q(x)$ and $S(x)$,

$$\begin{aligned} S(x) = -Q(\partial)\left(\sum_{l=0}^{2N-1} A_l|x|^{2l}\right) &= -\sum_{l=0}^{2N-1} A_l\sum_{j=1}^{N}\gamma_{2j}P_{2j}(\partial)\Delta^{N-j}(|x|^{2l}) \\ &= -\sum_{l=N+1}^{2N-1}\sum_{j=1}^{l-N}A_l\gamma_{2j}\frac{4^N l!(N-j)!}{(l-N-j)!}\binom{l-1+\frac{n}{2}}{N-j}P_{2j}(x)|x|^{2(l-N-j)} \\ &= \sum_{l=N+1}^{2N-1}\sum_{j=1}^{l-N}c_{l,j}P_{2j}(x)|x|^{2(l-N-j)}\,, \end{aligned}$$

where the last identity defines the $c_{l,j}$. In (59) we set

$$\begin{aligned}
\widehat{S\,\chi_B}(\xi) &= S(\imath\,\partial)\,\widehat{\chi_B}(\xi) \\
&= \sum_{l=N+1}^{2N-1}\sum_{j=1}^{l-N} c_{lj}\,(-1)^{l-N}\ P_{2j}(\partial)\,\triangle^{l-N-j}\,G_{\frac{n}{2}}(\xi) \\
&= \sum_{l=N+1}^{2N-1}\sum_{j=1}^{l-N}\sum_{k=0}^{l-N-j} c_{l,\,j,\,k}\ P_{2j}(\xi)\,|\xi|^{2(l-N-j-k)}G_{\frac{n}{2}+2l-2N-k}(\xi)\,.
\end{aligned}$$

Consequently,

$$\begin{aligned}
c_{l,j,k} &= c_{l,j}(-1)^{l-N}\frac{(-1)^k}{2^k}4^k\frac{(l-N-j)!}{(l-N-j-k)!}\binom{\frac{n}{2}+j+l-N-1}{k} \\
&= -(-1)^{l+k+N}A_l\gamma_{2j}\frac{4^N l!(N-j)!}{(l-N-j)!}\binom{l-1+\frac{n}{2}}{N-j} \\
&\quad\times 2^k\frac{(l-N-j)!}{(l-N-j-k)!}\binom{\frac{n}{2}+j+l-N-1}{k}\,.
\end{aligned}$$

Replacing A_l by the formula given in lemma 13 and performing some easy arithmetics we get

$$c_{l,j,k} = -\frac{(-1)^k}{V_n}\frac{\gamma_{2j}\binom{N+\frac{n}{2}-1}{N-1}2^k(N-j)!\binom{l-1+\frac{n}{2}}{N-j}\binom{\frac{n}{2}+j+l-N-1}{k}}{(l+\frac{n}{2}-N)(2N-l-1)!(l-N-j-k)!}\,. \quad (76)$$

The final computation of the C_{2j} is as follows.

$$C_{2j} = \sum_{l=N+j}^{2N-1} c_{l,j,l-N-j}\, G_{\frac{n}{2}+l-N+j}(0)$$

$$= \qquad \text{[by the explicit value of } G_p(0) \text{ given in (78) below]}$$

$$= \sum_{l=N+j}^{2N-1} c_{l,j,l-N-j} \frac{1}{2^{\frac{n}{2}+l-N+j}\Gamma(\frac{n}{2}+l-N+j+1)}$$

$$= \qquad \text{[by (76)]}$$

$$= -\sum_{l=N+j}^{2N-1} \frac{(-1)^{l-N-j}}{V_n} \frac{\gamma_{2j}\binom{N+\frac{n}{2}-1}{N-1} 2^{l-N-j}(N-j)!\binom{l-1+\frac{n}{2}}{N-j}\binom{\frac{n}{2}+j+l-N-1}{l-N-j}}{(l+\frac{n}{2}-N)(2N-l-1)!2^{\frac{n}{2}+l-N+j}\Gamma(\frac{n}{2}+l-N+j+1)}$$

$$= \qquad \text{[substituting the value given in (7) in } \gamma_{2j}\text{]}$$

$$= -\frac{\pi^{\frac{n}{2}}\Gamma(j)\binom{N+\frac{n}{2}-1}{N-1}(N-j)!}{V_n\Gamma(j+\frac{n}{2})2^{\frac{n}{2}+2j}}$$

$$\times \sum_{l=N+1}^{2N-1} \frac{(-1)^{l+N}\binom{l-1+\frac{n}{2}}{N-j}\binom{\frac{n}{2}+j+l-N-1}{l-N-j}}{(l+\frac{n}{2}-N)(2N-l-1)!\Gamma(\frac{n}{2}+l-N+j+1)}$$

$$= \qquad \text{[setting } l = i+N+j\text{]}$$

$$= -\frac{\pi^{\frac{n}{2}}\Gamma(j)\binom{N+\frac{n}{2}-1}{N-1}(N-j)!}{V_n\Gamma(j+\frac{n}{2})2^{\frac{n}{2}+2j}}$$

$$\times \sum_{i=0}^{N-1-j} \frac{(-1)^{i+j}\binom{i+N+j-1+\frac{n}{2}}{N-j}\binom{\frac{n}{2}+2j+i-1}{i}}{(i+j+\frac{n}{2})(N-i-j-1)!\Gamma(\frac{n}{2}+i+2j+1)}$$

$$= \qquad \text{[because } \Gamma(\frac{n}{2}+i+2j+1) = \Gamma(2j+\frac{n}{2})\prod_{k=0}^{i}(\frac{n}{2}+2j+i-k)\text{]}$$

$$= -\frac{(-1)^j \pi^{\frac{n}{2}} \Gamma(j) \binom{N+\frac{n}{2}-1}{N-1} (N-j)!}{V_n \Gamma(j+\frac{n}{2}) 2^{\frac{n}{2}+2j} \Gamma(2j+\frac{n}{2})}$$

$$\times \sum_{i=0}^{N-1-j} \frac{(-1)^i \binom{i+N+j-1+\frac{n}{2}}{N-j} \binom{\frac{n}{2}+2j+i-1}{i}}{(i+j+\frac{n}{2})(N-i-j-1)! \prod_{k=0}^{i}(\frac{n}{2}+2j+i-k)}$$

$$= \qquad \text{[using Lemma 14 below]}$$

$$= -\frac{(-1)^j \pi^{\frac{n}{2}} \Gamma(j) \binom{N+\frac{n}{2}-1}{N-1} (N-j)!}{V_n \Gamma(j+\frac{n}{2}) 2^{\frac{n}{2}+2j} \Gamma(2j+\frac{n}{2})} \binom{N-1}{j-1} \frac{\Gamma(j+\frac{n}{2})}{j\,\Gamma(N+\frac{n}{2})}$$

$$= \frac{-\pi^{\frac{n}{2}}}{V_n 2^{\frac{n}{2}} \Gamma(\frac{n}{2}+1)} \, \frac{(-1)^j}{j\, 4^j \Gamma(2j+\frac{n}{2})} .$$

□

Lemma 14. *For each $j = 1, \ldots, N-1$*

$$\sum_{i=0}^{N-1-j} \frac{(-1)^i \binom{i+N+j-1+\frac{n}{2}}{N-j} \binom{\frac{n}{2}+2j+i-1}{i}}{(i+j+\frac{n}{2})(N-i-j-1)! \prod_{k=0}^{i}(\frac{n}{2}+2j+i-k)} = \binom{N-1}{j-1} \frac{\Gamma(j+\frac{n}{2})}{j\Gamma(N+\frac{n}{2})} .$$

Proof. Divide the left hand side by the right hand side and denote the quotient by A. We have to prove that $A = 1$. Using elementary arithmetics

$$\frac{\binom{i+N+j-1+\frac{n}{2}}{N-j} \binom{\frac{n}{2}+2j+i-1}{i} \Gamma(N+\frac{n}{2})}{(i+j+\frac{n}{2}) \prod_{k=0}^{i}(\frac{n}{2}+2j+i-k)\; \Gamma(j+\frac{n}{2})}$$

$$= \binom{N+i+j+\frac{n}{2}-1}{N-j-1} \binom{N+\frac{n}{2}-1}{N-i-j-1} \frac{(N-i-j-1)!}{N-j} \binom{\frac{n}{2}+i+j-1}{i} ,$$

and so

$$A = \frac{j}{\binom{N-1}{j-1}(N-j)} \sum_{i=0}^{N-1-j} (-1)^i \binom{N+i+j+\frac{n}{2}-1}{N-j-1} \binom{N+\frac{n}{2}-1}{N-i-j-1} \binom{\frac{n}{2}+i+j-1}{i}$$

$$= \qquad \left[\text{because } \binom{a+i}{i} = (-1)^i \binom{-a-1}{i}\right]$$

$$= \frac{j}{\binom{N-1}{j-1}(N-j)} \sum_{i=0}^{N-1-j} \binom{N+i+j+\frac{n}{2}-1}{N-j-1} \binom{N+\frac{n}{2}-1}{N-i-j-1} \binom{-\frac{n}{2}-j}{i}$$

$= $ [by the triple-binomial identity (5.28) of ([GKP, p. 171]), see below]

$$= \frac{j}{\binom{N-1}{j-1}(N-j)} \binom{\frac{n}{2}+N+j-1}{0} \binom{N-1}{N-j-1} = 1\,.$$

For the reader's convenience and later reference we state the triple-binomial identity [GKP, (5.28), p. 171]:

$$\sum_{k=0}^{n} \binom{m-r+s}{k} \binom{n+r-s}{n-k} \binom{r+k}{m+n} = \binom{r}{m} \binom{s}{n}\,, \tag{77}$$

where m and n are non-negative integers. □

Our next task is to prove Lemma 11 and Lemma 12. Setting $\xi = r\,\xi_0$, $|\xi_0| = 1$, in (59) we obtain

$$\widehat{S_N\,\chi_B}(r\xi_0) = \sum_{l=N+1}^{2N-1} \sum_{j=1}^{l-N} \sum_{k=0}^{l-N-j} c_{l,\,j,\,k}\; P_{2j}(r\xi_0)\; |r\xi_0|^{2(l-N-j-k)} G_{\frac{n}{2}+2l-2N-k}(r\xi_0)$$

$$[\text{make the change of indexes } l = N+s]$$

$$= \sum_{s=1}^{N-1} \sum_{j=1}^{s} \sum_{k=0}^{s-j} c_{N+s,\,j,\,k}\, P_{2j}(\xi_0)\, r^{2(s-k)} G_{\frac{n}{2}+2s-k}(r)$$

$$= \sum_{j=1}^{N-1} \sum_{s=j}^{N-1} \sum_{k=0}^{s-j} c_{N+s,\,j,\,k}\, P_{2j}(\xi_0)\, r^{2(s-k)} G_{\frac{n}{2}+2s-k}(r)$$

$$:= \sum_{p=1}^{\infty} a_{2p}^{N}(\xi_0) r^{2p}\,.$$

In order to compute the coefficients $a_{2p}^{N}(\xi_0)$ we substitute the power series expansion of $G_q(r)$ [Gr, A-8], namely,

$$G_q(r) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!\,\Gamma(q+i+1)} \frac{r^{2i}}{2^{2i+q}}\,, \tag{78}$$

in the last triple sum above.

Proof of Lemma 11. We are assuming that $1 \le p \le N-1$. It is important to remark that, for this range of p, after introducing (78) in the triple sum above, only the values of the index j satisfying $1 \le j \le p$ are involved in the expression for a_{2p}^N. Once (78) has been introduced in the triple sum one should sum, in principle, on the four indexes i, j, s and k. But since we are looking at the coefficient of r^{2p} we have the relation $2(s-k)+2i = 2p$, which actually leaves us with three indexes. The range of each of these indexes is easy to determine and one gets

$$a_{2p}^N = \sum_{j=1}^{p} P_{2j}(\xi_0) \sum_{i=0}^{p-j} \sum_{s=p-i}^{N-1} c_{N+s,\,j,\,s-(p-i)} \times \text{coefficient of } r^{2i} \text{ from } G_{\frac{n}{2}+s+p-i}(r)\,.$$

In view of (78)

$$\begin{aligned}
a_{2p}^N &= \sum_{j=1}^{p} P_{2j}(\xi_0) \sum_{i=0}^{p-j} \sum_{s=p-i}^{N-1} c_{N+s,\,j,\,s-(p-i)} \frac{(-1)^i}{i! 2^{i+\frac{n}{2}+s+p} \Gamma(\frac{n}{2}+s+p+1)} \\
&= \qquad [\text{by the expression (76) for } c_{l,j,k}] \\
&= \sum_{j=1}^{p} P_{2j}(\xi_0) \sum_{i=0}^{p-j} \sum_{s=p-i}^{N-1} \frac{(-1)^{i+1}}{i! 2^{i+\frac{n}{2}+s+p} \Gamma(\frac{n}{2}+s+p+1)} \frac{(-1)^{s-(p-i)}}{V_n} \\
&\qquad \times \frac{\gamma_{2j} \binom{N+\frac{n}{2}-1}{N-1} 2^{s-(p-i)} (N-j)! \binom{N+s-1+\frac{n}{2}}{N-j} \binom{\frac{n}{2}+j+s-1}{s-(p-i)}}{(s+\frac{n}{2})(N-s-1)!(p-i-j)!} \\
&= (-1)^{p+1} \sum_{j=1}^{p} P_{2j}(\xi_0) \frac{(-1)^j \pi^{\frac{n}{2}} \Gamma(j)(N-j)!}{V_n \Gamma(\frac{n}{2}+j) 2^{\frac{n}{2}+2p}} \sum_{i=0}^{p-j} \frac{\binom{N+\frac{n}{2}-1}{N-1}}{i!(p-i-j)!} \\
&\qquad \times \sum_{s=p-i}^{N-1} \frac{(-1)^s \binom{N+s-1+\frac{n}{2}}{N-j} \binom{\frac{n}{2}+j+s-1}{s-(p-i)}}{(s+\frac{n}{2})(N-s-1)!\Gamma(\frac{n}{2}+s+p+1)}\,.
\end{aligned}$$

In Lemma 15 below we give a useful compact form for the last sum. Using it we obtain

$$a_{2p}^N = (-1)^{p+1} \sum_{j=1}^{p} P_{2j}(\xi_0) \frac{(-1)^j \pi^{\frac{n}{2}} \Gamma(j)(N-j)!}{V_n \Gamma(\frac{n}{2}+j) 2^{\frac{n}{2}+2p}} \sum_{i=0}^{p-j} \frac{\binom{N+\frac{n}{2}-1}{N-1}}{i!(p-i-j)!}$$

$$\times \frac{(-1)^{p-i}(N-p-1)! \binom{N-1}{p}}{(N-j)!\Gamma(N+\frac{n}{2}) j! \binom{\frac{n}{2}+p-i+j-1}{j}} .$$

Easy arithmetics with binomial coefficients gives

$$\binom{N+\frac{n}{2}-1}{N-1} (N-p-1)! \binom{N-1}{p} \frac{1}{\Gamma(N+\frac{n}{2})} = \frac{1}{p!\, \Gamma(\frac{n}{2}+1)} .$$

We finally get the extremely surprising identity

$$a_{2p}^N = \frac{-\pi^{\frac{n}{2}}}{V_n 2^{\frac{n}{2}+2p} p! \Gamma(\frac{n}{2}+1)} \sum_{j=1}^{p} \frac{(-1)^j \Gamma(j) P_{2j}(\xi_0)}{\Gamma(\frac{n}{2}+j)} \sum_{i=0}^{p-j} \frac{(-1)^i}{i!(p-i-j)! j! \binom{\frac{n}{2}+p-i+j-1}{j}} , \quad (79)$$

in which N has miraculously disappeared. Thus Lemma 11 is proved. □

Proof of Lemma 12. We start by proving the inequality (68), so that $1 \le p \le N-1$. We roughly estimate $a_{2p} = a_{2p}^N$ by putting the absolute value inside the sums in (79). The absolute value of each term in the innermost sum in (79) is obviously not greater than 1 and there are at most p terms. The factor in front of $P_{2j}(\xi_0)$ is again not greater than 1 in absolute value. Denoting by C the terms that depend only on n we obtain the desired inequality (68).

We turn now to the proof of inequality (69). Recall that

$$\widehat{S_N \chi_B}(r\xi_0) = \sum_{p=1}^{\infty} a_{2p}^N(\xi_0) r^{2p} = \sum_{j=1}^{N-1} \sum_{s=j}^{N-1} \sum_{k=0}^{s-j} c_{N+s,j,k}\ P_{2j}(\xi_0)\ r^{2(s-k)} G_{\frac{n}{2}+2s-k}(r) .$$

Replacing $G_{\frac{n}{2}+2s-k}(r)$ by the expression given by (78) we obtain, as before, a sum with four indexes. Now we eliminate the index i of (78) using $s-k+i=p$. Hence

$$a_{2p}^N = \sum_{j=1}^{N-1} P_{2j}(\xi_0) \sum_{s=j}^{N-1} \sum_{k=0}^{s-j} c_{N+s,\,j,\,k} \times \text{coefficient of } r^{2(p-s+k)} \text{ from } G_{\frac{n}{2}+2s-k}(r)$$

$$= \sum_{j=1}^{N-1} P_{2j}(\xi_0) \sum_{s=j}^{N-1} \sum_{k=0}^{s-j} c_{N+s,\,j,\,k} \frac{(-1)^{p-s+k}}{(p-s+k)!\Gamma(\frac{n}{2}+p+s-1)2^{2p+\frac{n}{2}+k}}$$

$$= \sum_{j=1}^{N-1} P_{2j}(\xi_0) \sum_{k=0}^{N-1-j} \sum_{s=j+k}^{N-1} c_{N+s,\,j,\,k} \frac{(-1)^{p-s+k}}{(p-s+k)!\Gamma(\frac{n}{2}+p+s-1)2^{2p+\frac{n}{2}+k}}$$

$$= \frac{(-1)^p}{V_n 4^p 2^{\frac{n}{2}}} \binom{N+\frac{n}{2}-1}{N-1} \sum_{j=1}^{N-1} \gamma_{2j} P_{2j}(\xi_0) \sum_{k=0}^{N-1-j} \sum_{s=j+k}^{N-1}$$

$$\times \frac{(-1)^s (N-j)! \binom{N+s-1+\frac{n}{2}}{N-j} \binom{\frac{n}{2}+j+s-1}{k}}{(p-s+k)!\Gamma(\frac{n}{2}+p+s+1)(s+\frac{n}{2})(N-s-1)!(s-j-k)!}\,.$$

The second identity is just (78). The third is a change of the order of summation and the latest follows from the formula (76) for the constants $c_{l,j,k}$.

In view of the elementary fact that

$$(N-j)! \binom{N+s-1+\frac{n}{2}}{N-j} \binom{\frac{n}{2}+j+s-1}{k} = \frac{\Gamma(s+\frac{n}{2}+N)}{k!\Gamma(s+\frac{n}{2}+j-k)}$$

we get

$$\left| \sum_{k=0}^{N-1-j} \sum_{s=j+k}^{N-1} \frac{(-1)^s (N-j)! \binom{N+s-1+\frac{n}{2}}{N-j} \binom{\frac{n}{2}+j+s-1}{k}}{(p-s+k)!\Gamma(\frac{n}{2}+p+s+1)(s+\frac{n}{2})(N-s-1)!(s-j-k)!} \right|$$

$$\leq \sum_{k=0}^{N-1-j} \frac{1}{k!}$$

$$\times \sum_{s=j+k}^{N-1} \frac{1}{\Gamma(s+\frac{n}{2}+j-k)(p-s+k)!(\frac{n}{2}+p+s)(s+\frac{n}{2})(N-s-1)!(s-j-k)!}$$

$$\leq \sum_{k=0}^{N-1-j} \frac{1}{k!} \sum_{s=j+k}^{N-1} \frac{1}{(s-j-k)!} \leq e^2\,,$$

where in the first inequality we used that, since $N \leq p$,

$$\frac{\Gamma(s+\frac{n}{2}+N)}{\Gamma(\frac{n}{2}+p+s+1)} \leq \frac{1}{\frac{n}{2}+p+s}\,.$$

The proof of (69) is complete. □

Lemma 15. *Let $N-1 \geq L \geq j \geq k \geq 0$ be integers. Then*

$$\sum_{s=j}^{N-1} \frac{(-1)^s \binom{\frac{n}{2}+N+s-1}{N-k} \binom{\frac{n}{2}+k+s-1}{s-j}}{(s+\frac{n}{2})(N-s-1)!\Gamma(\frac{n}{2}+s+L+1)}$$

$$= (-1)^j \frac{(N-L-1)!}{(N-k)!} \binom{N-1}{L} \frac{1}{k!\,\Gamma(\frac{n}{2}+N)} \frac{1}{\binom{\frac{n}{2}+k+j-1}{k}}\,.$$

Proof. To simplify notation set

$$A = \sum_{s=j}^{N-1} \frac{(-1)^s \binom{\frac{n}{2}+N+s-1}{N-k} \binom{\frac{n}{2}+k+s-1}{s-j}}{(s+\frac{n}{2})(N-s-1)!\Gamma(\frac{n}{2}+s+L+1)}\,.$$

Making the change of index of summation $i = s - j$ and using repeatedly the identity $\Gamma(x+1) = x\,\Gamma(x)$ we have

$$\begin{aligned} A &= \frac{(-1)^j}{\Gamma(\frac{n}{2}+j+L)} \sum_{i=0}^{N-1-j} \frac{(-1)^i \binom{\frac{n}{2}+N+i+j-1}{N-k} \binom{\frac{n}{2}+k+i+j-1}{i}}{(i+j+\frac{n}{2})(N-i-j-1)!\{\prod_{p=0}^{i}(\frac{n}{2}+i+j+L-p)\}} \\ &= \frac{(-1)^j}{\Gamma(\frac{n}{2}+j+L)}\; B\,, \end{aligned}$$

where the last identity defines B. To compute B we need to rewrite the terms in a more convenient way so that we may apply well known equalities, among which the triple-binomial identity (77). Notice that

$$\frac{\binom{\frac{n}{2}+N+i+j-1}{N-k} \binom{\frac{n}{2}+k+i+j-1}{i}}{\{\prod_{p=0}^{i}(\frac{n}{2}+i+j+L-p)\}}$$

$$= \frac{(N-L-1)!(L-k)!}{i!(N-k)!} \binom{\frac{n}{2}+N+i+j-1}{N-L-1} \binom{\frac{n}{2}+j+L-1}{L-k}\,.$$

Hence

$$B = \sum_{i=0}^{N-1-j} \frac{(-1)^i (N-L-1)!(L-k)!}{i!(N-k)!(N-i-j-1)!(i+j+\frac{n}{2})} \binom{\frac{n}{2}+N+i+j-1}{N-L-1} \binom{\frac{n}{2}+j+L-1}{L-k}\,.$$

Since clearly

$$\frac{\Gamma(\frac{n}{2}+N)}{\Gamma(\frac{n}{2}+j)} \frac{1}{(i+j+\frac{n}{2})(N-i-j-1)!} = \binom{\frac{n}{2}+N-1}{N-j-i-1} \binom{\frac{n}{2}+j+i-1}{i} i!\,,$$

B can be written as

$$
\begin{aligned}
B &= \frac{\Gamma(\frac{n}{2}+j)(N-L-1)!(L-k)!}{\Gamma(\frac{n}{2}+N)(N-k)!}\binom{\frac{n}{2}+j+L-1}{L-k} \\
&\quad \times \sum_{i=0}^{N-1-j}(-1)^i\binom{\frac{n}{2}+N-1}{N-j-i-1}\binom{\frac{n}{2}+j+i-1}{i}\binom{\frac{n}{2}+N+i+j-1}{N-L-1} \\
&= \frac{\Gamma(\frac{n}{2}+j)(N-L-1)!(L-k)!}{\Gamma(\frac{n}{2}+N)(N-k)!}\binom{\frac{n}{2}+j+L-1}{L-k}C\,,
\end{aligned}
$$

where the last identity defines C.

To simplify notation set $L = m+j$, where $0 \le m \le L-j$. Using the elementary identity

$$
\binom{\frac{n}{2}+N-1}{N-j-i-1}\binom{\frac{n}{2}+i+j-1}{i} = \frac{(N-j)!}{i!(N-i-j-1)!(\frac{n}{2}+i+j)}\binom{\frac{n}{2}+N-1}{N-j},
$$

we get

$$
C = \binom{\frac{n}{2}+N-1}{N-j}(N-j)!\sum_{i=0}^{N-1-j}\frac{(-1)^i\binom{\frac{n}{2}+N+i+j-1}{N-j-m-1}}{i!\,(N-i-j-1)!\,(\frac{n}{2}+i+j)}\,.
$$

The only task left is the computation of the sum

$$
D(j,m) = \sum_{i=0}^{N-1-j}\frac{(-1)^i\binom{\frac{n}{2}+N+i+j-1}{N-j-m-1}}{i!\,(N-i-j-1)!\,(\frac{n}{2}+i+j)}\,.
$$

The identity

$$
\frac{1}{\frac{n}{2}+i+j} = \frac{1}{\frac{n}{2}+j}\left(1-\frac{i}{\frac{n}{2}+i+j}\right),
$$

yields the expression

$$
\begin{aligned}
D(j,m) &= \sum_{i=0}^{N-1-j}\frac{(-1)^i}{(\frac{n}{2}+j)i!(N-i-j-1)!}\binom{\frac{n}{2}+i+j+N-1}{N-j-m-1} \\
&\quad - \sum_{i=0}^{N-1-j}\frac{(-1)^i i}{i!(\frac{n}{2}+j)(\frac{n}{2}+j+i)(N-i-j-1)!}\binom{\frac{n}{2}+i+j+N-1}{N-j-m-1}.
\end{aligned}
$$

The first sum in the above expression for $D(j,m)$ turns out to vanish for $m \ge 1$.

This is because

$$\sum_{i=0}^{N-1-j} \frac{(-1)^i}{(\frac{n}{2}+j)i!(N-i-j-1)!}\binom{\frac{n}{2}+i+j+N-1}{N-j-m-1}$$

$$= \frac{1}{(N-j-1)!\,(\frac{n}{2}+j)} \sum_{i=0}^{N-1-j}(-1)^i\binom{N-j-1}{i}\binom{\frac{n}{2}+i+j+N-1}{N-j-m-1}$$

$$= (-1)^{N-j-1}\binom{\frac{n}{2}+j+N-1}{-m}$$

$$= 0\,,$$

where the next to the last equality follows from an identity proven in [GKP, (5.24), p. 169] and the last equality follows from the fact that $\binom{s}{k} = 0$ provided k is a negative integer. Hence, setting $s = i-1$,

$$D(j,m) = -\sum_{i=0}^{N-1-j} \frac{(-1)^i\, i}{i!(\frac{n}{2}+j)(\frac{n}{2}+j+i)(N-i-j-1)!}\binom{\frac{n}{2}+i+j+N-1}{N-j-m-1}$$

$$= \frac{1}{\frac{n}{2}+j}\sum_{s=0}^{N-(j+1)-1} \frac{(-1)^s}{s!(\frac{n}{2}+s+(j+1))}\frac{\binom{\frac{n}{2}+(j+1)+N+s-1}{N-(j+1)-(m-1)-1}}{(N-(j+1)-s-1)!}$$

$$= \frac{1}{(\frac{n}{2}+j)}D(j+1,m-1)\,.$$

Repeating the above argument m times we obtain that

$$D(j,m) = \frac{1}{(\frac{n}{2}+j)(\frac{n}{2}+j+1)\cdots(\frac{n}{2}+j+m-1)}\;D(L,0)\,.$$

To compute $D(L,0)$ we use the elementary identity

$$\frac{1}{(\frac{n}{2}+L+s)} = \frac{s!(N-L-s-1)!\Gamma(\frac{n}{2}+L)}{\Gamma(\frac{n}{2}+N)}\binom{\frac{n}{2}+L+s-1}{s}\binom{\frac{n}{2}+N-1}{N-L-s-1},$$

from which we get

$$D(L,0) = \sum_{s=0}^{N-1-L}(-1)^s\frac{\Gamma(\frac{n}{2}+L)}{\Gamma(\frac{n}{2}+N)}\binom{\frac{n}{2}+L+s-1}{s}\binom{\frac{n}{2}+N-1}{N-L-s-1}\binom{\frac{n}{2}+L+N+s-1}{N-L-1}$$

$$= \frac{\Gamma(\frac{n}{2}+L)}{\Gamma(\frac{n}{2}+N)}\sum_{s=0}^{N-1-L}\binom{-\frac{n}{2}-L}{s}\binom{\frac{n}{2}+N-1}{N-L-s-1}\binom{\frac{n}{2}+L+N+s-1}{N-L-1}$$

$$= \frac{\Gamma(\frac{n}{2}+L)}{\Gamma(\frac{n}{2}+N)}\binom{N-1}{L},$$

where in the second identity we applied [GKP, (5.14), p. 164] and the latest equality is consequence of the triple-binomial identity [GKP, (5.28), p. 171] (for $n = N-L-1$, $m = 0$, $r = \frac{n}{2} + N + L - 1$ and $s = N - 1$). Consequently,

$$
\begin{aligned}
D(j,m) &= \frac{\Gamma(\frac{n}{2}+L)}{\Gamma(\frac{n}{2}+N)} \binom{N-1}{L} \frac{1}{(\frac{n}{2}+j)\cdots(\frac{n}{2}+L-1)} \\
&= \frac{\Gamma(\frac{n}{2}+j)}{\Gamma(\frac{n}{2}+N)} \binom{N-1}{L}.
\end{aligned}
$$

Hence

$$
C = \binom{\frac{n}{2}+N-1}{N-j} (N-j)! \, \frac{\Gamma(\frac{n}{2}+j)}{\Gamma(\frac{n}{2}+N)} \binom{N-1}{L},
$$

and

$$
\begin{aligned}
B = \frac{\Gamma(\frac{n}{2}+j)}{\Gamma(\frac{n}{2}+N)} \frac{(N-L-1)!(L-k)!}{(N-k)!} \binom{\frac{n}{2}+j+L-1}{L-k} \binom{\frac{n}{2}+N-1}{N-j} \\
\times \binom{\frac{n}{2}+N-1}{N-j} (N-j)! \, \frac{\Gamma(\frac{n}{2}+j)}{\Gamma(\frac{n}{2}+N)} \binom{N-1}{L}.
\end{aligned}
$$

Finally, after appropriate simplifications,

$$
\begin{aligned}
A &= \frac{(-1)^j}{\Gamma(\frac{n}{2}+j+L)} \, B \\
&= (-1)^j \frac{(N-L-1)!}{(N-k)!} \frac{\binom{N-1}{L}}{\Gamma(\frac{n}{2}+N) k! \binom{\frac{n}{2}+j+k-1}{k}},
\end{aligned}
$$

which completes the proof of the lemma. □

8 Examples and questions

Example 1. Consider the polynomial operator $T = T_\lambda$ in $\mathbb{R}^2$ determined by

$$
\Omega(x,y) = \frac{xy}{|z|^2} + \lambda \frac{x^3y - xy^3}{|z|^4},
$$

where $z = x + iy$. We claim that the inequality $\|T_\lambda^\star f\|_2 \le C\|T_\lambda f\|_2$, $f \in L^2(\mathbb{R}^2)$, holds if and only if $|\lambda| < 2$. This follows from the Theorem because the multiplier of the operator T_λ is

$$
\frac{\xi\eta}{\xi^2+\eta^2} \left(-\pi + \lambda \frac{\pi}{2} \frac{(\xi^2-\eta^2)}{\xi^2+\eta^2} \right),
$$

$\frac{\xi\eta}{\xi^2+\eta^2}$ is the multiplier of a second order Riesz Transform and the function $-\pi + \lambda\frac{\pi}{2}\frac{(\xi^2-\eta^2)}{\xi^2+\eta^2}$ has no zeroes on the unit circle if and only if $|\lambda| < 2$.

Example 2. Let B stand for the Beurling transform and consider the operator

$$T_\lambda = B - \lambda B^2, \quad \lambda \in \mathbb{C},$$

which corresponds to

$$\Omega(z) = -\frac{1}{\pi}\left(\frac{\overline{z}^2|z|^2 + 2\lambda\overline{z}^4}{|z|^4}\right).$$

In this case the multiplier is

$$\frac{\overline{\xi}^3\xi - \lambda\overline{\xi}^4}{|\xi|^4} = \frac{\overline{\xi}}{\xi}\left(1 - \lambda\frac{\overline{\xi}}{\xi}\right), \quad \xi \in \mathbb{C},$$

which is the multiplier of B times a function that vanishes on the unit circle if and only if $|\lambda| = 1$. Then the Theorem tells us that $\|T_\lambda^\star f\|_2 \le C\|T_\lambda f\|_2$, $f \in L^2(\mathbb{R}^2)$, if and only if $|\lambda| \ne 1$. Indeed, strictly speaking, the necessary condition of the Theorem, i.e. (iii) implies (i), applies only to *real* homogeneous polynomials. However, in the case at hand the factorization results one needs can be checked by direct inspection.

Example 3. We give an example of a polynomial operator T which is of the form $T = R \circ U$, where R is an even higher order Riesz Transform and U is invertible, but the L^2 inequality $\|T^\star f\|_2 \le C\|Tf\|_2$, $f \in L^2(\mathbb{R}^n)$, does not hold. The operator T is associated with the homogenous polynomial of degree 8

$$P(x,y) = \frac{1}{\gamma_2} P_2(x,y)(x^2+y^2)^3 + \varepsilon\left(\frac{1}{\gamma_4} P_4(x,y)(x^2+y^2)^2 - \frac{1}{\gamma_8} P_8(x,y)\right),$$

where

$P_2(x,y) = xy$,
$P_4(x,y) = x^4 - 6x^2y^2 + y^4$,

and

$P_8(x,y) = x^8 + y^8 - 28x^6y^2 - 28x^2y^6 + 70x^4y^4$

are harmonic polynomials and ε is small enough. Notice that P_2 does not divide P_4 nor P_8. Therefore, by the Theorem, there is no control of $T^\star$ in terms of T. On the other hand, P_4 and P_8 have been chosen so that $P_4(\xi_1,\xi_2)|\xi|^4 - P_8(\xi_1,\xi_2)$ is divisible by P_2, and so the multiplier of T is

$$\frac{P_2(\xi_1,\xi_2)}{|\xi|^2} + \varepsilon\left(\frac{P_4(\xi_1,\xi_2)}{|\xi|^4} - \frac{P_8(\xi_1,\xi_2)}{|\xi|^8}\right) = \frac{P_2(\xi_1,\xi_2)}{|\xi|^2}\left(1 + \varepsilon\frac{Q(\xi_1,\xi_2)}{|\xi|^6}\right),$$

where Q is a homogeneous polynomial of degree 6. Define R as the higher order Riesz transform whose multiplier is $\frac{P_2(\xi_1,\xi_2)}{|\xi|^2}$ and U as the operator whose multiplier is $1 + \varepsilon\frac{Q(\xi_1,\xi_2)}{|\xi|^6}$. If ε is small enough, then U is invertible.

Example 4. We show here that condition (iii) in the Theorem is very restrictive. Take $n = 2$. Since $\Omega(e^{i\theta})$ is an even function it has, modulo a rotation, a Fourier series expansion of the type

$$\Omega(e^{i\theta}) = \sum_{j\geq 1} a_j \sin(2j\theta)\,. \tag{80}$$

Thus the harmonic polynomials P_{2j} are

$$P_{2j}(z) = a_j \frac{z^{2j} - \overline{z}^{2j}}{2i}\,.$$

If $a_j \neq 0$, then P_{2j} vanishes exactly on $2j$ straight lines through the origin uniformly distributed and containing the two axis. Each such line is determined by a pair of opposed $4j$-th roots of the unity. Assume now that the operator determined by $\Omega(e^{i\theta})$ satisfies condition *(iii)* of the Theorem. Let j_0 be the first positive integer with $a_{j_0} \neq 0$. Then only the a_j with j a multiple of j_0 may be non-zero, owing to the particular structure of the zero set of P_{2j}. In other words, the first non-zero P_{2j} determines all the others, modulo the constants a_j.

Example 5. We show now a method to construct even kernels in $\mathbb{R}^3$ that satisfy condition *(iii)* in the Theorem. It can be easily adapted to any dimension. The kernel is determined by the function

$$\Omega(x,y,z) = xy \sum_{j\geq 0} \epsilon_j\, Q_{2j}(x,y,z)$$

where the sequence (ϵ_j) is chosen so that Ω is in $C^\infty(S^2)$ and the Q_{2j} are defined by

$$Q_{2j}(x,y,z) = \sum_{k=0}^{2j} c_k\, y^{2k}\, z^{2j-2k}\,.$$

The c_k are determined by a recurrent formula obtained by requiring that $xy\, Q_{2j}(x,y,z)$ is harmonic. Computing its Laplacean we get the recurrent condition

$$c_k = -c_{k-1} \frac{2k(2k+1)}{(2j-2k+1)\,(2j-2k+2)}, \quad 1 \leq k \leq j\,,$$

where c_0 may be freely chosen.

We would like to close the paper by asking a couple of questions which we have not been able to answer.

Question 1. Since our methods are very much dependent on the Fourier transform we do not know whether either the weak type inequality

$$\|T^\star f\|_{1,\infty} \leq C \|Tf\|_1\,,$$

or the L^p inequality with $1 < p < \infty$, $p \neq 2$,

$$\|T^\star f\|_p \leq C\|Tf\|_p, \quad f \in L^p(\mathbb{R}^n)\,,$$

imply the L^2 inequality

$$\|T^\star f\|_2 \leq C\|Tf\|_2, \quad f \in L^2(\mathbb{R}^n)\,.$$

As suggested by Carlos Pérez, this might be related to interpolation results for couples of sub-linear operators.

Question 2. How far may the smoothness assumption on Ω be weakened ? More concretely, does the Theorem still hold true for Ω of class $C^m(S^{n-1})$ for some positive integer m ?

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