The maximal singular integral: estimates in terms of the singular integral

Joan Verdera

Abstract

This paper considers estimates of the maximal singular integral T^*f in terms of the singular integral Tf only. The most basic instance of the estimates we look for is the $L^2(\mathbb{R}^n)$ inequality $\|T^*f\|_2 \leq C\,\|Tf\|_2$. We present the complete characterization, recently obtained by Mateu, Orobitg, Pérez and the author, of the smooth homogeneous convolution Calderón–Zygmund operators for which such inequality holds. We focus attention on special cases of the general statement to convey the main ideas of the proofs in a transparent way, as free as possible of the technical complications inherent to the general case. Particular attention is devoted to higher Riesz transforms.

1 Introduction

In this expository paper we consider the problem of estimating the Maximal Singular Integral T^*f only in terms of the Singular Integral Tf. In other words, the function f should appear in the estimates only through Tf. The context is that of classical Calderón–Zygmund theory: we deal with smooth homogeneous convolution singular integral operators of the type

$$Tf(x) = p.v.\int f(x-y)\,K(y)\,dy \equiv \lim_{\epsilon\to 0} T^\epsilon f(x)\,, \tag{1}$$

where

$$T^\epsilon f(x) = \int_{|y-x|>\epsilon} f(x-y)K(y)\,dy$$

is the truncated integral at level ϵ. The kernel K is

$$K(x) = \frac{\Omega(x)}{|x|^n}, \quad x \in \mathbb{R}^n \setminus \{0\}\,, \tag{2}$$

where Ω is a (real valued) homogeneous function of degree 0 whose restriction to the unit sphere S^{n-1} is of class $C^\infty(S^{n-1})$ and satisfies the cancellation property

$$\int_{|x|=1} \Omega(x)\,d\sigma(x) = 0\,, \tag{3}$$

σ being the normalized surface measure on S^{n-1}. The maximal singular integral is

$$T^\star f(x) = \sup_{\epsilon>0} |T^\epsilon f(x)|, \quad x \in \mathbb{R}^n .$$

As we said before, the problem we are envisaging consists in estimating $T^\star f$ in terms of Tf only. The well known Cotlar's inequality

$$T^\star f(x) \leqslant C\,(M(Tf)(x) + Mf(x))\,, \quad x \in \mathbb{R}^n\,, \tag{4}$$

is of no use because it contains the term f besides Tf. The most basic form of the estimate we are looking for is the L^2 inequality

$$\|T^\star f\|_2 \leqslant C\|Tf\|_2, \quad f \in L^2(\mathbb{R}^n)\,. \tag{5}$$

This problem arose when the author was working at the David–Semmes problem ([2, p.139, first paragraph]). It was soon discovered ([7]) that the parity of the kernel plays an essential role. Some years after, a complete characterization of the even operators for which (5) holds was presented in [5] and afterwards the case of odd kernels was solved in [6]. Unfortunately there does not seem to be a way of adapting the techniques of those papers to the Ahlfors regular context in which the David–Semmes problem was formulated.

The proof of the main result in [5] and [6] is long and technically involved. It is the purpose of this paper to describe the main steps of the argument in the most transparent way possible. We give complete proofs of particular instances of the main results of the papers mentioned, so that the reader may grasp, in a simple situation, the idea behind the proof of the general cases. Thus, in a sense, the present paper could serve as an introduction to [5] and [6].

Notice that (5) is true whenever T is a continuous isomorphism of $L^2(\mathbb{R}^n)$ onto itself. Indeed a classical estimate, which follows from Cotlar's inequality, states that

$$\|T^\star f\|_2 \leqslant C\,\|f\|_2, \quad f \in L^2(\mathbb{R}^n)\,, \tag{6}$$

which combined with the assumption that T is an isomorphism gives (5). Thus (5) is true for the Hilbert Transform and for the Beurling Transform. The first non-trivial case is a scalar Riesz transform in dimension 2 or higher. Recall that the j-th Riesz transform is the Calderón–Zygmund operator with kernel

$$\frac{x_j}{|x|^{n+1}}, \quad x \in \mathbb{R}^n \setminus \{0\}, \quad 1 \leqslant j \leqslant n.$$

The first non trivial case for even operators is any second order Riesz transform. For example, the second order Riesz transform with kernel

$$\frac{x_1 x_2}{|x|^{n+2}}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

In Section 2 we prove the L^2 estimate (5) for the second order Riesz transform above and in Section 4 for the $j-th$ Riesz transform. Indeed, in both cases we prove a stronger pointwise estimate which works for all higher Riesz transforms. Recall that a higher Riesz transform is a smooth homogeneous convolution singular integral operator with kernel of the type

$$\frac{P(x)}{|x|^{n+d}}, \quad x \in \mathbb{R}^n \setminus \{0\},$$

where P is a harmonic homogeneous polynomial of degree $d \geqslant 1$. The mean value property of harmonic functions combined with homogeneity yields the cancellation property (3). One has the following ([5])

Theorem 1. *If T is an even higher Riesz transform, then*

$$T^* f(x) \leqslant C\, M(Tf)(x), \quad x \in \mathbb{R}^n, \quad f \in L^2(\mathbb{R}^n), \tag{7}$$

where M is the maximal Hardy-Littlewood operator.

Indeed, for a second order Riesz transform S one has that the truncation at level ϵ is a mean of $S(f)$ on a ball. More precisely one has

$$S^\epsilon(f)(x) = \frac{1}{|B(x,\epsilon)|} \int_{B(x,\epsilon)} S(f)(y)\, dy \tag{8}$$

A weighted variant of the preceding identity works for a general even higher Riesz transform. Of course, (5) for even higher Riesz transforms follows immediately from (7). It turns out that, as we explain in Section 3, (7) does not hold for odd Riesz transforms, not even for the Hilbert transform. But we can prove the following substitute result ([6]), which obviously takes care of (5) for odd higher Riesz transforms.

Theorem 2. *If T is an odd higher Riesz transform, then*

$$T^* f(x) \leqslant C\, M^2(Tf)(x), \quad x \in \mathbb{R}^n, \quad f \in L^2(\mathbb{R}^n), \tag{9}$$

where $M^2 = M \circ M$ is the iteration of the maximal Hardy- Littlewood operator.

Without any harmonicity assumption the L^2 estimate (5) does not hold. The simplest example involves the Beurling transform B, which is the singular integral operator in the plane with complex valued kernel

$$-\frac{1}{\pi}\frac{1}{z^2} = -\frac{1}{\pi}\frac{\overline{z}^2}{|z|^4} = -\frac{1}{\pi}\frac{x^2-y^2}{|z|^4} + i\frac{1}{\pi}\frac{2xy}{|z|^4}.$$

The Fourier transform of the tempered distribution $p.v.(-\frac{1}{\pi}\frac{1}{z^2})$ is the function $\frac{\overline{\xi}}{\xi}$, so that B is an isometry of $L^2(\mathbb{R}^2)$ onto itself. It turns out that the singular integral

$$T = B + B^2 = B(I+B)$$

does not satisfy the L^2 control (5). The reason for that, as we will see later on in this Section, is that the operator $I + B$ is not invertible in $L^2(\mathbb{R}^2)$.

One way to explain the difference between the even and odd cases is as follows. Theorem 1 concerns an even higher Riesz transform determined by a harmonic homogeneous polynomial of degree, say, d. In its proof one is lead to consider the operator $(-\triangle)^{d/2}$, which is a differential operator. Instead, in Theorem 2, d is odd and thus $(-\triangle)^{d/2}$ is only a pseudo-differential operator. The effect of this is that in the odd case certain functions are not compactly supported and are not bounded. Nevertheless, they still satisfy a BMO condition, which is the key fact in obtaining the second iteration of the maximal operator.

The search for a description of those singular integrals T of a given parity for which (5) holds begun just after [7] was published. The final answer was given in [5] and [6]. To state the result denote by A the Calderón–Zygmund algebra consisting of the operators of the form $\lambda I + T$, where T is a smooth homogeneous convolution singular integral operator and λ a real number.

Theorem 3. *Let T be an even smooth homogeneous convolution singular integral operator with kernel $\Omega(x)/|x|^n$. Then the following are equivalent.*

(i)

$$T^* f(x) \leqslant C\, M(T f)(x), \quad x \in \mathbb{R}^n, \quad f \in L^2(\mathbb{R}^n),$$

where M is the Hardy-Littlewood maximal operator.

(ii)

$$\int |T^* f|^2 \leqslant C \int |T f|^2, \quad f \in L^2(\mathbb{R}^n).$$

(iii) If the spherical harmonics expansion of Ω is

$$\Omega(x) = P_2(x) + P_4(x) + \cdots, \quad |x| = 1,$$

then there exist an even harmonic homogeneous polynomial P of degree d, such that P divides P_{2j} (in the ring of all polynomials in n variables with real coeficients) for all j, $T = R_P \circ U$, where R_P is the higher Riesz transform with kernel $P(x)/|x|^{n+d}$, and U is an invertible operator in the Calderón–Zygmund algebra A.

Several remarks are in order. First, it is surprising that the L^2 control we are looking for, that is, condition (ii) above, is equivalent to the apparently much stronger pointwise inequality (i). We do not know any proof of this fact which does not go through the structural condition (iii). Second, condition (iii) on the spherical harmonics expansion of Ω is purely algebraic and easy to check in practice on the Fourier transform side. Observe that if condition (iii) is satisfied, then the polynomial P must be a scalar multiple of the first non-zero spherical harmonic P_{2j} in the expansion of Ω. We illustrate this with an example.

Example.

Let $P(x,y) = -\frac{1}{\pi}xy$ and denote by R_P the second order Riesz transform in the plane associated with the harmonic homogeneous polynomial P. Its kernel is

$$-\frac{1}{\pi}\frac{xy}{|z|^4}, \quad z = x + iy \in \mathbb{C} \setminus \{0\}. \tag{10}$$

According to a well known formula [9, p.73] the Fourier transform of the principal value distribution associated with this kernel is

$$\frac{uv}{|\xi|^2}, \quad \xi = u + iv \in \mathbb{C} \setminus \{0\}.$$

This is also the symbol (or Fourier multiplier) of R_P, in the sense that

$$\widehat{R_P(f)}(\xi) = \frac{uv}{|\xi|^2}\hat{f}(\xi), \quad \xi \neq 0, \quad f \in L^2(\mathbb{R}^n).$$

Similarly, the Fourier multiplier of the fourth order Riesz transform with kernel

$$\frac{2}{\pi}\frac{x^3y - xy^3}{|z|^6}, \quad z \neq 0,$$

is

$$\frac{u^3v - uv^3}{|\xi|^4}, \quad \xi \neq 0.$$

Given a real number λ let T be the singular integral with kernel

$$-\frac{1}{\pi}\frac{2xy}{|z|^4} + \lambda\,\frac{2}{\pi}\frac{x^3y - xy^3}{|z|^6}.$$

Its symbol is

$$\frac{uv}{|\xi|^2}\left(1 + \lambda\frac{u^2 - v^2}{|\xi|^2}\right).$$

We clearly have

$$T = R_P \circ U,$$

U being the bounded operator on $L^2(\mathbb{R}^n)$ with symbol $1 + \lambda\frac{u^2-v^2}{|\xi|^2}$. Notice that the multiplier $1 + \lambda\frac{u^2-v^2}{|\xi|^2}$ vanishes at some point of the unit sphere if and only if $|\lambda| \geqslant 1$. Therefore condition (iii) of Theorem 3 is satisfied if and only if $|\lambda| < 1$. For instance, taking $\lambda = 1$ one gets an operator for which neither the L^2 estimate (ii) nor the pointwise inequality (i) hold.

To grasp the subtlety of the division condition in (iii) it is instructive to consider the special case of the plane. The function Ω, which is real, has a Fourier series expansion

$$\Omega(e^{i\theta}) = \sum_{n=-\infty}^{\infty} c_n\, e^{in\theta} = \sum_{n=1}^{\infty} c_n\, e^{in\theta} + \overline{c_n}\, e^{-in\theta}$$

$$= \sum_{n=1}^{\infty} 2\,\mathrm{Re}(c_n\, e^{in\theta})$$

The expression $2\,\mathrm{Re}(c_n\, e^{i\,n\theta})$ is the general form of the restriction to the unit circle of a harmonic homogeneous polynomial of degree n on the plane. There are exactly $2n$ zeroes of $2\,\mathrm{Re}(c_n\, e^{i\,n\theta})$ on the circle, which are uniformly distributed. They are the $2n$-th roots of unity if and only if c_n is purely imaginary.

Since Ω is even, only the Fourier coefficients with even index may be non-zero and so

$$\Omega(e^{i\theta}) = \sum_{n=1}^{\infty} 2\,\mathrm{Re}(c_{2\,n}\, e^{i\,2\,n\theta}).$$

Replacing θ by $\theta + \alpha$ we obtain

$$\Omega(e^{i(\theta+\alpha)}) = \sum_{n=N}^{\infty} 2\,\mathrm{Re}(c_{2\,n}\, e^{i\,2\,n\alpha}\, e^{i\,2\,n\theta}),$$

where $c_{2\,N} \neq 0$. Take α so that $c_{2\,N}\, e^{i\,2\,N\,\alpha}$ is purely imaginary. Set $\gamma_{2\,n} = c_{2\,n}\, e^{i\,2\,n\,\alpha}$. Then

$$\Omega(e^{i(\theta+\alpha)}) = \sum_{n=N}^{\infty} 2\,\mathrm{Re}(\gamma_{2\,n}\, e^{i\,2\,n\theta}).$$

If $\mathrm{Re}(\gamma_{2\,N}\, e^{i\,2\,N\theta})$ divides $\mathrm{Re}(\gamma_{2\,n}\, e^{i\,2\,n\theta})$, then , for some positive integer k,

$$k\frac{\pi}{4\,n} = \frac{\pi}{4\,N},$$

or $n = k\,N$. This means that only the Fourier coefficients with index a multiple of $2\,N$ may be non-zero :

$$\Omega(e^{i(\theta+\alpha)}) = \sum_{p=1}^{\infty} 2\,\mathrm{Re}(\gamma_{2\,N\,p}\; e^{i\,2\,N\,p\,\theta}).$$

Moreover γ_{2Np} must be purely imaginary, that is, $\gamma_{2\,N\,p} = r_{2\,N\,p}\, i$, with $r_{2\,N\,p}$ real. Replacing $\theta + \alpha$ by θ we get

$$\begin{aligned}\Omega(e^{i\theta}) &= \sum_{p=1}^{\infty} 2\,\mathrm{Re}(r_{2Np}\, i\; e^{-i2Np\alpha}\; e^{i2Np\theta}),\\ &= \sum_{p=1}^{\infty} r_{2Np}\, i\; e^{-i2Np\alpha}\; e^{i2Np\theta} - r_{2Np}\, i\; e^{i2Np\alpha}\; e^{-i2Np\theta}.\end{aligned}$$

As it is well-known the sequence of the $r_{2Np}, p = 1, 2, \ldots$ is rapidly decreasing, because $\Omega(e^{i\theta})$ is infinitely differentiable. Therefore the division property in condition (iii) of Theorem 1 can be reformulated as a statement about the arguments and the support of the Fourier coefficients of $\Omega(e^{i\theta})$.

For odd operators the statement of Theorem 3 must be slightly modified ([6]).

Theorem 4. *Let T be an odd smooth homogeneous convolution singular integral operator with kernel $\Omega(x)/|x|^n$. Then the following are equivalent.*

(i)

$$T^* f(x) \leqslant C\, M^2(Tf)(x), \quad x \in \mathbb{R}^n, \quad f \in L^2(\mathbb{R}^n),$$

$M^2 = M \circ M$ being the iterated Hardy-Littlewood maximal operator.

(ii)

$$\int |T^* f|^2 \leqslant C \int |Tf|^2, \quad f \in L^2(\mathbb{R}^n).$$

(iii) If the spherical harmonics expansion of Ω is

$$\Omega(x) = P_1(x) + P_3(x) + \cdots, \quad |x| = 1,$$

then there exist an odd harmonic homogeneous polynomial P of degree d, such that P divides P_{2j+1} (in the ring of all polynomials in n variables with real coeficients) for all j, $T = R_P \circ U$, where R_P is the higher Riesz transform with kernel $P(x)/|x|^{n+d}$, and U is an invertible operator in the Calderón–Zygmund algebra A.

Sections 2 and 4 contain , respectively, the proofs of Theorems 1 and 2 for the most simple kernels. In Section 3 we show that the Hilbert transform does not satisfy the pointwise inequality (7). In Section 5 we prove that condition (iii) in Theorem 3 is necessary and in Section 6 that it is sufficient, in both cases in particularly simple situations. Section 7 contains brief comments on the proof of the general case and a mention of a couple of open problems.

2 Proof of Theorem 1 for second order Riesz transforms.

For se sake of clarity we work only with the second order Riesz transform T with kernel

$$\frac{x_1 x_2}{|x|^{n+2}}.$$

The inequality to be proven, namely (7), is invariant by translations and by dilations, so that we only need to show that

$$|T^1 f(0)| \leq C\, M(Tf)(0), \tag{11}$$

where

$$T^1 f(0) = \int_{\mathbb{R}^n \setminus B} \frac{x_1 x_2}{|x|^{n+2}} f(x)\, dx$$

is the truncation at level 1 at the origin. Here B is the unit (closed) ball centered at the origin. A natural way to show (11) is to find a function b such that

$$\chi_{\mathbb{R}^n \setminus B}(x)\, \frac{x_1 x_2}{|x|^{n+2}} = T(b).$$

One should keep in mind that T is injective but not onto. Then there is no reason whatsoever for such a b to exist. If such a b exists then

$$T^1 f(0) = \int Tb(x)\, f(x)\, dx = \int b(x)\, T(f)(x)\, dx \tag{12}$$

If moreover b is in $L^\infty(\mathbb{R}^n)$ and is supported on B, we get

$$|T_1 f(0)| \le \|b\|_\infty |B| \frac{1}{|B|} \int_B |T(f)(x)|\, dx \le CM(T(f))(0).$$

Thus everything has been reduced to the following lemma.

Lemma 5. *There exists a bounded measurable function b supported on B such that*

$$\chi_{\mathbb{R}^n \setminus B}(x) \frac{x_1 x_2}{|x|^{n+2}} = T(b)(x), \quad \textit{for almost all} \quad x \in \mathbb{R}^n.$$

Proof. Let E be the standard fundamental solution of the Laplacian in $\mathbb{R}^n$. Then, for some dimensional constant c_n, we have that, in the distributions sense,

$$\partial_1 \partial_2 E = c_n \; p.v. \frac{x_1 x_2}{|x|^{n+2}}. \tag{13}$$

Let us define a function φ by

$$\varphi(x) = \begin{cases} E(x) & \text{on } \mathbb{R}^n \setminus B \\ A_0 + A_1 |x|^2 & \text{on } B \end{cases} \tag{14}$$

where the constants A_0 and A_1 are chosen so that φ and $\nabla\varphi$ are continuous on $\mathbb{R}^n$. This is possible because, for each i,

$$\partial_i \varphi(x) = \begin{cases} c_n \frac{x_i}{|x|^n}, & x \in \mathbb{R}^n \setminus B \\ 2A_1 x_i, & x \in B \end{cases}$$

and so, for an appropriate choice of A_1, the above two expressions coincide on ∂B for all i, or, equivalently, $\nabla\varphi$ is continuous. The continuity of φ is now just a matter of choosing A_0 so that $E(x) = A_0 + A_1|x|^2$ on ∂B, which is possible because E is radial.

The continuity of φ and $\nabla\varphi$ guaranties that we can compute a second order derivative of φ in the distributions sense by just computing it pointwise on B and on $\mathbb{R}^n \setminus B$. The reason is that no boundary terms will appear when applying Green-Stokes to compute the action of the second order derivative of φ under consideration on a test function. Therefore

$$\Delta\varphi = 2nA_1 \chi_B \equiv b,$$

where the last identity is the definition of b. Since $\varphi = E * \Delta\varphi$ we obtain, for some dimensional constant c_n,

$$\partial_1 \partial_2 \varphi = \partial_1 \partial_2 E * \Delta\varphi = c_n \; p.v. \frac{x_1 x_2}{|x|^{n+2}} * \Delta\varphi = c_n \, T(b).$$

On the other hand, by (14) and noticing that $\partial_1\partial_2|x|^2 = 0$, we get

$$\partial_1\partial_2\varphi = \chi_{\mathbb{R}^n\setminus B}(x)c_n\,\frac{x_1x_2}{|x|^{n+2}},$$

and the proof of Lemma 5 is complete. □

Notice that (12) together with the special form of the function b found in the proof of Lemma 5 yield the formula (8), namely, that a truncation at level ϵ at the point x of $S(f)$, S being a second order Riesz transform, is the mean of $S(f)$ on the ball $B(x,\epsilon)$.

3 The pointwise control of T^* by $M\circ T$ fails for the Hilbert transform

We show now that the inequality

$$H^*f(x) \le C\,M(Hf)(x)\,, \quad x\in\mathbb{R} \quad f\in L^2(\mathbb{R}), \tag{15}$$

where H is the Hilbert transform, fails. Replacing f by $H(f)$ in (15) and recalling that $H(Hf) = -f$, $f\in L^2(\mathbb{R})$, we see that (15) is equivalent to

$$H^*(H(f))(x) \le C\,M(f)(x)\,, \quad x\in\mathbb{R}, \quad f\in L^2(\mathbb{R}).$$

It turns out that the operator $H^*\circ H$ is not of weak type $(1,1)$.

Let us prove that if $f = \chi_{(0,1)}$, then there are positive constants m and C such that whenever $x > m$,

$$H^*(Hf)(x) \ge C\,\frac{\log x}{x} \tag{16}$$

This shows that $H^*\circ H$ is not of weak type $(1,1)$. Indeed, choosing $m > e$ if necessary, we have

$$\sup_{\lambda>0}\lambda\,|\{x\in\mathbb{R} : H^*(Hf)(x) > \lambda\}| \ge \sup_{\lambda>0}\lambda\,|\{x > m : \frac{\log x}{x} > C^{-1}\,\lambda\}|$$

$$= C\,\sup_{\lambda>0}\lambda\,|\{x > m : \frac{\log x}{x} > \lambda\}| \ge C\,\sup_{\lambda>0}\lambda\,(\varphi^{-1}(\lambda) - e),$$

where φ is the decreasing function $\varphi : (e,\infty) \to (0,e^{-1})$, given by $\varphi(x) = \frac{\log x}{x}$. To conclude observe that the right hand side of the estimate is unbounded as $\lambda\to 0$:

$$\lim_{\lambda\to 0}\lambda\varphi^{-1}(\lambda) = \lim_{\lambda\to\infty}\varphi(\lambda)\lambda = \infty.$$

To prove (16) we recall that for $f = \chi_{(0,1)}$

$$Hf(y) = \log\frac{|y|}{|y-1|}.$$

Let $m > 1$ big enough to be chosen later on. Take $x > m$. By definition of H^*

$$H^*(Hf)(x) \geq \left| \int_{|y-x|>m+x} \frac{1}{y-x} \; \log \frac{|y|}{|y-1|} \, dy \right|$$

and splitting the integral in the obvious way

$$\int_{-\infty}^{-m} \frac{1}{y-x} \log \frac{-y}{-y+1} \, dy + \int_{2x+m}^{\infty} \frac{1}{y-x} \log \frac{y}{y-1} \, dy$$

$$= \int_{m}^{\infty} \frac{1}{x+y} \log \frac{y+1}{y} \, dy + \int_{2x+m}^{\infty} \frac{1}{y-x} \log \frac{y}{y-1} \, dy = A(x) + B(x),$$

where both $A(x)$ and $B(x)$ are positive. Hence

$$H^*(Hf)(x) \geq A(x).$$

Since

$$\log(1 + \frac{1}{y}) \approx \frac{1}{y}, \qquad \text{as} \quad y \to \infty,$$

there is a constant $m > 1$ such that whenever $y > m$

$$\frac{1}{2} < \frac{\log(1 + \frac{1}{y})}{\frac{1}{y}} < \frac{3}{2}.$$

Hence, for this constant m we have

$$A(x) = \int_{m}^{\infty} \frac{1}{x+y} \log\left(1 + \frac{1}{y}\right) dy \approx \int_{m}^{\infty} \frac{1}{x+y} \frac{dy}{y} = \frac{1}{x} \log \frac{y}{x+y}\Big|_{m}^{\infty} \approx \frac{\log x}{x},$$

which proves (16).

Notice that the term $B(x)$ is better behaved :

$$B(x) \leq \int_{2x+m}^{\infty} \frac{1}{y-x} \log \frac{y}{y-1} \, dy \leq \int_{2x+m}^{\infty} \frac{2}{y} \frac{dy}{y} \leq \frac{1}{x}.$$

4 Proof of Theorem 2 for first order Riesz transforms

In this Section we prove that

$$R_j^*(f)(x) \leq C\, M^2(R_j(f)), \quad x \in \mathbb{R}^n, \tag{17}$$

where R_j is the j-th Riesz transform, namely, the Calderón–Zygmund operator with kernel

$$\frac{x_j}{|x|^{n+1}}, \quad x \in \mathbb{R}^n \setminus \{0\}, \quad 1 \leqslant j \leqslant n.$$

Recall that $M^2 = M \circ M$ and notice that for $n = 1$ we are dealing with the Hilbert transform. The inequality (17) for the Hilbert transform is, as far as we know, new. To have a glimpse at the difficulties we will encounter in proving (17) we start by discussing the case of the Hilbert transform.

As in the even case we want to find a function b such that

$$\frac{1}{x}\chi_{\mathbb{R}\setminus(-1,1)}(x) = H(b).$$

Since $H(-H) = I$

$$\begin{aligned} b(x) &= -H(\frac{1}{y}\chi_{\mathbb{R}\setminus(-1,1)}(y))(x) \\ &= \frac{1}{\pi}\int_{|y|>1} \frac{1}{y-x}\frac{1}{y}\,dy \\ &= \frac{1}{\pi x}\,\log\frac{|1+x|}{|1-x|}\,. \end{aligned}$$

We conclude that, unlike in the even case, the function b is unbounded and is not supported in the unit interval $(-1, 1)$. On the positive side, we see that b is a function in $BMO = BMO(\mathbb{R})$, the space of functions of bounded mean oscillation on te line. Since b decays at infinity as $1/x^2$, b is integrable on the whole line. However, the minimal decreasing majorant of the absolute value of b is not integrable, owing to the poles at ± 1. This prevents a pointwise estimate of $H^* f$ by a constant times $M(Hf)$. We can now proceed with the proof of (17) keeping in mind the kind of difficulties we will have to overcome.

We start with the analog of Lemma 5. We denote by BMO the space of functions of bounded mean oscillation on $\mathbb{R}^n$.

Lemma 6. *There exists a function* $b \in$ BMO *such that*

$$\chi_{\mathbb{R}^n\setminus B}(x)\,\frac{x_j}{|x|^{n+1}} = R_j(b)(x), \quad \textit{for almost all} \quad x \in \mathbb{R}^n, \quad 1 \leqslant j \leqslant n\,. \tag{18}$$

Proof. For an appropriate constant c_n the function

$$E(x) = c_n\,\frac{1}{|x|^{n-1}}, \quad 0 \neq x \in \mathbb{R}^n$$

satisfies

$$\widehat{E}(\xi) = \frac{1}{|\xi|}, \quad 0 \neq \xi \in \mathbb{R}^n.$$

Since the pseudo-differential operator $(-\Delta)^{1/2}$ is defined on the Fourier transform side as

$$\widehat{(-\Delta)^{1/2}\psi}(\xi) = |\xi|\hat{\psi}(\xi),$$

E may be understood as a fundamental solution of $(-\Delta)^{1/2}$. This will allow to structure our proof in complete analogy to that of Lemma 5 until new facts emerge. Consider the function φ that takes the value c_n on B and $E(x)$ on $\mathbb{R}^n \setminus B$. We have that $\varphi = E * (-\Delta)^{1/2}\varphi$ and we define b as $(-\Delta)^{1/2}\varphi$.

As it is well known

$$\partial_j E = -(n-1)c_n \, p.v. \frac{x_j}{|x|^{n+1}} \,,$$

in the distributions sense and, since φ is continuous on the boundary of B,

$$\partial_j \varphi = -(n-1)c_n \chi_{\mathbb{R}^n \setminus B}(x) \frac{x_j}{|x|^{n+1}} \tag{19}$$

also in the distributions sense. Then

$$\begin{aligned} -(n-1)c_n \chi_{\mathbb{R}^n \setminus B}(x) \frac{x_j}{|x|^{n+1}} &= \partial_j \varphi \\ &= \partial_j E * b \\ &= -(n-1)c_n \, p.v. \frac{x_j}{|x|^{n+1}} * b, \end{aligned}$$

which is (18). It remains to show that $b \in BMO$.

Checking on the Fourier transform side we easily see that

$$b = (-\Delta)^{1/2}\varphi = \gamma_n \sum_{k=1}^{n} R_k(\partial_k \varphi), \tag{20}$$

for some dimensional constant γ_n. Since $\partial_k \varphi$ is a bounded function by (19) and R_k maps L^∞ into BMO, b is in BMO and the proof is complete. □

Unfortunately b is not bounded and is not supported on $\mathbb{R}^n \setminus B$. Moreover one can check easily that b blows up at the boundary of B as the function $\log(1/|1-|x||)$. This entails that the the minimal decreasing majorant of the absolute value of b is not integrable, as in the one dimensional case.

We take up now the proof of (17). By translation and dilation invariance we only have to estimate the truncation of $R_j f$ at the point $x = 0$ and at level $\epsilon = 1$. By Lemma 6

$$\begin{aligned} R_j^1 f(0) &= -\int \chi_{\mathbb{R}^n \setminus B}(x) \, \frac{x_j}{|x|^{n+1}} \, f(x) \, dx = -\int R_j b(x) \, f(x) \, dx \\ &= \int b(x) \, R_j f(x) \, dx \,. \end{aligned}$$

Let b_{2B} denote the mean of b on the ball $2B$. We split the last integral above into three pieces

$$\begin{aligned} R_j^1 f(0) &= \int_{2B} (b(x) - b_{2B}) \, R_j f(x) \, dx + b_{2B} \int_{2B} R_j f(x) \, dx + \int_{\mathbb{R}^n \setminus 2B} b(x) \, R_j f(x) \, dx \\ &= I_1 + I_2 + I_3 \,. \end{aligned} \tag{21}$$

Since b_{2B} is a dimensional constant the term I_2 can be immediately estimated by $C\,M(R_j f)(0)$. The term I_3 can easily be estimated if we first prove that

$$|b(x)| \leqslant C \frac{1}{|x|^{n+1}}, \quad |x| \geqslant 2\,. \tag{22}$$

Indeed, the preceding decay inequality yields

$$|I_3| \leqslant C \int_{\mathbb{R}^n \setminus 2B} |R_j f(x)| \frac{1}{|x|^{n+1}}\, dx \leqslant C\, M(R_j f)(0)\,.$$

To prove (22) express b by means of (20)

$$\begin{aligned} \frac{b}{\gamma_n} = \sum_{k=1}^{n} R_k \star \chi_{\mathbb{R}^n \setminus B}(x)\, \frac{x_k}{|x|^{n+1}} &= \sum_{k=1}^{n} R_k \star R_k - \sum_{k=1}^{n} R_k \star \chi_B(x) \frac{x_k}{|x|^{n+1}} \\ &= \gamma_n'\, \delta_0 - \sum_{k=1}^{n} R_k(\chi_B(x) \frac{x_k}{|x|^{n+1}})\,, \end{aligned}$$

where γ_n' is a dimensional constant and δ_0 the dirac delta at the origin. The preceding formula for b looks magical and one may even think that some terms make no sense. For instance, the term $R_k \star R_k$ should not be thought as the action of the k-th Riesz transform of the distribution $p.v.\, x_k/|x|^{n+1}$. It is more convenient to look at it on the Fourier transform side, where you see immediately that it is $\gamma_n'\, \delta_0$. The term $R_k \star \chi_B(x) \frac{x_k}{|x|^{n+1}}$ should be thought as a distribution, which acts on a test function as one would expect via principal values (see below).

If $|x| > 1$ we have

$$\begin{aligned} R_k(\chi_B(x) \frac{x_k}{|x|^{n+1}})(x) &= \lim_{\epsilon \to 0} \int_{\epsilon < |y| < 1} \frac{x_k - y_k}{|x-y|^{n+1}} \frac{y_k}{|y|^{n+1}}\, dy \\ &= \lim_{\epsilon \to 0} \int_{\epsilon < |y| < 1} \left(\frac{x_k - y_k}{|x-y|^{n+1}} - \frac{x_k}{|x|^{n+1}} \right) \frac{y_k}{|y|^{n+1}}\, dy\,. \end{aligned}$$

Since

$$\left|\frac{x_k - y_k}{|x-y|^{n+1}} - \frac{x_k}{|x|^{n+1}}\right| \leqslant C \frac{|y|}{|x|^{n+1}}, \quad |x| \geqslant 2, \quad |y| \leqslant 1\,,$$

we obtain, for $|x| \geqslant 2$,

$$|R_k(\chi_B(x) \frac{x_k}{|x|^{n+1}})(x)| \leqslant C \int_{|y|<1} \frac{1}{|x|^{n+1}} \frac{1}{|y|^{n-1}}\, dy = \frac{C}{|x|^{n+1}}\,,$$

which gives (22).

We are left with the term I_1. Since b is in BMO it is exponentially integrable by John-Nirenberg's Theorem. We estimate I_1 by Holder's inequality associated with the "dual" Young functions $e^t - 1$ and $t + t\log^+ t$ ([4, p. 165]). We get

$$|I_1| \leqslant C\, \|b\|_{BMO} \|R_j f\|_{L \log L(2B)},$$

where, for an integrable function g on $2B$,

$$\|g\|_{L\log L(2B)} = \inf\{\lambda > 0 \;:\; \frac{1}{|2B|}\int_{2B}\left(\frac{|g(x)|}{\lambda} + \frac{|g(x)|}{\lambda}\log^+(\frac{|g(x)|}{\lambda})\right)dx \leqslant 1\}.$$

It is a nice fact (see [8] or [4, p.159]) that the maximal operator associated with $L\log L$, that is,

$$M_{L(\log L)}g(x) = \sup_{Q\ni x}\|f\|_{L(\log L),Q},$$

the supremum being over all balls Q, satisfies

$$M_{L(\log L)}f(x) \approx M^2 f(x), \quad x\in\mathbb{R}^n. \tag{23}$$

Thus

$$|I_1| \leq C\, M^2(R_j f)(0)$$

and the proof of (17) is complete.

5 Necessary conditions for the L^2 estimate of T^*f by Tf

In this Section we find the necessary conditions for the L^2 estimate

$$\|T^\star f\|_2 \leqslant C\|Tf\|_2, \quad f\in L^2(\mathbb{R}^n) \tag{24}$$

which are stated in (iii) of Theorem 3 for the case of even kernels. In particular, this will supply many even kernels for which the preceding estimate fails (and thus the pointwise estimate in (i) of Theorem 3 fails).

We will look at the simplest possible situation. The kernel of our operator T in the plane is of the form

$$K(z) = -\frac{1}{\pi}\frac{xy}{|z|^4} + \frac{2}{\pi}\frac{P_4(z)}{|z|^6}, \tag{25}$$

where $z = x+iy$ is the complex variable in the plane $\mathbb{C}$ and P_4 is a harmonic homogeneous polynomial of degree 4. The constants in front of the two terms are set so that the expression of the Fourier multiplier is the simplest. Indeed, the Fourier transform of the principal value tempered distribution associated with K is

$$\widehat{p.v.\,K}(\xi) = \frac{uv}{|\xi|^2} + \frac{P_4(\xi)}{|\xi|^4}, \quad 0\neq\xi = u+iv\in\mathbb{C},$$

by [9, p.73]. Our purpose is to find necessary conditions on P_4 so that the L^2 estimate (24) holds. Notice that the kernel K is not harmonic, except in the case $P_4 = 0$ which we ignore. The spherical harmonics expansion of K is reduced to the sum of the two terms in (25).

Let E be the standard fundamental solution of the bilaplacian Δ^2 in the plane. Thus

$$E(z) = \frac{1}{8\pi}|z|^2\log|z|, \qquad 0\neq z\in\mathbb{C},$$

and $\hat{E}(\xi) = |\xi|^{-4}, \quad 0 \neq \xi \in \mathbb{C}$. We have

$$(\partial_1\partial_2\Delta + P_4(\partial_1,\partial_2))\,(E) = p.v.K$$

as one easily checks on the Fourier transform side. Here we adopt the usual convention of denoting by $P_4(\partial_1,\partial_2)$ th differential operator obtained by replacing the variables x and y of P_4 by ∂_1 and ∂_2 respectively.

Define a function φ by

$$\varphi(z) = \begin{cases} E(z) & \text{on } \mathbb{C}\setminus B \\ A_0 + A_1|z|^2 + A_2|z|^4 + A_3|z|^6 & \text{on } B \end{cases}$$

where B is the ball centered at the origin of radius 1. The constants $A_j, 0 \leq j \leq 3$, are chosen so that all derivatives of φ of order not greater than 3 are continuous. This can be done because E is radial. With this choice to compute a fourth order derivative of φ in the distributions sense we only need to compute the corresponding pointwise derivative of φ in B and on its complement. Set $b = \Delta^2\varphi$, so that

$$\varphi = E * \Delta^2\varphi = E * b.$$

A straightforward computation yields

$$b = \Delta^2\varphi = \chi_B(z)(\alpha + \beta|z|^2),$$

for some constants α and β. Then, as in the proof of the L^2 estimate (24) for even second order Riesz transforms presented in Section 2, b is supported on the ball B and is bounded. Set

$$L = \partial_1\partial_2\Delta + P_4(\partial_1,\partial_2),$$

so that

$$L(\varphi) = L(E) * b = p.v.K * b = T(b).$$

On the other hand, by the definition of φ,

$$L(\varphi) = \chi_{\mathbb{C}\setminus B}(z)K(z) + L(A_0 + A_1|z|^2 + A_2|z|^4 + A_3|z|^6)\chi_B(z)\,.$$

Now the term $L(A_0 + A_1|z|^2 + A_2|z|^4 + A_3|z|^6)$ does not vanish. Indeed, one can see that for some constant c

$$L(A_0 + A_1|z|^2 + A_2|z|^4 + A_3|z|^6) = c\,xy\,.$$

The result follows from the following three facts :

$$(\partial_1\partial_2\Delta)\,(|z|^4) = 0,$$

$$(\partial_1\partial_2\Delta)\,(|z|^6) = c\,xy$$

and

$$P_4(|z|^4) = P_4(|z|^6) = 0.$$

The last identity is due to the fact that P_4 is a homogeneous harmonic polynomial of degree 4. Notice that a priori $P_4(|z|^4)$ is a constant and $P_4(|z|^6)$ is a homogeneous polynomial of degree 2. The reader can verify that they are both zero just by taking the Fourier transform and then checking their action on a test function.

The conclusion is that

$$T(b) = \chi_{\mathbb{C}\setminus B}(z)K(z) + cxy\chi_B(z). \tag{26}$$

The novelty with respect to the argument of Section 2 involving second order Riesz transforms is the second term in the right hand side of the preceding formula. Convolving (26) with a function f in $L^2(\mathbb{C})$ one gets

$$cxy\chi_B(z) * f = T(f) * b - T^1(f),$$

where $T^1 f$ is the truncation at level 1. Now, if (24) holds then, since $b \in L^1(\mathbb{C})$,

$$\|cxy\chi_B(z) * f\|_2 \leq C\,\|T(f)\|_2, \quad f \in L^2(\mathbb{C}),$$

and hence, passing to the multipliers,

$$|\widehat{cxy\chi_B}(z)(\xi)| \leq C\,\frac{|uv|\xi|^2 + P_4(\xi)|}{|\xi|^4}, \quad \xi \neq 0. \tag{27}$$

Our next task is to understand the left hand side of the above inequality to obtain useful relations between the zero sets of the various polynomials at hand. We should recall that the Fourier transform of the characteristic function of the unit ball in $\mathbb{R}^2$ is $J_1(\xi)/|\xi|$, where $J_1(\xi)$ is the Bessel function of order 1. Write $G_m(\xi) = J_m(\xi)/|\xi|^m$. The functions G_m are radial and so we can view them as depending on a non-negative real variable r. We have [3, p.425] the useful identity

$$\frac{1}{r}\frac{dG_m}{dr}(r) = -G_{m+1}(r), \quad 0 \leq r.$$

From this it is easy to obtain the formula

$$\begin{aligned}\widehat{xy\chi_B(z)}(\xi) &= -\partial_1\partial_2(G_1(|\xi|))\\ &= -uv\,G_3(|\xi|),\end{aligned}$$

which transforms (27) into

$$|uvG_3(|\xi|)| \leq C\,\frac{|uv|\xi|^2 + P_4(\xi)|}{|\xi|^4}, \quad \xi \neq 0. \tag{28}$$

Set

$$Q(\xi) = uv|\xi|^2 + P_4(\xi), \quad \xi \in \mathbb{C}.$$

Then (28) becomes, on the unit circle,

$$|uv| \leq C\,|Q(\xi)|, \quad |\xi| = 1. \tag{29}$$

The above inequality encodes valuable information on the zero set of P_4. Recall that our goal is to show that uv divides P_4.

Observe that Q is a real polynomial with zero integral on the unit circle, as sum of two non- constant homogeneous harmonic polynomials. Thus Q vanishes at some point $\xi = u + iv$ on the unit circle. Then $uv = 0$ by (29) and so $P_4(\xi) = 0$, owing to the definition of Q. We need now a precise expression for P_4. The general harmonic homogeneous polynomial of degree 4 is

$$\operatorname{Re}(\lambda\xi^4) = \alpha(u^3v - v^3u) + \beta(u^4 + v^4 - 6u^2v^2), \tag{30}$$

where λ is a complex number and α and β are real. Assume that P_4 is as above. We know that $u^2 + v^2 = 1$, $P_4(u, v) = 0$ and that $uv = 0$. If $u = 0$, then $\beta v^4 = 0$, which yields $\beta = 0$. If $v = 0$, then $\beta u^4 = 0$ and we conclude again that $\beta = 0$. Therefore

$$P_4(u, v) = \alpha(u^3v - v^3u) \tag{31}$$

and uv divides $P_4(u, v)$. We immediately conclude that the operator T with kernel

$$K(z) = \frac{xy}{|z|^4} + \frac{x^4 + y^4 - 6x^2y^2}{|z|^6}, \quad 0 \neq z \in \mathbb{C},$$

is an example in which the L^2 inequality (24) fails. Before going on we remark that a key step in proving the division property has been that Q has at least one zero on the circle. This is also a central fact in the proof of the general case.

We can easily deduce now another necessary condition for (24) . Substituting (31) in (29) and simplifying the common factor uv we get

$$0 < |G_3(1)| \leq C\,(1 + \alpha(u^2 - v^2)), \quad |\xi| = 1,$$

which means that the right hand side cannot vanish on the unit circle, namely, $|\alpha| < 1$. Therefore we get the structural condition

$$T = R_P \circ U,$$

where R_P is the Riesz transform associated with the polynomial $P(x, y) = -(1/\pi)\, xy$ and U is an invertible operator in the Calderón–Zygmund algebra A.

Taking $\alpha = 1$ we get an operator T for which (24) fails but whose kernel

$$K(z) = -\frac{1}{\pi}\frac{xy}{|z|^4} + \frac{2}{\pi}\frac{x^3y - x, y^3}{|z|^6}, \quad 0 \neq z \in \mathbb{C},$$

satisfies the division property of Theorem 3 part (iii).

6 Sufficient conditions for the L^2 estimate of $T^* f$ by Tf

In this Section we show how condition (iii) in Theorem 3 yields the pointwise inequality

$$T^* f(z) \leqslant C\, M(Tf)(z), \quad z \in \mathbb{C}\,. \tag{32}$$

As in the previous Section, we work in the particularly simple case in which the spherical harmonics expansion of the kernel is reduced to two terms. The first is a harmonic homogeneous polynomial of degree 2, which for definiteness is taken to be

$$P(z) = -\frac{1}{\pi} xy.$$

The second term is a fourth degree harmonic homogeneous polynomial. The division assumption in (iii) of Theorem 3 is that P divides this second term. In view of the general form of a fourth degree harmonic homogeneous polynomial (30) we conclude that our kernel must be of the form

$$K(z) = -\frac{1}{\pi}\frac{xy}{|z|^4} + \frac{2}{\pi}\alpha\frac{x^3y - xy^3}{|z|^6}, \quad 0 \neq z \in \mathbb{C}, \quad \alpha \in \mathbb{R}.$$

The second assumption in (iii) of Theorem 3 is that T is of the form $T = R_P \circ U$, where R_P is the second order Riesz transform determined by P and U is an invertible operator in the Calderón–Zygmund algebra A. This is equivalent, as one can easily check looking at multipliers in the Fourier transform side, to $|\alpha| < 1$.

In the simple context we have just set the two assumptions of condition (iii) of Theorem 3 are not independent. The reader can easily check that the structural condition $T = R_P \circ U$ implies the division property, that is, that P divides the fourth degree term. We will point out later on where this simplifies the argument.

We start now the proof of (32). Recall that, as we showed in the preceding Section, there exists a bounded mesurable function b supported on the unit ball B and a constant c such that

$$T(b) = \chi_{\mathbb{C}\setminus B}(z)K(z) + cxy\chi_B(z). \tag{33}$$

Our goal is to express the second term in the right hand side above as

$$cxy\chi_B(z) = T(\beta)(z), \quad \text{for almost all } z \in \mathbb{C}, \tag{34}$$

where β is a bounded measurable function such that

$$|\beta(z)| \leqslant \frac{C}{|z|^3}, \quad |z| \geqslant 2\,. \tag{35}$$

We first show that this is enough for (32). The only difficulty is that β is not supported in B, but the decay inequality (35) is an excellent substitute. Set $\gamma = b - \beta$. Then (dA is planar Lebesgue measure)

$$\begin{aligned} T^1 f(0) &= \int \chi_{\mathbb{C}\setminus B}(z) K(z)\, f(z)\, dA(z) \\ &= \int T(\gamma)(z)\, f(z)\, dA(z) \\ &= \int \gamma(z)\, T f(z)\, dA(z) \\ &= \int_{2B} \gamma(z)\, T f(z)\, dz + \int_{\mathbb{C}\setminus 2B} \gamma(z)\, T f(z)\, dA(z). \end{aligned}$$

The first term is clearly less that a constant times $M(TF)(0)$, because γ is bounded, and the second too, because of (35) with β replaced by γ.

The proof of (34) is divided into two steps. The first step consists in showing that there exists a function β_0 such that

$$cxy\chi_B(z) = R(\beta_0)(z), \quad \text{for almost all } z \in \mathbb{C},$$

where $R = R_P$. To find β_0 let us look for a function ψ such that

$$P(\partial)\psi = cxy\chi_B(z). \tag{36}$$

Assume that we have found ψ and that it is regular enough so that

$$\psi = E * \Delta\psi,$$

where E is the standard fundamental solution of the Laplacian. Then

$$\begin{aligned} cxy\chi_B(z) &= P(\partial)\psi = P(\partial)E \star \Delta\psi \\ &= c\, p.v. \frac{P(x)}{|z|^4} \star \Delta\psi = R(\beta_0)\,, \end{aligned}$$

where $\beta_0 = c\,\Delta\psi$.

Taking the Fourier transform in (36) gives

$$P(\xi)\hat{\psi}(\xi) = c\, \partial_1\partial_2\widehat{\chi_B}(\xi) = c\, uv\, G_3(|\xi|).$$

For the definition of G_3 see the paragraph below (27). Hence

$$\hat{\psi}(\xi) = c\, G_3(\xi),$$

where c is some constant. It is a well known fact in the elementary theory of Bessel functions [3, p.429] that

$$c\, G_3(\xi) = (1 - \widehat{|z|^2)^2}\chi_B(z)(\xi)\,.$$

In other words,

$$\psi(z) = c\,(1 - |z|^2)^2\chi_B(z)\,.$$

Clearly ψ and its first order derivatives are continuous functions supported on the closed unit ball B. The second order derivatives of ψ are supported on B and on B they are polynomials. In particular, we get that $\beta_0 = c\,\Delta\psi$ is a function supported on B, which satisfies a Lipschitz condition on B and satisfies the cancellation property $\int \beta_0 = c \int \Delta\psi = 0$.

It is worth remarking that in the general case, where the spherical harmonic expansion of the kernel contains many terms, one has to resort to the division assumption of (iii) in Theorem 3 to complete the proof of the first step.

We proceed now with the second step. Since $T = R \circ U$ we have

$$c\,xy\chi_B(z) = R(\beta_0)(z) = T(U^{-1}(\beta_0))(z).$$

Set $\beta = U^{-1}(\beta_0)$, so that (34) is satisfied. We are left with the task of showing that β is bounded and satisfies the decay estimate (35).

The inverse of U is an operator in the Calderón–Zygmund algebra A. Thus

$$\beta = U^{-1}(\beta_0) = (\lambda I + V)(\beta_0) = \lambda\beta_0 + V(\beta_0),$$

where λ is a real number and V an even convolution smooth homogeneous Calderón–Zygmund operator. The desired decay estimate for β now follows readily, because β_0 is supported in the closed ball B and has zero integral. It remains to show that $V(\beta_0)$ is bounded. At first glance this is quite unlikely because V is a general even convolution smooth homogeneous Calderón–Zygmund operator and β_0 has no global smoothness properties in the plane. Indeed, although β_0 is Lipschitz on B, it has a jump at the boundary of B. Assume for a moment that $\beta_0 = \chi_B$. It is then known that $V(\chi_B)$ is a bounded function because V is an even Calderón–Zygmund operator and the boundary of B is smooth. Here the fact that the operator is even is crucial as one can see by considering the action of the Hilbert transform on the interval $(-1, 1)$. We are not going to present the nice argument for the proof that $V(\beta_0)$ is bounded [5]. Let us only mention that this result for the Beurling transform and smoothly bounded domains plays a basic role in the regularity theory of certain solutions of the Euler equation in the plane [1].

7 The proof in the general case and final comments

The proof of Theorems 3 and 4 in the general case proceeds in two stages. First one proves the Theorems in the case in which the spherical harmonics expansion of the kernel contains finitely many non-zero terms. Then one has to truncate the expansion of the kernel and see that some of the estimates obtained in the first step do not depend on the number of terms. This is a delicate issue at some moments, but necessary to perform a final compactness argument. In both steps there are difficulties of various types to be overcome and a major computational issue, lengthly and involved, which very likely can be substantially simplified by a more clever argument.

A final word on the proof for the necessity of the division condition. To show that a polynomial with complex coefficients divides another, one often resorts to Hilbert's Nullstellensatz, the zero set theorem of Hilbert, which states that if P is a prime polynomial with complex coefficients and finitely many variables, to show that P divides another such polynomial Q one has to check only that Q vanishes on the zeros of P. This fails for real polynomials, as simple examples show. Now, since we are working with real polynomials we cannot straightforwardly apply Hilbert's theorem. What saves us is that our real polynomials have a fairly substantial amount of zeroes, just because they have zero integral on the unit sphere. We can then jump to the complex case and come back to the real by checking that the Hausdorff dimension of the zero set of certain polynomials is big enough.

There are several questions about Theorems 3 and 4 that deserve further study. The first is a potential application to the David–Semmes problem mentioned in the introduction, which was the source of the question. Another is the smoothness of the kernels. It is not known how to prove the analogs of Theorems 3 and 4 for kernels of moderate smoothness, say of class C^m for some positive integer m. Finally it is has recently been shown by Bosch, Mateu and Orobitg that

$$\|T^\star f\|_p \leqslant C\|Tf\|_p, \quad f \in L^p(\mathbb{R}^n)\,, \quad 1 < p < \infty\,,$$

implies any of the three equivalent conditions in Theorems 3 and 4.

Acknowledgements. The author is grateful to the organizers of the XXXI Conference in Harmonic Analysis, held in Rome at the beginning of June 2011 in honor of Professor A.Figà-Talamanca, for the kind invitation to participate and for their efficient work. The author has been partially supported by the grants MTM2010-15657 and 2009SGR420.

References

[1] Chemin, J.Y. Fluides parfaits incompressibles. Astérisque, **2**30. Société Mathématique de France, Paris (1995).

[2] David, G., Semmes, S. : Singular integrals and rectifiable sets in $\mathbb{R}^n$: Au-delàs des graphes lipschitziens. Astérisque **193**. Soc. Math. France, Paris (1991).

[3] Grafakos, L. : Classical Fourier Analysis. Graduate Texts in Mathematics **249**, Springer-Verlag, Berlin, Second Edition (2008).

[4] Grafakos, L. : Modern Fourier Analysis. Graduate Texts in Mathematics **250**, Springer-Verlag, Berlin, Second Edition (2008).

[5] Mateu, J., Orobitg, J., Verdera, J. : Estimates for the maximal singular integral in terms of the singular integral: the case of even kernels. Annals of Math. **174**, 1429–1483 (2011)

[6] Mateu, J., Orobitg, J., Pérez, C., Verdera, J. : New Estimates for the Maximal Singular Integral. Int. Math. Research Notices **2010**, 3658–3722 (2010)

[7] Mateu, J., Verdera J. : L^p and weak L^1 estimates for the maximal Riesz transform and the maximal Beurling transform. Math. Res. Lett. **13**, 957–966 (2006)

[8] Pérez, C. Endpoint estmates for commutators of singular integral operators. J. Funct. Anal. **128**, 163–185 (1995).

[9] Stein, E.M. : Singular integrals and differentiability properties of functions. Princeton University Press, Princeton, (1970).

Joan Verdera
Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra, Barcelona, Catalonia
E-mail: jvm@mat.uab.cat