CHARACTERIZING LYAPUNOV DOMAINS VIA RIESZ TRANSFORMS ON HÖLDER SPACES

DORINA MITREA, MARIUS MITREA, AND JOAN VERDERA

Abstract. Under mild geometric measure theoretic assumptions on an open set $\Omega \subset \mathbb{R}^n$, we show that the Riesz transforms on its boundary are continuous mappings on the Hölder space $\mathscr{C}^\alpha(\partial\Omega)$ if and only if Ω is a Lyapunov domain of order α (i.e., a domain of class $\mathscr{C}^{1+\alpha}$). In the category of Lyapunov domains we also establish the boundedness on Hölder spaces of singular integral operators with kernels of the form $P(x-y)/|x-y|^{n-1+l}$, where P is any odd homogeneous polynomial of degree l in $\mathbb{R}^n$. This family of singular integral operators, which may be thought of as generalized Riesz transforms, includes the boundary layer potentials associated with basic PDE's of mathematical physics, such as the Laplacian, the Lamé system, and the Stokes system. We also consider the limiting case $\alpha = 0$ (with $\mathrm{VMO}(\partial\Omega)$ as the natural replacement of $\mathscr{C}^\alpha(\partial\Omega)$), and discuss an extension to the scale of Besov spaces.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open set. Singular integral operators mapping functions on $\partial\Omega$ into functions defined either on $\partial\Omega$, or in Ω, arise naturally in many branches of mathematics and engineering. From the work of G. David and S. Semmes (cf. [7], [8]) we know that uniformly rectifiable (UR) sets make up the most general context in which Calderón-Zygmund like operators are bounded on Lebesgue spaces L^p, with $p \in (1,\infty)$ (see Theorem 3.1 in the body of the paper for a concrete illustration of the scope of this theory from the perspective of boundary integral methods). David and Semmes have also proved that, under the background assumption of Ahlfors regularity, uniform rectifiability is implied by the simultaneous L^2 boundedness of all integral convolution type operators on $\partial\Omega$, whose kernels are smooth, odd, and satisfy standard growth conditions (cf. [8, Definition 1.20, p. 11]). In fact, a remarkable recent result proved by F. Nazarov, X. Tolsa, and A. Volberg in [37] states that the L^2-boundedness of the Riesz transforms alone, given for $j \in \{1,\dots,n\}$ by

$$R_j f(x) := \lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \frac{x_j - y_j}{|x-y|^n} f(y)\, d\mathcal{H}^{n-1}(y), \qquad x \in \partial\Omega, \tag{1.1}$$

yields uniform rectifiability (here and elsewhere $\mathcal{H}^{n-1}$ stands for the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$ and ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$). The corresponding result in the plane was proved much earlier in [30].

The above discussion points to uniform rectifiability as being intimately connected with the boundedness of a large class of Calderón-Zygmund like operators on Lebesgue spaces. This being said, uniform rectifiability is far too weak to guarantee by itself analogous boundedness properties in other functional analytic contexts, such as the scale of Hölder spaces $\mathscr{C}^\alpha$, with $\alpha \in (0,1)$. The goal of this paper is to identify the category of domains for which the Riesz transforms (1.1) are bounded on Hölder spaces as the class of Lyapunov domains (cf. Definition 2.1), and also show that, in fact, a much larger family of singular integral operators (generalizing the Riesz transforms) act naturally in this setting. On this note we

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wish to remark that the trade-mark property of Lyapunov domains is the Hölder continuity of their outward unit normals. Alternative characterizations, of a purely geometric flavor, may be found in [2]. The issue of boundedness of singular integral operators on Hölder spaces has a long history, with early work focused on Cauchy-type operators in the plane (cf. [36], [15], and the references therein). More recently this topic has been considered in [9], [10], [12], [14], [16], [27], [29], [31, Chapter X, § 4], [44], [46].

As it turns out, much geometrical information is encapsulated into the action of the Riesz transforms on the constant function 1. At least heuristically, the flatter $\partial\Omega$ is near a point $x_o \in \partial\Omega$ the more regular (e.g., bounded, or continuous) the functions $R_j 1$ are expected to be near x_o. At the low regularity spectrum, a result of X. Tolsa (cf. [45]) states that if $\Sigma \subset \mathbb{R}^n$ has $\mathcal{H}^{n-1}(\Sigma) < +\infty$ then

$$\Sigma \text{ is countably rectifiable (of dimension } n-1) \iff$$
$$\left\{\begin{array}{l} \text{for every } \ j \in \{1,\dots,n\} \ \text{ one has} \\ \lim\limits_{\varepsilon\to 0^+} \displaystyle\int_{\substack{y\in\Sigma \\ |y-x|>\varepsilon}} \frac{x_j - y_j}{|x-y|^n}\, d\mathcal{H}^{n-1}(y) \text{ exists for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Sigma. \end{array}\right. \tag{1.2}$$

In particular, having $\partial\Omega$ countably rectifiable (of dimension $n-1$) ensures that the functions $R_j 1$, $1 \le j \le n$, can be defined in a pointwise fashion $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$. On the other hand, assuming that $\partial\Omega$ is Ahlfors regular, the Nazarov, Tolsa, Volberg recent main result in [37] mentioned earlier may be rephrased in light of the $T(1)$ theorem[1] for singular integral operators associated with odd standard kernels as

$$\partial\Omega \ \text{ uniformly rectifiable set } \iff R_j 1 \in \mathrm{BMO}(\partial\Omega) \quad \text{for each } \ j \in \{1,\dots,n\}. \tag{1.3}$$

Hence, the membership of the $R_j 1$'s to the John-Nirenberg space $\mathrm{BMO}(\partial\Omega)$ characterizes the uniform rectifiability of $\partial\Omega$. From this perspective, one of the issues addressed by our first main result is that of extracting more geometric regularity for Ω if more analytic regularity for the $R_j 1$'s is available. This fits into the paradigm of describing geometric characteristics (such as regularity of a certain nature) of a given set in terms of properties of suitable analytical entities (such as singular integral operators) associated with this environment. Specifically, we have the following theorem (for all relevant definitions the reader is referred to §2).

Theorem 1.1. *Assume $\Omega \subseteq \mathbb{R}^n$ is an open set with a compact Ahlfors regular boundary, satisfying $\partial\Omega = \partial(\overline{\Omega})$ and $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and define $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$. Then for each $\alpha \in (0,1)$ the following claims are equivalent:*

(a) *Ω is a domain of class $\mathscr{C}^{1+\alpha}$ (or Lyapunov domain of order α);*

(b) *the Riesz transforms, defined as in (1.1), satisfy*
$$R_j 1 \in \mathscr{C}^{\alpha}(\partial\Omega) \quad \textit{for each } \ j \in \{1,\dots,n\}; \tag{1.4}$$

(c) *given any odd homogeneous polynomial P of degree $l \ge 1$ in $\mathbb{R}^n$, the singular integral operator*
$$Tf(x) := \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \frac{P(x-y)}{|x-y|^{n-1+l}} f(y)\, d\sigma(y), \qquad x \in \partial\Omega, \tag{1.5}$$
maps $\mathscr{C}^{\alpha}(\partial\Omega)$ boundedly into itself;

(d) *Ω is a* UR *domain and one has*
$$\mathscr{R}_j^{\pm} 1 \in \mathscr{C}^{\alpha}(\Omega_{\pm}) \quad \textit{for each } \ j \in \{1,\dots,n\} \tag{1.6}$$

[1]for spaces of homogeneous type; cf., e.g., [3, Theorem 12.3], [5, Chapter IV], [17, Theorem 5.56, p. 166]

where, for $j \in \{1, \dots, n\}$,

$$\mathscr{R}_j^{\pm} f(x) := \frac{1}{\omega_{n-1}} \int\limits_{\partial\Omega} \frac{x_j - y_j}{|x-y|^n} f(y)\, d\sigma(y), \qquad x \in \Omega_{\pm}; \tag{1.7}$$

(e) Ω *is a* UR *domain and, for each odd homogeneous polynomial* P *of degree* $l \geq 1$ *in* $\mathbb{R}^n$*, the integral operators*

$$\mathbb{T}_{\pm} f(x) := \int\limits_{\partial\Omega} \frac{P(x-y)}{|x-y|^{n-1+l}} f(y)\, d\sigma(y), \qquad x \in \Omega_{\pm}, \tag{1.8}$$

map $\mathscr{C}^\alpha(\partial\Omega)$ *boundedly into* $\mathscr{C}^\alpha\big(\Omega_{\pm}\big)$.

Moreover, if Ω *is a* $\mathscr{C}^{1+\alpha}$ *domain for some* $\alpha \in (0,1)$*, there exists a finite constant* $C > 0$*, depending only on* n*,* α*,* $\mathrm{diam}(\partial\Omega)$*, the upper Ahlfors regularity constant of* $\partial\Omega$*, and* $\|\nu\|_{\mathscr{C}^\alpha(\partial\Omega)}$ *(where* ν *is the outward unit normal to* Ω*) with the property that*

$$\big\|\mathbb{T}_{\pm} f\big\|_{\mathscr{C}^\alpha\left(\overline{\Omega_{\pm}}\right)} \leq C^l 2^{l^2} \|P\|_{L^2(S^{n-1})} \|f\|_{\mathscr{C}^\alpha(\partial\Omega)}, \qquad \forall\, f \in \mathscr{C}^\alpha(\partial\Omega), \tag{1.9}$$

and

$$\big\|Tf\big\|_{\mathscr{C}^\alpha(\partial\Omega)} \leq C^l 2^{l^2} \|P\|_{L^2(S^{n-1})} \|f\|_{\mathscr{C}^\alpha(\partial\Omega)}, \qquad \forall\, f \in \mathscr{C}^\alpha(\partial\Omega). \tag{1.10}$$

As a corollary of the above theorem, given an open set $\Omega \subseteq \mathbb{R}^n$ with a compact Ahlfors regular boundary satisfying $\partial\Omega = \partial(\overline{\Omega})$ and $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$, along with some $\alpha \in (0,1)$, it follows that Ω is a domain of class $\mathscr{C}^{1+\alpha}$ if and only if the Riesz transforms (1.1) are bounded operators on $\mathscr{C}^\alpha(\partial\Omega)$.

The operators described in (1.5) may be thought of as generalized Riesz transforms since they correspond to (1.5) with

$$P(x) := x_j/\omega_{n-1} \quad \text{for} \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n, \quad 1 \leq j \leq n. \tag{1.11}$$

For the same choices of the polynomials, the claim in part (e) of Theorem 1.1 implies that the harmonic single-layer operator (cf. (5.63) for a definition) is well-defined, linear, and bounded as a mapping

$$\mathscr{S} : \mathscr{C}^\alpha(\partial\Omega) \longrightarrow \mathscr{C}^{1+\alpha}(\Omega_{\pm}). \tag{1.12}$$

In dimension two, there is a variant of Theorem 1.1 starting from the demand that the boundary of the domain in question is an upper Ahlfors regular Jordan curve and, in lieu of the Riesz transforms, using the following version of the classical Cauchy integral operator:

$$\mathfrak{C} f(z) := \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int\limits_{\substack{\zeta \in \partial\Omega \\ |z-\zeta| > \varepsilon}} \frac{f(\zeta)}{\zeta - z}\, d\mathcal{H}^1(\zeta), \qquad z \in \partial\Omega. \tag{1.13}$$

Theorem 1.2. *Let* $\Omega \subseteq \mathbb{C}$ *be a bounded open set whose boundary is an upper Ahlfors regular Jordan curve and fix* $\alpha \in (0,1)$*. Then* Ω *is a domain of class* $\mathscr{C}^{1+\alpha}$ *if and only if the operator* (1.13) *satisfies* $\mathfrak{C}\,1 \in \mathscr{C}^\alpha(\partial\Omega)$.

Under the initial background hypotheses on Ω made in Theorem 1.1, Ω being a $\mathscr{C}^1$ domain is equivalent with $\nu \in \mathscr{C}^0(\partial\Omega, \mathbb{R}^n)$ (cf. [19] in this regard). This being said, the limiting case $\alpha = 0$ of the equivalence (a)$\Leftrightarrow$(b) in Theorem 1.1 requires replacing the space of continuous functions by the (larger) Sarason space VMO, of functions of vanishing mean oscillations (on $\partial\Omega$, viewed as a space of homogeneous type, in the sense of Coifman-Weiss, when equipped with the measure σ and the Euclidean distance). Specifically, the following result holds.

Theorem 1.3. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set with a compact Ahlfors regular boundary, satisfying $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$, and denote by ν the geometric measure theoretic outward unit normal to Ω. Then*

$$\left.\begin{array}{r} \nu \in \mathrm{VMO}(\partial\Omega, \mathbb{R}^n) \ \ \textit{and} \\ \partial\Omega \ \textit{uniformly rectifiable set} \end{array}\right\} \Longleftrightarrow R_j 1 \in \mathrm{VMO}(\partial\Omega) \ \ \textit{for all} \ \ j \in \{1, \dots, n\}. \tag{1.14}$$

Equivalence (1.14) should be contrasted with (1.3). In the present context, the additional background assumption that $\mathcal{H}^{n-1}(\partial\Omega\setminus\partial_*\Omega) = 0$ merely ensures that the geometric measure theoretic outward unit normal ν to Ω is well-defined σ-a.e. on $\partial\Omega$.

The collection of all geometric conditions entering Theorem 1.3, i.e., that $\Omega \subseteq \mathbb{R}^n$ is an open set with an Ahlfors regular boundary, satisfying $\mathcal{H}^{n-1}(\partial\Omega\setminus\partial_*\Omega) = 0$, and such that $\partial\Omega$ is a uniformly rectifiable set, amounts to saying that Ω is a UR domain (cf. Definition 2.6). Concerning this class of domains, it has been noted in [20, Corollary 3.9, p. 2633] that

> if $\Omega \subset \mathbb{R}^n$ is an open set satisfying a two-sided corkscrew condition (in the sense of Jerison-Kenig [23]) and whose boundary is Ahlfors regular, then Ω is a UR domain. (1.15)

In fact, the same circle of techniques yielding Theorem 1.3 also allows us to characterize the class of regular SKT domains, originally introduced in [20, Definition 4.8, p. 2690] by demanding: δ-Reifenberg flatness for some sufficiently small $\delta > 0$ (cf. Definition 7.6), Ahlfors regular boundary, and vanishing mean oscillations for the geometric measure theoretic outward unit normal. Specifically, combining (1.15), Theorem 1.3, Theorem 7.7, and [20, Theorem 4.21, p. 2711] gives the following theorem.

Theorem 1.4. *If $\Omega \subseteq \mathbb{R}^n$ is an open set with a compact Ahlfors regular boundary, satisfying a two-sided John condition as described in Definition 7.3 (which, in particular, implies the two-sided corkscrew condition) then*

$$\begin{gathered} R_j 1 \in \mathrm{VMO}(\partial\Omega) \ \textit{for every} \ j \in \{1, \dots, n\} \\ \textit{if and only if} \ \Omega \ \textit{is a regular} \ \mathrm{SKT} \ \textit{domain.} \end{gathered} \tag{1.16}$$

It turns out that the equivalence (a)$\Leftrightarrow$(b) in Theorem 1.1 essentially self-extends to the larger scale of Besov spaces $B^{p,p}_s(\partial\Omega)$ with $p \in [1, \infty]$ and $s \in (0, 1)$ satisfying $sp > n - 1$, for which the Hölder spaces occur as a special, limiting case, corresponding to $p = \infty$. For a precise statement, see Theorem 7.11.

The category of singular integral operators falling under the scope of Theorem 1.1 already includes boundary layer potentials associated with basic PDE's of mathematical physics, such as the Laplacian, the Helmholtz operator, the Lamé system, the Stokes system, and even higher-order elliptic systems (cf., e.g., [6], [21], [32], [33]). This being said, granted the estimates established in the last part of Theorem 1.1, the method of spherical harmonics then allows us to prove the following result, dealing with a more general class of operators.

Theorem 1.5. *Let Ω be a $\mathscr{C}^{1+\alpha}$ domain, $\alpha \in (0, 1)$, with compact boundary, and let $k \in \mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})$ be an odd function satisfying $k(\lambda x) = \lambda^{1-n} k(x)$ for all $\lambda \in (0, \infty)$ and $x \in \mathbb{R}^n \setminus \{0\}$. In addition, assume that there exists a sequence $\{m_l\}_{l\in\mathbb{N}_0} \subseteq \mathbb{N}_0$ for which*

$$\sum_{l=0}^{\infty} 4^{l^2} l^{-2m_l} \big\| (\Delta_{S^{n-1}})^{m_l} \big(k\big|_{S^{n-1}}\big) \big\|_{L^2(S^{n-1})} < +\infty, \tag{1.17}$$

where $\Delta_{S^{n-1}}$ is the Laplace-Beltrami operator on the unit sphere S^{n-1} in $\mathbb{R}^n$.

Then the singular integral operators

$$\mathbb{T} f(x) := \int_{\partial\Omega} k(x - y) f(y)\, d\sigma(y), \qquad x \in \Omega, \tag{1.18}$$

$$Tf(x) := \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} k(x-y)f(y)\,d\sigma(y), \qquad x\in\partial\Omega, \tag{1.19}$$

induce linear and bounded mappings

$$\mathbb{T} : \mathscr{C}^\alpha(\partial\Omega) \longrightarrow \mathscr{C}^\alpha(\overline{\Omega}), \qquad T : \mathscr{C}^\alpha(\partial\Omega) \longrightarrow \mathscr{C}^\alpha(\partial\Omega). \tag{1.20}$$

We wish to note that Theorem 1.5 refines the implication (a) ⇒(e) in Theorem 1.1 since, as explained in Remark 6.1, condition (1.17) is satisfied whenever the kernel k is of the form $P(x)/|x|^{n-1+l}$ for some homogeneous polynomial P of degree $l\in 2\mathbb{N}-1$ in $\mathbb{R}^n$. In fact, condition (1.17) holds for kernels k that are real-analytic away from 0 with lacunary Taylor series (involving sufficiently large gaps between the non-zero coefficients of their expansions, depending on n, α, diam $(\partial\Omega)$, $\|\nu\|_{\mathscr{C}^\alpha(\partial\Omega)}$, and the upper Ahlfors regularity constant of $\partial\Omega$). Thus, the conclusions in Theorem 1.5 are valid for such kernels which are also odd and positive homogeneous of degree $1-n$.

Even though the statement does not reflect it, the proof of Theorem 1.1 makes essential use of the Clifford algebra $\mathcal{C}\ell_n$, a highly non-commutative generalization of the field of complex numbers to n-dimensions, which also turns out to be geometrically sensitive. Indeed, this is a tool which has occasionally emerged at the core of a variety of problems at the interface between geometry and analysis. For us, one key aspect of this algebraic setting is the close relationship between the Riesz transforms and the principal value[2] Cauchy-Clifford integral operator $\mathbf{C}$ (defined in (5.2)). For the purpose of this introduction we single out the remarkable formula

$$\nu = -4\mathbf{C}\Big(\sum_{j=1}^{n}(R_j 1)e_j\Big) \ \ \sigma\text{-a.e. on } \ \partial\Omega \tag{1.21}$$

expressing the (geometric measure theoretic) outward unit normal to Ω as the Clifford algebra "cocktail" $\sum_{j=1}^{n}(R_j 1)e_j$ of Riesz transforms acting on 1 coupled with the imaginary units e_j in $\mathcal{C}\ell_n$, distorted through the action of the Cauchy-Clifford operator $\mathbf{C}$. Identity (1.21) is derived from a version of the Poincaré-Bertrand formula, and plays a basic role in the proof of (b) ⇒ (a) in Theorem 1.1, together with a higher-dimensional generalization in a rough setting of the classical Plemelj-Privalov theorem stating that the principal value Cauchy integral operator on a piecewise smooth Jordan curve without cusps in the plane is bounded on Hölder spaces (cf. [38], [39], [40]; cf. also [22] for a higher dimensional version for Lyapunov domains with compact boundaries). Specifically, in Theorem 5.6 we show that whenever $\Omega\subset\mathbb{R}^n$ is a Lebesgue measurable set whose boundary is compact, upper Ahlfors regular, and satisfies $\mathcal{H}^{n-1}(\partial\Omega\setminus\partial_*\Omega)=0$, it follows that for each $\alpha\in(0,1)$ the principal value Cauchy-Clifford operator $\mathbf{C}$ induces a well-defined, linear, and bounded mapping

$$\mathbf{C} : \mathscr{C}^\alpha(\partial\Omega)\otimes\mathcal{C}\ell_n \longrightarrow \mathscr{C}^\alpha(\partial\Omega)\otimes\mathcal{C}\ell_n. \tag{1.22}$$

The strategy employed in the proof of (a) ⇒ (e) in Theorem 1.1 is somewhat akin to that of establishing a "$T(1)$-theorem" in the sense that matters are reduced to checking that $\mathbb{T}_\pm$ act reasonably on the constant function 1 (see (3.41) in this regard). In turn, this is accomplished via a proof by induction on $l\in 2\mathbb{N}-1$, the degree of the homogeneous polynomial P. The base case $l=1$, corresponding to linear combinations of polynomials as in (1.11), is dealt with by viewing $(x_j-y_j)/|x-y|^n$ as a dimensional multiple of $\partial_j E_\Delta(x-y)$ where E_Δ is the standard fundamental solution for the Laplacian in $\mathbb{R}^n$. As such, the key cancellation property that eventually allows us to establish the desired Hölder estimate in this base case may be ultimately traced back to the PDE satisfied by $(x_j-y_j)/|x-y|^n$. In carrying out the inductive step we make essential use of elements of Clifford analysis permitting us to relate $\mathbb{T}_\pm 1$ to the action of certain integral operators constructed as in

[2]in the standard sense of removing balls centered at the singularity and taking the limit as the radii shrink to zero

(1.8) but relative to lower degree polynomials acting on components of the outward unit normal ν to Ω. In this scenario, what allows the use of the induction hypothesis is the fact that, since Ω is a domain of class $\mathscr{C}^{1+\alpha}$, the said components belong to $\mathscr{C}^{\alpha}(\partial\Omega)$.

The layout of the paper is as follows. Section 2 contains a discussion of background material of geometric measure theoretic nature, along with some auxiliary lemmas which are relevant in our future endeavors. In Section 3 we first recall a version of the Calderón-Zygmund theory for singular integral operators on Lebesgue space in UR domains, and then proceed to establish several useful preliminary estimates for general singular integral operators. Next, Section 4 is reserved for a presentation of those aspects of Clifford analysis which are relevant for the present work. Section 5 is devoted to a study of Cauchy-Clifford integral operators (both of boundary-to-domain and boundary-to-boundary type) in the context of Hölder spaces. In contrast with the Calderón-Zygmund theory for singular integrals in UR domains reviewed in the first part of Section 3, the novelty here is the consideration of a much larger category of domains (see Theorem 5.6 for details). In the last part of Section 5 we also discuss the harmonic single and double layer potentials (involved in the initial induction step in the proof of the implication (a) $\Rightarrow$ (e) in Theorem 1.1). Finally, in Section 6, the proofs of Theorems 1.1, 1.2, 1.5 are presented, while Section 7 contains the proofs of Theorem 1.3, and the Besov space version of the equivalence (a)$\Leftrightarrow$(b) in Theorem 1.1 (see Theorem 7.11), and also a more general version of (1.16) in Theorem 7.7.

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2. Geometric Measure Theoretic Preliminaries

Throughout, $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$ and we shall denote by $\mathbf{1}_E$ the characteristic function of a set E. For $\alpha\in(0,1)$ and $U\subseteq\mathbb{R}^n$ arbitrary set (implicitly assumed to have cardinality ≥ 2), define the `homogeneous Hölder space of order` α on U as

$$\dot{\mathscr{C}}^{\alpha}(U) := \big\{u : U\to\mathbb{C} :\, [u]_{\dot{\mathscr{C}}^{\alpha}(U)} < +\infty\big\}, \tag{2.1}$$

where $[\,\cdot\,]_{\dot{\mathscr{C}}^{\alpha}(U)}$ stands for the seminorm

$$[u]_{\dot{\mathscr{C}}^{\alpha}(U)} := \sup_{\substack{x,y\in U\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}}. \tag{2.2}$$

The `inhomogeneous Hölder space of order` α on U is then defined as

$$\mathscr{C}^{\alpha}(U) := \big\{u\in\dot{\mathscr{C}}^{\alpha}(U) :\, u \text{ is bounded in } U\big\}, \tag{2.3}$$

as is equipped with the norm

$$\|u\|_{\mathscr{C}^{\alpha}(U)} := \sup_{U}|u| + [u]_{\dot{\mathscr{C}}^{\alpha}(U)}, \qquad \forall\, u\in\mathscr{C}^{\alpha}(U). \tag{2.4}$$

Moreover, if $\mathcal{O}$ is an open, nonempty, subset of $\mathbb{R}^n$, then for $\alpha\in(0,1)$ given define

$$\mathscr{C}^{1+\alpha}(\mathcal{O}) := \big\{u\in\mathscr{C}^{1}(\mathcal{O}) :\, \|u\|_{\mathscr{C}^{1+\alpha}(\mathcal{O})} < +\infty\big\}, \tag{2.5}$$

where

$$\|u\|_{\mathscr{C}^{1+\alpha}(\mathcal{O})} := \sup_{x\in\mathcal{O}}|u(x)| + \sup_{x\in\mathcal{O}}|(\nabla u)(x)| + \sup_{\substack{x,y\in\mathcal{O}\\ x\neq y}} \frac{\big|(\nabla u)(x)-(\nabla u)(y)\big|}{|x-y|^{\alpha}}. \tag{2.6}$$

The following observations will be tacitly used in the sequel. For each set $U\subseteq\mathbb{R}^n$ and any $\alpha\in(0,1)$, we have that $\mathscr{C}^{\alpha}(U)$ is an algebra, the spaces $\dot{\mathscr{C}}^{\alpha}(U)$ and $\mathscr{C}^{\alpha}(U)$ are contained in the space of uniformly continuous functions on U, and $\dot{\mathscr{C}}^\alpha(U) = \dot{\mathscr{C}}^\alpha(\overline{U})$, $\mathscr{C}^\alpha(U) = \mathscr{C}^\alpha(\overline{U})$. Moreover, $\dot{\mathscr{C}}^\alpha(U) = \mathscr{C}^\alpha(U)$ if U is bounded.

Definition 2.1. *A nonempty, open, proper subset Ω of $\mathbb{R}^n$ is called a* `domain of class` $\mathscr{C}^{1+\alpha}$ *for some $\alpha \in (0,1)$ (or a* `Lyapunov domain of order` *α), if there exist r, $h > 0$ with the following significance. For every point $x_0 \in \partial\Omega$ one can find a coordinate system $(x_1, \dots, x_n) = (x', x_n)$ in $\mathbb{R}^n$ which is isometric to the canonical one and has origin at x_0, along with a real-valued function $\varphi \in \mathscr{C}^{1+\alpha}(\mathbb{R}^{n-1})$ such that*

$$\Omega \cap \mathcal{C}(r,h) = \big\{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \, |x'| < r \;\; \text{and} \; \varphi(x') < x_n < h\big\}, \tag{2.7}$$

where $\mathcal{C}(r,h)$ stands for the cylinder

$$\big\{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \, |x'| < r \; \text{and} \; -h < x_n < h\big\}. \tag{2.8}$$

Strictly speaking, the traditional definition of a Lyapunov[3] domain $\Omega \subseteq \mathbb{R}^n$ of order α requires that $\partial\Omega$ is locally given by the graph of a differentiable function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ whose normal ν to its graph Σ has the property that the acute angle $\theta_{x,y}$ between $\nu(x)$ and $\nu(y)$ for two arbitrary points $x, y \in \Sigma$ satisfies $\theta_{x,y} \leq C|x-y|^\alpha$; see, e.g., [22, Définition 2.1, p. 301]. This being said, it is easy to see that the latter condition implies that ν is Hölder continuous of order α and, ultimately, that Ω is a domain of class $\mathscr{C}^{1+\alpha}$ in the sense of our Definition 2.1.

We shall now present a brief summary of a number of definitions and results from geometric measure theory which are relevant for the current work (cf. H. Federer [13], W. Ziemer [47], L. Evans and R. Gariepy [11] for more details). Call a Lebesgue measurable set $\Omega \subset \mathbb{R}^n$ of `locally finite perimeter` provided $\nabla \mathbf{1}_\Omega$ is a locally finite Borel regular $\mathbb{R}^n$-valued measure. Given an open set $\Omega \subset \mathbb{R}^n$ of locally finite perimeter we denote by σ the total variation measure of $\nabla \mathbf{1}_\Omega$. Then σ is a locally finite positive measure, supported on $\partial\Omega$. Clearly, each component of $\nabla \mathbf{1}_\Omega$ is absolutely continuous with respect to σ so from the Radon-Nikodym theorem it follows that

$$\nabla \mathbf{1}_\Omega = -\nu\sigma, \tag{2.9}$$

where

$$\begin{gathered} \nu \text{ is an } \mathbb{R}^n\text{-valued function with components in } L^\infty(\partial\Omega, \sigma), \\ \text{satisfying } |\nu(x)| = 1, \text{ for } \sigma\text{-a.e. point } x \in \partial\Omega. \end{gathered} \tag{2.10}$$

Above and elsewhere, by $L^p(\partial\Omega, \sigma)$, $0 < p \leq \infty$, we denote the usual scale of Lebesgue spaces on $\partial\Omega$ (with respect to the measure σ). In the sequel we shall frequently identify σ with its restriction to $\partial\Omega$, with no special mention. We shall refer to ν and σ, respectively, as the (geometric measure theoretic) `outward unit normal` and the `surface measure` on $\partial\Omega$.

Next, denote by $\mathscr{L}^n$ the Lebesgue measure in $\mathbb{R}^n$ and recall that the `measure-theoretic boundary` $\partial_*\Omega$ of a Lebesgue measurable set $\Omega \subseteq \mathbb{R}^n$ is defined by

$$\begin{aligned} \partial_*\Omega := \Big\{x \in \partial\Omega : \, &\limsup_{r \to 0^+} \frac{\mathscr{L}^n(B(x,r) \cap \Omega)}{r^n} > 0 \\ &\text{and} \quad \limsup_{r \to 0^+} \frac{\mathscr{L}^n(B(x,r) \setminus \Omega)}{r^n} > 0\Big\}. \end{aligned} \tag{2.11}$$

Also, the `reduced boundary` $\partial^*\Omega$ of Ω is defined as

$$\partial^*\Omega := \big\{x \in \partial\Omega : \, |\nu(x)| = 1\big\}. \tag{2.12}$$

As is well-known, (cf. [47, Lemma 5.9.5 on p. 252] and [11, p. 208]) one has

$$\partial^*\Omega \subseteq \partial_*\Omega \subseteq \partial\Omega \quad \text{and} \quad \mathcal{H}^{n-1}(\partial_*\Omega \setminus \partial^*\Omega) = 0, \tag{2.13}$$

[3] Also spelled as Liapunov

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$. Also,

$$\sigma = \mathcal{H}^{n-1}\lfloor\partial^*\Omega. \tag{2.14}$$

Hence, if Ω has locally finite perimeter, it follows from (2.13) that the outward unit normal is defined σ-a.e. on $\partial_*\Omega$. In particular, if

$$\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0, \tag{2.15}$$

then from (2.12)-(2.13) we see that the outward unit normal ν is defined σ-a.e. on $\partial\Omega$, and (2.14) becomes $\sigma = \mathcal{H}^{n-1}\lfloor\partial\Omega$. Works of Federer and of De Giorgi also give that

$$\partial^*\Omega \text{ is } \texttt{countably rectifiable} \text{ (of dimension } n-1), \tag{2.16}$$

in the sense that it is a countable disjoint union

$$\partial^*\Omega = N \cup \Big(\bigcup_{k\in\mathbb{N}} M_k\Big), \tag{2.17}$$

where each M_k is a compact subset of an $(n-1)$-dimensional $\mathscr{C}^1$ surface in $\mathbb{R}^n$ and $\mathcal{H}^{n-1}(N) = 0$. It then happens that ν is normal to each such surface, in the usual sense. For further reference let us remark here that, as is apparent from (2.16), (2.13), and (2.17),

$$\begin{array}{c}\text{if } \Omega \subset \mathbb{R}^n \text{ has locally finite perimeter and (2.15) holds}\\ \text{then } \partial\Omega \text{ is countably rectifiable (of dimension } n-1).\end{array} \tag{2.18}$$

The following characterization of the class of $\mathscr{C}^{1+\alpha}$ domains from [19] is going to play an important role for us here.

Theorem 2.2. *Assume that Ω is a nonempty, open, proper subset of $\mathbb{R}^n$, of locally finite perimeter, with compact boundary, for which*

$$\partial\Omega = \partial\big(\overline{\Omega}\big), \tag{2.19}$$

and denote by ν the geometric measure theoretic outward unit normal to $\partial\Omega$, as defined in (2.9)-(2.10). Also, fix $\alpha \in (0,1)$. Then Ω is a $\mathscr{C}^{1+\alpha}$ domain if and only if, after altering ν on a set of σ-measure zero, one has $\nu \in \mathscr{C}^\alpha(\partial\Omega, \mathbb{R}^n)$.

Condition (2.19) expresses the fact that the domain Ω sits on just one side of its topological boundary, and is designed to preclude pathological happenstances such as a slit disk. By the Jordan-Brower separation theorem (cf. [1, Theorem 1, p. 284]), (2.19) is automatically satisfied if $\partial\Omega$ is a compact, connected, $(n-1)$-dimensional topological manifold without boundary (since in this scenario $\mathbb{R}^n \setminus \partial\Omega$ consists precisely of two components, each with boundary $\partial\Omega$; see [2] for details).

Changing topics, we remind the reader that a set $\Sigma \subset \mathbb{R}^n$ is called `Ahlfors regular` provided it is closed, nonempty, and there exists $C \in (1,\infty)$ such that

$$C^{-1}\, r^{n-1} \leq \mathcal{H}^{n-1}\big(B(x,r)\cap\Sigma\big) \leq C\, r^{n-1}, \tag{2.20}$$

for each $x \in \Sigma$ and $r \in (0, \operatorname{diam}\Sigma)$. When considered by itself, the second inequality above will be referred to as `upper Ahlfors regularity`. In this vein, we wish to remark that (cf. [11, Theorem 1, p. 222])

$$\begin{array}{c}\text{any Lebesgue measurable subset of } \mathbb{R}^n \text{ with an upper}\\ \text{Ahlfors regular boundary is of locally finite perimeter.}\end{array} \tag{2.21}$$

Later on, the following result is going to be of significance to us.

Proposition 2.3. *Let $\Sigma \subseteq \mathbb{R}^n$ be an Ahlfors regular set which is countably rectifiable (of dimension $n-1$). Define $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$ and consider an arbitrary function $f \in L^1_{\rm loc}(\Sigma, \sigma)$.*

Then for each $j \in \{1, \dots, n\}$,

$$\lim_{\varepsilon\to 0^+}\left\{\sup_{r\in(\varepsilon/2,\,\varepsilon)}\left|\int\limits_{\substack{y\in\Sigma\\ \varepsilon/4<|y-x|\leq r}}\frac{x_j-y_j}{|x-y|^n}f(y)\,d\sigma(y)\right|\right\}=0\quad \textit{for } \sigma\textit{-a.e. } x\in\Sigma. \tag{2.22}$$

Proof. Fix $j \in \{1, \dots, n\}$ and pick some large $R > 0$. For each $\varepsilon \in (0, 1)$, $r \in (\varepsilon/2, \varepsilon)$, and $x \in \Sigma \cap B(0, R)$ split

$$\int\limits_{\substack{y\in\Sigma\\ \varepsilon/4<|y-x|\leq r}}\frac{x_j-y_j}{|x-y|^n}f(y)\,d\sigma(y)=I_{\varepsilon,r}+II_{\varepsilon,r} \tag{2.23}$$

where

$$I_{\varepsilon,r}:=\int\limits_{\substack{y\in\Sigma\\ \varepsilon/4<|y-x|\leq r}}\frac{x_j-y_j}{|x-y|^n}[f(y)-f(x)]\,d\sigma(y), \tag{2.24}$$

$$II_{\varepsilon,r}:=f(x)\Bigg\{\int\limits_{\substack{y\in\Sigma\cap B(0,R+1)\\ \varepsilon/4<|y-x|<1}}\frac{x_j-y_j}{|x-y|^n}\,d\sigma(y)-\int\limits_{\substack{y\in\Sigma\cap B(0,R+1)\\ r<|y-x|<1}}\frac{x_j-y_j}{|x-y|^n}\,d\sigma(y)\Bigg\}. \tag{2.25}$$

The left-to-right implication in (1.2), used for the set $\Sigma \cap B(0, R+1)$, gives that σ-a.e. point $x \in \Sigma \cap B(0, R)$ has the property that for each $\delta > 0$ there exists $\theta_\delta \in (0, 1)$ such that for each $\theta_1, \theta_2 \in (0, \theta_\delta)$ we have

$$\left|\int\limits_{\substack{y\in\Sigma\cap B(0,R+1)\\ \theta_1<|y-x|<1}}\frac{x_j-y_j}{|x-y|^n}\,d\sigma(y)-\int\limits_{\substack{y\in\Sigma\cap B(0,R+1)\\ \theta_2<|y-x|<1}}\frac{x_j-y_j}{|x-y|^n}\,d\sigma(y)\right|<\delta. \tag{2.26}$$

In turn, this readily yields

$$\lim_{\varepsilon\to 0^+}\left\{\sup_{r\in(\varepsilon/2,\varepsilon)}\big|II_{\varepsilon,r}\big|\right\}=0\quad\text{for } \sigma\text{-a.e. } x\in\Sigma\cap B(0,R). \tag{2.27}$$

Next, thanks to the upper Ahlfors regularity condition satisfied by Σ, we may estimate

$$\big|I_{\varepsilon,r}\big|\leq\Big(\frac{4}{\varepsilon}\Big)^{n-1}\int_{\Sigma\cap B(x,\varepsilon)}|f(y)-f(x)|\,d\sigma(y)\leq c⨍_{\Sigma\cap B(x,\varepsilon)}|f(y)-f(x)|\,d\sigma(y), \tag{2.28}$$

where the barred integral indicates mean average. Hence, on the one hand,

$$\lim_{\varepsilon\to 0^+}\left\{\sup_{r\in(\varepsilon/2,\varepsilon)}\big|I_{\varepsilon,r}\big|\right\}=0\quad\text{if } x \text{ is a Lebesgue point for } f. \tag{2.29}$$

On the other hand, the triplet $(\Sigma, |\cdot - \cdot|, \sigma)$ is a space of homogeneous type and the underlying measure is Borel regular. As such, Lebesgue's Differentiation Theorem gives that σ-a.e. point in Σ is a Lebesgue point for f. Bearing this in mind, the desired conclusion now follows from (2.23), (2.27), and (2.29). □

In the treatment of the principal value Cauchy-Clifford integral operator in §5, the following lemma plays a significant role.

Lemma 2.4. *Let* $\Omega \subset \mathbb{R}^n$ *be a nonempty open set of locally finite perimeter such that* (2.15) *holds. Then, for each* $x \in \partial^*\Omega$, *there exists a Lebesgue measurable set* $\mathcal{O}_x \subset (0, 1)$ *of density* 1 *at* 0, *i.e., satisfying*

$$\lim_{\varepsilon\to 0^+}\frac{\mathscr{L}^1\big(\mathcal{O}_x\cap(0,\varepsilon)\big)}{\varepsilon}=1, \tag{2.30}$$

with the property that

$$\lim_{\mathcal{O}_x\ni r\to 0^+}\frac{\mathcal{H}^{n-1}\big(\Omega\cap\partial B(x,r)\big)}{\omega_{n-1}r^{n-1}}=\frac{1}{2}. \tag{2.31}$$

Proof. We largely follow [43]. Given $x\in\partial^*\Omega$, there exists an approximate tangent plane π to Ω at x (cf. the discussion on [20, p. 2627]) and we denote by $\pi^\pm$ the two half-spaces into which π divides $\mathbb{R}^n$ (with the convention that the outward unit normal to π^- is $\nu(x)$). For each $r>0$, set $\partial^\pm B(x,r):=\partial B(x,r)\cap\pi^\pm$ and introduce

$$W(x,r):=\partial^- B(x,r)\triangle\big[\Omega\cap\partial B(x,r)\big] \tag{2.32}$$

where, generally speaking, $U\triangle V$ denotes the symmetric difference $(U\setminus V)\cup(V\setminus U)$. With this notation, in the proof of Proposition 3.3 on p. 2628 of [20] it has been shown that

$$\int_0^R\mathcal{H}^{n-1}\big(W(x,r)\big)\,dr=o(R^n)\quad\text{as}\quad R\to 0^+. \tag{2.33}$$

Thus, if we consider the function

$$\phi:(0,1)\to[0,\infty)\ \text{ given by }\ \phi(r):=r^{1-n}\mathcal{H}^{n-1}\big(W(x,r)\big)\ \text{ for each }\ r\in(0,1), \tag{2.34}$$

it follows from (2.33) that

$$\int_{R/2}^R\phi(r)\,dr\le\Big(\frac{R}{2}\Big)^{1-n}\int_0^R\mathcal{H}^{n-1}\big(W(x,r)\big)\,dr=o(R)\quad\text{as}\quad R\to 0^+. \tag{2.35}$$

Bring in the dyadic intervals $I_k:=\big[2^{-(k+1)},2^{-k}\big]$ for $k\in\mathbb{N}_0$ and note that (2.35) entails

$$\delta_k:=⨍_{I_k}\phi(r)\,dr\longrightarrow 0^+\quad\text{as}\quad k\to\infty. \tag{2.36}$$

For each $k\in\mathbb{N}_0$ split

$$I_k=A_k\cup B_k,\quad\text{with}\quad B_k:=\big\{r\in I_k:\,\phi(r)>\sqrt{\delta_k}\big\}\ \text{ and }\ A_k:=I_k\setminus B_k. \tag{2.37}$$

Then Chebychev's inequality permits us to estimate

$$\frac{\mathscr{L}^1(B_k)}{\mathscr{L}^1(I_k)}\le\frac{1}{\sqrt{\delta_k}}⨍_{I_k}\phi(r)\,dr=\sqrt{\delta_k},\qquad\forall\,k\in\mathbb{N}_0. \tag{2.38}$$

In light of (2.36), this implies that if we now define

$$\mathcal{O}_x:=\bigcup_{k\in\mathbb{N}_0}A_k\subset(0,1), \tag{2.39}$$

then

$$\lim_{\mathcal{O}_x\ni r\to 0^+}\phi(r)=0. \tag{2.40}$$

We claim that (2.30) also holds for this choice of $\mathcal{O}_x$. To see that this is the case, assume that some arbitrary $\theta>0$ has been fixed. For each $\varepsilon\in(0,1)$, let $N_\varepsilon\in\mathbb{N}_0$ be such that $2^{-N_\varepsilon-1}<\varepsilon\le 2^{-N_\varepsilon}$. Since $N_\varepsilon\to\infty$ as $\varepsilon\to 0^+$, it follows from (2.36) that there exists $\varepsilon_\theta>0$ with the property that

$$\delta_k\le\theta^2\ \text{whenever}\ 0<\varepsilon<\varepsilon_\theta\ \text{and}\ k\ge N_\varepsilon. \tag{2.41}$$

Assuming that $0<\varepsilon<\varepsilon_\theta$ we may then estimate

$$\begin{aligned}0&\le\frac{\varepsilon-\mathscr{L}^1\big(\mathcal{O}_x\cap(0,\varepsilon)\big)}{\varepsilon}=\varepsilon^{-1}\mathscr{L}^1\big((0,\varepsilon)\setminus\mathcal{O}_x\big)\\&\le\varepsilon^{-1}\mathscr{L}^1\big((0,2^{-N_\varepsilon})\setminus\mathcal{O}_x\big)=\varepsilon^{-1}\sum_{k=N_\varepsilon}^\infty\mathscr{L}^1(B_k)\\&\le\varepsilon^{-1}\sum_{k=N_\varepsilon}^\infty\mathscr{L}^1(I_k)\sqrt{\delta_k}\le\varepsilon^{-1}\theta 2^{-N_\varepsilon}\le\theta/2.\end{aligned} \tag{2.42}$$

This finishes the proof of (2.30). At this stage there remains to observe that since for ecah $r \in (0,1)$ we have

$$\Big|\frac{\mathcal{H}^{n-1}\big(\Omega\cap\partial B(x,r)\big)}{\omega_{n-1}r^{n-1}}-\frac{1}{2}\Big|\leq\frac{\mathcal{H}^{n-1}\big(W(x,r)\big)}{\omega_{n-1}r^{n-1}}=\frac{1}{\omega_{n-1}}\phi(r), \tag{2.43}$$

formula (2.31) is a consequence of (2.40). □

Following G. David and S. Semmes [7] we now make the following definition.

Definition 2.5. *Call a subset Σ of $\mathbb{R}^n$ a* `uniformly rectifiable set` *provided it is Ahlfors regular and the following holds. There exist ε, $M\in(0,\infty)$ such that for each $x\in\Sigma$ and $R\in(0,\operatorname{diam}\Sigma)$, there is a Lipschitz map $\varphi: B_R^{n-1}\to\mathbb{R}^n$ (where B_R^{n-1} is a ball of radius R in $\mathbb{R}^{n-1}$) with Lipschitz constant $\leq M$, such that*

$$\mathcal{H}^{n-1}\big(\Sigma\cap B(x,R)\cap\varphi(B_R^{n-1})\big)\geq\varepsilon R^{n-1}. \tag{2.44}$$

Informally speaking, uniform rectifiability is about the ability of identifying big pieces of Lipschitz images inside the given set (in a uniform, scale invariant, fashion) and can be thought of as a quantitative version of countable rectifiability. Following [20], we shall also make the following definition.

Definition 2.6. *Call a nonempty open subset Ω of $\mathbb{R}^n$ a* `UR (uniformly rectifiable) domain` *provided $\partial\Omega$ is a uniformly rectifiable set and (2.15) holds.*

For further use, it is useful to point out that, as is apparent from definitions,

$$\begin{aligned}&\text{if }\ \Omega\subset\mathbb{R}^n \text{ is a UR domain with } \partial\Omega=\partial(\overline{\Omega})\ \text{ then}\\ &\mathbb{R}^n\setminus\overline{\Omega}\text{ is a UR domain, with the same boundary.}\end{aligned} \tag{2.45}$$

We now turn to the notion of nontangential boundary trace of functions defined in a nonempty, proper, open set $\Omega\subset\mathbb{R}^n$. Fix $\kappa>0$ and for each boundary point $x\in\partial\Omega$ introduce the nontangential approach region

$$\Gamma_\kappa(x):=\{y\in\Omega:\ |x-y|<(1+\kappa)\operatorname{dist}(y,\partial\Omega)\}. \tag{2.46}$$

It should be noted that, under the current hypotheses, it could happen that $\Gamma_\kappa(x)=\emptyset$ for points $x\in\partial\Omega$ (as is the case if, e.g., Ω has a suitable cusp with vertex at x). Next, given $u:\Omega\to\mathbb{R}$, we wish to consider $\lim\limits_{\Gamma_\kappa(x)\ni y\to x}u(y)$ for points $x\in\partial\Omega$. For this definition to be pointwise σ-a.e. meaningful (where, as usual, $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$), it is necessary that

$$x\in\overline{\Gamma_\kappa(x)}\quad\text{for }\sigma\text{-a.e.}\quad x\in\partial\Omega. \tag{2.47}$$

We shall call an open set $\Omega\subseteq\mathbb{R}^n$ satisfying (2.47) above `weakly accessible`. Assuming that this is the case, it is then meaningful to consider

$$u\Big|_{\partial\Omega}^{\rm n.t.}(x):=\lim_{\Gamma_\kappa(x)\ni y\to x}u(y)\quad\text{for }\sigma\text{-a.e. }\ x\in\partial\Omega. \tag{2.48}$$

For future use, let us also define the nontangential maximal operator of u as

$$\big(\mathcal{N}u\big)(x):=\sup_{y\in\Gamma_\kappa(x)}|u(y)|,\qquad x\in\partial\Omega, \tag{2.49}$$

with the convention that that $\big(\mathcal{N}u\big)(x):=0$ whenever $x\in\partial\Omega$ is such that $\Gamma_\kappa(x)=\emptyset$.

The following result has been proved in [20, Proposition 2.9, p. 2588].

Proposition 2.7. *Let $\Omega\subseteq\mathbb{R}^n$ be an open set with an Ahlfors regular boundary, satisfying (2.15). Then Ω is weakly accessible. As a corollary, any* UR *domain is weakly accessible.*

We continue by recording the definition of the class of uniform domains introduced by O. Martio and J. Sarvas in [28].

Definition 2.8. *Call a nonempty, proper, open set* $\Omega \subseteq \mathbb{R}^n$ *a* `uniform domain` *if there exists* $c > 0$ *with the property that*

$$\begin{gathered}\forall\, x, y \in \Omega \quad \exists\, \gamma : [0,1] \to \Omega \;\; \text{rectifiable curve joining } x \text{ and } y, \\ \text{such that } \; \mathrm{length}(\gamma) \leq c|x-y|, \quad \text{and which has the property that} \\ \min\big\{\mathrm{length}(\gamma_{x,z})\,,\,\mathrm{length}(\gamma_{z,y})\big\} \leq c\,\mathrm{dist}(z,\partial\Omega) \;\; \text{for all } \; z \in \gamma([0,1]),\end{gathered} \tag{2.50}$$

where $\gamma_{x,z}$ *and* $\gamma_{z,y}$ *are the two connected components of the path* $\gamma([0,1])$ *joining* x *with* z *and* z *with* y*, respectively.*

Condition (2.50) asserts that the length of $\gamma([0,1])$ is comparable to the distance between its endpoints and that, away from its endpoints, the curve γ stays correspondingly far from $\partial\Omega$. Hence, heuristically, condition (2.50) implies that points in Ω can be joined in Ω by a curvilinear (or twisted) double cone which is neither too crocked nor too thin. Here we wish to note that, given an open nonempty subset Ω of $\mathbb{R}^n$ with compact boundary along with some $\alpha \in (0,1)$, the following implication holds:

$$\Omega \text{ is a } \mathscr{C}^{1+\alpha} \text{ domain} \Longrightarrow \Omega \text{ is a uniform domain.} \tag{2.51}$$

Throughout, we make the convention that, given a nonempty, proper subset Ω of $\mathbb{R}^n$, we abbreviate

$$\rho(z) := \mathrm{dist}(z,\partial\Omega) \quad \text{for every} \quad z \in \Omega. \tag{2.52}$$

Lemma 2.9. *Let* $\Omega \subset \mathbb{R}^n$ *be a uniform domain. Then for each* $\alpha \in (0,1)$ *there exists a finite constant* $C > 0$*, depending only on* α *and* Ω*, such that the estimate*

$$[u]_{\dot{\mathscr{C}}^\alpha(\Omega)} \leq C \sup_{x\in\Omega} \Big\{\rho(x)^{1-\alpha}|\nabla u(x)|\Big\} \tag{2.53}$$

holds for every function $u \in \mathscr{C}^1(\Omega)$*.*

Proof. Consider $c > 0$ such that condition (2.50) is satisfied. Let then $x, y \in \Omega$ be two arbitrary points and assume that γ is as in Definition 2.8. Denote by L and s, respectively, the length of the curve $\gamma^* := \gamma([0,1])$ and the arc-length parameter on γ^*, $s \in [0,L]$. Also, let $[0,\,L] \ni s \mapsto \gamma(s) \in \gamma^*$ be the canonical arc-length parametrization of γ^*. In particular, $s \mapsto \gamma(s)$ is absolutely continuous, $\left|\frac{d\gamma}{ds}\right| = 1$ for almost every s, and for every continuous function f in Ω

$$\int_{\gamma^*} f := \int_0^L f(\gamma(s))\,ds. \tag{2.54}$$

Thus, from (2.50) and (2.54), for each $\alpha \in (0,1)$ we have

$$\begin{aligned}\int_{\gamma^*}\rho^{\alpha-1} &= \int_0^L \rho(\gamma(s))^{\alpha-1}\,ds \leq c^{1-\alpha}\int_0^L \Big[\min\{s, L-s\}\Big]^{\alpha-1}\,ds \\ &\leq 2c^{1-\alpha}\int_0^{L/2} s^{\alpha-1}\,ds = C(c,\alpha)L^\alpha \leq C(c,\alpha)|x-y|^\alpha.\end{aligned} \tag{2.55}$$

Then, since $\left|\frac{d\gamma}{ds}\right| = 1$ for almost every s, for every $u \in \mathscr{C}^1(\Omega)$ we may write

$$\begin{aligned}|u(x)-u(y)| &= \Big|\int_0^L \frac{d}{ds}\big[u(\gamma(s))\big]\,ds\Big| \\ &\leq \int_0^L \big|(\nabla u)(\gamma(s))\big|\,ds = \int_{\gamma^*}|\nabla u| \\ &\leq \sup_{\gamma^*}\Big\{|\nabla u|\,\rho^{1-\alpha}\Big\}\int_{\gamma^*}\rho^{\alpha-1}\end{aligned}$$

$$\leq C|x-y|^{\alpha}\Big\||\nabla u|\,\rho^{1-\alpha}\Big\|_{L^\infty(\Omega)}, \tag{2.56}$$

finishing the proof of (2.53). □

Recall that for each $k \in \mathbb{N}$ we let $\mathscr{L}^k$ stand for the k-dimensional Lebesgue measure in $\mathbb{R}^k$. Also, we shall let $\langle \cdot, \cdot \rangle$ denote the standard inner product of vectors in $\mathbb{R}^n$.

Lemma 2.10. *Assume that $D \subseteq \mathbb{R}^n$ is a set of locally finite perimeter. Denote by ν its geometric measure theoretic outward unit normal and define $\sigma := \mathcal{H}^{n-1}\lfloor\partial_* D$. Also, suppose that $\vec{F} \in \mathscr{C}^1_0(\mathbb{R}^n, \mathbb{R}^n)$. Then for each $x \in \mathbb{R}^n$,*

$$\int_{D\cap B(x,r)} \mathrm{div}\vec{F}\, d\mathscr{L}^n = \int_{\partial_* D\cap B(x,r)} \langle \vec{F}, \nu\rangle\, d\sigma + \int_{D\cap\partial B(x,r)} \langle \vec{F}, \nu\rangle\, d\mathcal{H}^{n-1} \tag{2.57}$$

and

$$\int_{D\setminus B(x,r)} \mathrm{div}\vec{F}\, d\mathscr{L}^n = \int_{\partial_* D\setminus B(x,r)} \langle \vec{F}, \nu\rangle\, d\sigma - \int_{D\cap\partial B(x,r)} \langle \vec{F}, \nu\rangle\, d\mathcal{H}^{n-1} \tag{2.58}$$

for $\mathscr{L}^1$-a.e. $r \in (0,\infty)$, where ν in each of the last integrals in the above right hand-sides is the outward unit normal to $B(x,r)$.

Proof. Identity (2.57) is simply [11, Lemma 1, p. 195]. Then (2.58) follows by combining this with the Gauss-Green formula from [11, Theorem 1, p. 209]. □

We conclude this section by recording the following two-dimensional result which is going to be relevant when dealing with the proof of Theorem 1.2.

Proposition 2.11. *Let $\Omega \subseteq \mathbb{C}$ be a bounded open set whose boundary is an upper Ahlfors regular Jordan curve. Then Ω is a simply connected* UR *domain satisfying $\partial\Omega = \partial(\overline{\Omega})$. Hence, in particular, $\mathcal{H}^1(\partial\Omega \setminus \partial_*\Omega) = 0$ and $\mathbb{C}\setminus\overline{\Omega}$ is also a* UR *domain with the same boundary as Ω.*

Moreover, the curve $\partial\Omega$ is rectifiable and if L denotes its length and $[0,L] \ni s \mapsto z(s) \in \Sigma$ is its arc-length parametrization, then

$$\mathscr{H}^1(E) = \mathscr{L}^1(z^{-1}(E)), \qquad \forall\, E \subseteq \partial\Omega \ \text{ measurable set} \tag{2.59}$$

where $\mathscr{L}^1$ is the one-dimensional Lebesgue measure, and if ν denotes the geometric measure theoretic outward unit normal to Ω then

$$\nu(z(s)) = -i\, z'(s) \quad \text{for } \mathscr{L}^1\text{-a.e. } s \in [0,L]. \tag{2.60}$$

A proof of Proposition 2.11 may be found in [34].

3. Background and Preparatory Estimates for Singular Integrals

The proofs of the main results require a number of prerequisites, and this section collects several useful estimates for singular integral operators. The following theorem, essentially amounting to a version of the Calderón-Zygmund theory for singular integrals in UR domains, is contained in [20, Theorem 3.33, p. 2669] (where a more general version applicable to variable coefficient operators can be found).

Theorem 3.1. *There exists a positive integer $N = N(n)$ with the following significance. Suppose that $\Omega \subseteq \mathbb{R}^n$ is a* UR *domain and denote by $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and ν, respectively, the surface measure and outward unit normal on $\partial\Omega$. Next, consider a function*

$$\begin{aligned} &k \in \mathscr{C}^N(\mathbb{R}^n\setminus\{0\}) \ \text{ satisfying } \ k(-x) = -k(x) \ \text{ for each } \ x \in \mathbb{R}^n \\ &\text{and so that } \ k(\lambda x) = \lambda^{-(n-1)}k(x) \ \text{ for all } \ \lambda > 0, \ \ x \in \mathbb{R}^n\setminus\{0\}, \end{aligned} \tag{3.1}$$

and define the integral operators

$$\mathcal{T}f(x) := \int_{\partial\Omega} k(x-y)f(y)\, d\sigma(y). \qquad x \in \Omega. \tag{3.2}$$

Also, for each $\varepsilon > 0$, consider the truncated singular integral operator

$$T_\varepsilon f(x) := \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} k(x-y)f(y)\,d\sigma(y), \qquad x \in \partial\Omega, \tag{3.3}$$

and define the maximal operator T_* by setting

$$T_* f(x) := \sup_{\varepsilon>0} |T_\varepsilon f(x)|, \qquad x \in \partial\Omega. \tag{3.4}$$

Then for each $p \in (1,\infty)$ one can select a constant $C \in (0,\infty)$ depending only on p along with the Ahlfors regularity and UR *constants of $\partial\Omega$, for which*

$$\|T_* f\|_{L^p(\partial\Omega,\sigma)} \le C\|k|_{S^{n-1}}\|_{\mathscr{C}^N}\|f\|_{L^p(\partial\Omega,\sigma)} \tag{3.5}$$

for every $f \in L^p(\partial\Omega,\sigma)$. Furthermore, for each $p \in [1,\infty)$, $f \in L^p(\partial\Omega,\sigma)$, the limit

$$Tf(x) := \lim_{\varepsilon\to 0^+} T_\varepsilon f(x) \tag{3.6}$$

exists for σ-a.e. $x \in \partial\Omega$ and the induced operators

$$T : L^p(\partial\Omega,\sigma) \longrightarrow L^p(\partial\Omega,\sigma), \quad p \in (1,\infty), \tag{3.7}$$

$$T : L^1(\partial\Omega,\sigma) \longrightarrow L^{1,\infty}(\partial\Omega,\sigma), \tag{3.8}$$

are well-defined, linear and bounded.

In addition, for each $p \in (1,\infty)$ there exists a finite constant $C = C(p,\Omega)$ such that

$$\|\mathcal{N}(\mathcal{T}f)\|_{L^p(\partial\Omega,\sigma)} \le C\|k|_{S^{n-1}}\|_{\mathscr{C}^N}\|f\|_{L^p(\partial\Omega,\sigma)} \tag{3.9}$$

and, corresponding to $p = 1$,

$$\|\mathcal{N}(\mathcal{T}f)\|_{L^{1,\infty}(\partial\Omega,\sigma)} \le C(\Omega,k,\kappa)\|f\|_{L^1(\partial\Omega,\sigma)}. \tag{3.10}$$

Also, the jump-formula

$$\Big(\mathcal{T}f\big|^{\rm n.t.}_{\partial\Omega}\Big)(x) = \lim_{\substack{\Omega\ni z\to x\\ z\in\Gamma_\kappa(x)}} \mathcal{T}f(z) = \frac{1}{2i}\widehat{k}(\nu(x))f(x) + Tf(x) \tag{3.11}$$

is valid at σ-a.e. point $x \in \partial\Omega$ (where 'hat' denotes the Fourier transform in $\mathbb{R}^n$ and $i := \sqrt{-1} \in \mathbb{C}$). Finally, for each $p \in (1,\infty)$, the adjoint of the operator T acting on $L^p(\partial\Omega,\sigma)$ is $-T$ acting on $L^{p'}(\partial\Omega,\sigma)$ with $1/p + 1/p' = 1$.

The Fourier transform in $\mathbb{R}^n$ employed in (3.11) is

$$\widehat{\phi}(\xi) := \int_{\mathbb{R}^n} e^{-i\langle x,\,\xi\rangle}\phi(x)\,dx, \qquad \xi \in \mathbb{R}^n. \tag{3.12}$$

Let us also remark here that the hypotheses (3.1) imposed on the kernel k imply that $|k(x)| \le \|k\|_{L^\infty(S^{n-1})}|x|^{1-n}$ for each $x \in \mathbb{R}^n \setminus \{0\}$. Hence, k is a tempered distribution in $\mathbb{R}^n$ and $\widehat{k}$, originally considered in the class of tempered distributions in $\mathbb{R}^n$, satisfies

$$\begin{gathered} \widehat{k} \in \mathscr{C}^m(\mathbb{R}^n\setminus\{0\}) \ \text{ if } N \in \mathbb{N} \text{ is even} \\ \text{and } m \in \mathbb{N}_0 \text{ is such that } m < N-1 \end{gathered} \tag{3.13}$$

(cf. [32, Exercise 4.60, p. 133]). In particular, (3.13) ensures that $\widehat{k}(\nu(x))$ is meaningfully defined in (3.11) for σ-a.e. $x \in \partial\Omega$ whenever $N \ge 2$.

Lemma 3.2. *Suppose Ω is a nonempty proper open subset of $\mathbb{R}^n$ with a compact boundary, satisfying an upper Ahlfors regularity condition with constant $c \in (0,\infty)$. In this setting, define $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and consider an integral operator*

$$\mathscr{T}f(x) := \int_{\partial\Omega} k(x,y)f(y)\,d\sigma(y), \qquad x \in \Omega, \tag{3.14}$$

whose kernel $k : \Omega \times \partial\Omega \to \mathbb{R}$ *has the property that there exists some finite positive constant* C_0 *such that*

$$|k(x,y)| \leq \frac{C_0}{|x-y|^{n-1}} \tag{3.15}$$

for each $x \in \Omega$ *and* σ*-a.e.* $y \in \partial\Omega$*. Also, suppose that*

$$\sup_{x\in\Omega} \big|\mathscr{T}1(x)\big| < +\infty. \tag{3.16}$$

Then for every $\alpha \in (0,1)$ *one has*

$$\begin{aligned} \sup_{x\in\Omega} |\mathscr{T}f(x)| &\leq c\,C_0 \frac{2^{2n-2+\alpha}}{2^\alpha - 1}\big(1 + [\mathrm{diam}(\partial\Omega)]^\alpha\big)[f]_{\dot{\mathscr{C}}^\alpha(\partial\Omega)} \\ &\quad + \Big(\|\mathscr{T}1\|_{L^\infty(\Omega)} + c\,C_0\,\big[\mathrm{diam}\,(\partial\Omega)\big]^{n-1}\Big)\|f\|_{L^\infty(\partial\Omega)}, \end{aligned} \tag{3.17}$$

for every $f \in \mathscr{C}^\alpha(\partial\Omega)$*.*

Proof. Pick an arbitrary $f \in \mathscr{C}^\alpha(\partial\Omega)$ and fix any $x \in \Omega$. Consider first the case when $\mathrm{dist}(x,\partial\Omega) \geq 1$, in which scenario we may directly estimate

$$|\mathscr{T}f(x)| \leq C_0\,\sigma(\partial\Omega)\|f\|_{L^\infty(\partial\Omega)} \leq c\,C_0\,\big[\mathrm{diam}\,(\partial\Omega)\big]^{n-1}\|f\|_{L^\infty(\partial\Omega)}. \tag{3.18}$$

In the case when $\mathrm{dist}(x,\partial\Omega) < 1$, select a point $x_* \in \partial\Omega$ such that

$$|x - x_*| = \mathrm{dist}(x,\partial\Omega) =: r \in (0,1) \tag{3.19}$$

and split $\mathscr{T}f(x) = I + II + III$, where

$$I := \int_{\partial\Omega\cap B(x_*,2r)} k(x,y)\big[f(y) - f(x_*)\big]\,d\sigma(y), \tag{3.20}$$

$$II := \int_{\partial\Omega\setminus B(x_*,2r)} k(x,y)\big[f(y) - f(x_*)\big]\,d\sigma(y), \tag{3.21}$$

and

$$III := (\mathscr{T}1)(x)f(x_*). \tag{3.22}$$

Note that

$$\begin{aligned} |I| &\leq \int_{\partial\Omega\cap B(x_*,2r)} |k(x,y)|\,|f(y) - f(x_*)|\,d\sigma(y) \\ &\leq C_0[f]_{\dot{\mathscr{C}}^\alpha(\partial\Omega)} \int_{\partial\Omega\cap B(x_*,2r)} \frac{|y-x_*|^\alpha}{|x-y|^{n-1}}\,d\sigma(y) \\ &\leq C_0[f]_{\dot{\mathscr{C}}^\alpha(\partial\Omega)} \frac{(2r)^\alpha}{r^{n-1}}\,\sigma\big(\partial\Omega\cap B(x_*,2r)\big), \end{aligned} \tag{3.23}$$

where the third inequality comes from the fact that $|y - x_*|^\alpha \leq (2r)^\alpha$ on the domain of integration, and the fact that $1/|x-y| \leq 1/|x-x_*| = 1/r$, for all $y \in \partial\Omega$. Hence,

$$|I| \leq 2^{n-1+\alpha} c\,C_0[f]_{\dot{\mathscr{C}}^\alpha(\partial\Omega)}, \tag{3.24}$$

bearing in mind (3.19) and the upper Ahlfors regularity of $\partial\Omega$. Also,

$$\big|II\big| \leq C_0[f]_{\dot{\mathscr{C}}^\alpha(\partial\Omega)} \int_{\partial\Omega\setminus B(x_*,2r)} \frac{|y-x_*|^\alpha}{|x-y|^{n-1}}\,d\sigma(y). \tag{3.25}$$

Note that if $y \in \partial\Omega \setminus B(x_*,2r)$ then

$$|y - x_*| \leq |y-x| + |x-x_*| \quad \text{and} \quad r \leq \frac{|y-x_*|}{2} \implies |y-x_*| \leq 2|y-x|. \tag{3.26}$$

Hence, $1/|x-y|^{n-1} \leq 2^{n-1}/|y-x_*|^{n-1}$ on the domain of integration $\partial\Omega \setminus B(x_*, 2r)$. Also, if we introduce

$$N := \Big[\log_2\big(\tfrac{\mathrm{diam}(\partial\Omega)}{r}\big)\Big] \in \mathbb{N}, \tag{3.27}$$

then $\partial\Omega \setminus B(x_*, 2^k r) = \emptyset$ for each integer $k > N$. Together, these observations and (3.25) allow us to estimate

$$\begin{aligned} |II| &\leq 2^{n-1} C_0 [f]_{\dot{\mathscr{C}}^\alpha(\partial\Omega)} \int_{\partial\Omega \setminus B(x_*, 2r)} \frac{|y-x_*|^\alpha}{|y-x_*|^{n-1}}\, d\sigma(y) \\ &\leq 2^{n-1} C_0 [f]_{\dot{\mathscr{C}}^\alpha(\partial\Omega)} \sum_{k=1}^{N} \int_{\partial\Omega \cap [B(x_*, 2^{k+1} r) \setminus B(x_*, 2^k r)]} \frac{1}{|y-x_*|^{n-1-\alpha}}\, d\sigma(y) \\ &\leq 2^{n-1} C_0 [f]_{\dot{\mathscr{C}}^\alpha(\partial\Omega)} \sum_{k=1}^{N} (2^k r)^{-(n-1-\alpha)} \sigma\big(\partial\Omega \cap B(x_*, 2^{k+1} r)\big). \end{aligned} \tag{3.28}$$

Thus, by the upper Ahlfors regularity condition,

$$\begin{aligned} |II| &\leq 2^{n-1} C_0 [f]_{\dot{\mathscr{C}}^\alpha(\partial\Omega)} \sum_{k=1}^{N} (2^k r)^{-(n-1-\alpha)} c (2^{k+1} r)^{n-1} \\ &= 2^{2n-2} c\, C_0 r^\alpha [f]_{\dot{\mathscr{C}}^\alpha(\partial\Omega)} \sum_{k=1}^{N} (2^\alpha)^k \\ &\leq 2^{2n-2+\alpha} c\, C_0 r^\alpha [f]_{\dot{\mathscr{C}}^\alpha(\partial\Omega)} \frac{(2^N)^\alpha}{2^\alpha - 1} \\ &\leq \frac{2^{2n-2+\alpha}}{2^\alpha - 1} c\, C_0 [f]_{\dot{\mathscr{C}}^\alpha(\partial\Omega)} \big[\mathrm{diam}\,(\partial\Omega)\big]^\alpha. \end{aligned} \tag{3.29}$$

Since, clearly, $|III| \leq \|\mathscr{T} 1\|_{L^\infty(\Omega)} \|f\|_{L^\infty(\partial\Omega)}$, the desired conclusion follows. □

Lemma 3.3. *Retain the same assumptions on Ω as in Lemma 3.2 and consider an integral operator*

$$\mathscr{Q} f(x) := \int_{\partial\Omega} q(x,y) f(y)\, d\sigma(y), \qquad x \in \Omega, \tag{3.30}$$

whose kernel $q : \Omega \times \partial\Omega \to \mathbb{R}$ is assumed to satisfy

$$|q(x,y)| \leq \frac{C_1}{|x-y|^n}, \qquad \forall\, x \in \Omega, \quad \forall\, y \in \partial\Omega, \tag{3.31}$$

for some finite positive constant C_1. Also, with ρ as in (2.52), suppose

$$C_2 := \sup_{x \in \Omega} \Big\{ \rho(x)^{1-\alpha} |(\mathscr{Q} 1)(x)| \Big\} < +\infty. \tag{3.32}$$

Then for every $\alpha \in (0,1)$ one has

$$\sup_{x \in \Omega} \Big\{ \rho(x)^{1-\alpha} |\mathscr{Q} f(x)| \Big\} \leq \frac{2^{2n-1+\alpha}}{1 - 2^{\alpha-1}} c\, C_1 [f]_{\dot{\mathscr{C}}^\alpha(\partial\Omega)} + C_2 \|f\|_{L^\infty(\partial\Omega)}, \tag{3.33}$$

for every $f \in \mathscr{C}^\alpha(\partial\Omega)$.

Proof. Select an arbitrary $f \in \mathscr{C}^\alpha(\partial\Omega)$. Pick some $x \in \Omega$ and choose $x_* \in \partial\Omega$ such that $|x - x_*| = \rho(x) =: r$. Split $\mathscr{Q} f(x) = I + II + III$, where

$$I := \int_{\partial\Omega \cap B(x_*, 2r)} q(x,y) \big[f(y) - f(x_*)\big]\, d\sigma(y), \tag{3.34}$$

$$II := \int_{\partial\Omega \setminus B(x_*, 2r)} q(x,y) \big[f(y) - f(x_*)\big]\, d\sigma(y), \tag{3.35}$$

and

$$III := (\mathscr{Q}1)(x)f(x_*). \tag{3.36}$$

Then

$$\begin{aligned}|I| &\le \int_{\partial\Omega\cap B(x_*,2r)} |q(x,y)|\,|f(y)-f(x_*)|\,d\sigma(y)\\ &\le C_1[f]_{\dot{\mathscr{C}}^\alpha(\partial\Omega)}\int_{\partial\Omega\cap B(x_*,2r)}\frac{|y-x_*|^\alpha}{|x-y|^n}\,d\sigma(y)\\ &\le C_1[f]_{\dot{\mathscr{C}}^\alpha(\partial\Omega)}\frac{(2r)^\alpha}{r^n}\,\sigma\big(\partial\Omega\cap B(x_*,2r)\big)\\ &\le 2^{n-1+\alpha}c\,C_1\rho(x)^{\alpha-1}[f]_{\dot{\mathscr{C}}^\alpha(\partial\Omega)}.\end{aligned} \tag{3.37}$$

Next, keeping in mind that $1/|x-y|^n \le 2^n/|y-x_*|^n$ on $\partial\Omega\setminus B(x_*,2r)$ (cf. (3.26)), we may estimate

$$\begin{aligned}|II| &\le C_1[f]_{\dot{\mathscr{C}}^\alpha(\partial\Omega)}\int_{\partial\Omega\setminus B(x_*,2r)}\frac{|y-x_*|^\alpha}{|x-y|^n}\,d\sigma(y).\\ &\le 2^nC_1[f]_{\dot{\mathscr{C}}^\alpha(\partial\Omega)}\int_{\partial\Omega\setminus B(x_*,2r)}\frac{|y-x_*|^\alpha}{|y-x_*|^n}\,d\sigma(y)\\ &\le 2^nC_1[f]_{\dot{\mathscr{C}}^\alpha(\partial\Omega)}\sum_{k=1}^\infty\int_{\partial\Omega\cap[B(x_*,2^{k+1}r)\setminus B(x_*,2^kr)]}\frac{1}{|y-x_*|^{n-\alpha}}\,d\sigma(y)\\ &\le 2^nC_1[f]_{\dot{\mathscr{C}}^\alpha(\partial\Omega)}\sum_{k=1}^\infty(2^kr)^{-(n-\alpha)}\sigma\big(\partial\Omega\cap B(x_*,2^{k+1}r)\big)\\ &\le 2^nC_1[f]_{\dot{\mathscr{C}}^\alpha(\partial\Omega)}\sum_{k=1}^\infty(2^kr)^{-(n-\alpha)}c(2^{k+1}r)^{n-1}\\ &= 2^{2n-1}c\,C_1r^{\alpha-1}[f]_{\dot{\mathscr{C}}^\alpha(\partial\Omega)}\sum_{k=1}^\infty(2^{\alpha-1})^k\\ &= \frac{2^{2n-2+\alpha}}{1-2^{\alpha-1}}c\,C_1\rho(x)^{\alpha-1}[f]_{\dot{\mathscr{C}}^\alpha(\partial\Omega)}.\end{aligned} \tag{3.38}$$

Given that $\rho(x)^{1-\alpha}|III| \le C_2\|f\|_{L^\infty(\partial\Omega)}$, estimate (3.33) is established. □

Lemma 3.4. *Let Ω be a nonempty open proper subset of $\mathbb{R}^n$ whose boundary is compact and satisfies an upper Ahlfors regularity condition with constant $c\in(0,\infty)$. In this setting, define $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and consider an integral operator*

$$\mathcal{T}f(x) := \int_{\partial\Omega}K(x,y)f(y)\,d\sigma(y), \qquad x\in\Omega, \tag{3.39}$$

whose kernel $K:\Omega\times\partial\Omega\to\mathbb{R}$ has the property that there exists a finite constant $B>0$ such that

$$|K(x,y)| + |x-y||\nabla_xK(x,y)| \le \frac{B}{|x-y|^{n-1}} \tag{3.40}$$

for each $x\in\Omega$ and σ-a.e. $y\in\partial\Omega$. Fix some $\alpha\in(0,1)$ and suppose that

$$A := \sup_{x\in\Omega}\big|(\mathcal{T}1)(x)\big| + \sup_{x\in\Omega}\Big\{\rho(x)^{1-\alpha}\big|\nabla(\mathcal{T}1)(x)\big|\Big\} < +\infty. \tag{3.41}$$

Then for every $f \in \mathscr{C}^\alpha(\partial\Omega)$ one has

$$\begin{aligned}\sup_{x\in\Omega} |\mathcal{T}f(x)| + \sup_{x\in\Omega} \Big\{\rho(x)^{1-\alpha}\big|\nabla(\mathcal{T}f)(x)\big|\Big\} \\ \leq c\,B\,C_{n,\alpha}\big(2+[\mathrm{diam}(\partial\Omega)]^\alpha\big)[f]_{\dot{\mathscr{C}}^\alpha(\partial\Omega)} \\ + \big(2A + c\,B[\mathrm{diam}(\partial\Omega)]^{n-1}\big)\|f\|_{L^\infty(\partial\Omega)},\end{aligned} \tag{3.42}$$

where

$$C_{n,\alpha} := 2^{2n-2-\alpha}\max\big\{(2^\alpha-1)^{-1}\,,\,2(1-2^{\alpha-1})^{-1}\big\}. \tag{3.43}$$

As a consequence, there exists a finite constant $C_{n,\alpha,\Omega} > 0$ with the property that for every $f \in \mathscr{C}^\alpha(\partial\Omega)$ one has

$$\sup_{x\in\Omega} |\mathcal{T}f(x)| + \sup_{x\in\Omega} \Big\{\rho(x)^{1-\alpha}\big|\nabla(\mathcal{T}f)(x)\big|\Big\} \leq C_{n,\alpha,\Omega}(A+B)\|f\|_{\mathscr{C}^\alpha(\partial\Omega)}. \tag{3.44}$$

Proof. This is an immediate consequence of Lemma 3.2 and Lemma 3.3. □

4. Clifford Analysis

A key tool for us is Clifford analysis, and here we elaborate on those aspects used in the proof of Theorem 1.1. To begin, the `Clifford algebra` with n imaginary units is the minimal enlargement of $\mathbb{R}^n$ to a unitary real algebra $(\mathcal{C}\ell_n, +, \odot)$, which is not generated (as an algebra) by any proper subspace of $\mathbb{R}^n$, and such that

$$x \odot x = -|x|^2 \quad \text{for any } \; x \in \mathbb{R}^n \hookrightarrow \mathcal{C}\ell_n. \tag{4.1}$$

This identity readily implies that, if $\{e_j\}_{1\leq j\leq n}$ is the standard orthonormal basis in $\mathbb{R}^n$, then

$$e_j \odot e_j = -1 \quad \text{and} \quad e_j \odot e_k = -e_k \odot e_j \;\; \text{whenever} \;\; 1 \leq j \neq k \leq n. \tag{4.2}$$

In particular, identifying the canonical basis $\{e_j\}_{1\leq j\leq n}$ from $\mathbb{R}^n$ with the n imaginary units generating $\mathcal{C}\ell_n$, yields the embedding[4]

$$\mathbb{R}^n \hookrightarrow \mathcal{C}\ell_n, \qquad \mathbb{R}^n \ni x = (x_1, \dots, x_n) \equiv \sum_{j=1}^n x_j e_j \in \mathcal{C}\ell_n. \tag{4.3}$$

Also, any element $u \in \mathcal{C}\ell_n$ can be uniquely represented in the form

$$u = \sum_{l=0}^n {\sum_{|I|=l}}' u_I\, e_I, \quad u_I \in \mathbb{R}. \tag{4.4}$$

Here e_I stands for the product $e_{i_1} \odot e_{i_2} \odot \cdots \odot e_{i_l}$ if $I = (i_1, i_2, \dots, i_l)$ and $e_0 := e_\emptyset := 1$ is the multiplicative unit. Also, $\sum'$ indicates that the sum is performed only over strictly increasing multi-indices, i.e., $I = (i_1, i_2, \dots, i_l)$ with $1 \leq i_1 < i_2 < \cdots < i_l \leq n$. We endow $\mathcal{C}\ell_n$ with the natural Euclidean metric

$$|u| := \Big\{\sum_I |u_I|^2\Big\}^{1/2} \;\; \text{for each} \;\; u = \sum_I u_I e_I \in \mathcal{C}\ell_n. \tag{4.5}$$

The Clifford conjugation on $\mathcal{C}\ell_n$, denoted by 'bar', is defined as the unique real-linear involution on $\mathcal{C}\ell_n$ for which $\overline{e_I}e_I = e_I\overline{e_I} = 1$ for any multi-index I. More specifically,

[4]As the alerted reader might have noted, for $n = 2$ the identification in (4.3) amounts to embedding $\mathbb{R}^2$ into quaternions, i.e., $\mathbb{R}^2 \hookrightarrow \mathbb{H} := \{x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} : \, x_0, x_1, x_2, x_3 \in \mathbb{R}\}$ via $\mathbb{R}^2 \ni (x_1, x_2) \equiv x_1\mathbf{i} + x_2\mathbf{j} \in \mathbb{H}$. The reader is reassured that this is simply a matter of convenience, and we might as well have arranged so that the embedding (4.3) comes down, when $n = 2$, to perhaps the more familiar identification $\mathbb{R}^2 \equiv \mathbb{C}$, by taking $\mathbb{R}^n \ni x = (x_0, x_1, \dots, x_{n-1}) \equiv x_0 + x_1e_1 + \dots x_{n-1}e_{n-1} \in \mathcal{C}\ell_{n-1}$. The latter choice leads to a parallel theory to the one presented here, entailing only minor natural alterations.

given $u=\sum_I u_I e_I \in \mathcal{C}\!\ell_n$ we set $\overline{u} := \sum_I u_I \overline{e_I}$ where, for each $I=(i_1,i_2,\dots,i_l)$ with $1\le i_1<i_2<\cdots<i_l\le n$,

$$\overline{e_I} = (-1)^l e_{i_l}\odot e_{i_{l-1}}\odot\cdots\odot e_{i_1}. \tag{4.6}$$

Let us also define the scalar part of $u=\sum_I u_I e_I \in \mathcal{C}\!\ell_n$ as $u_0 := u_\emptyset$, and endow $\mathcal{C}\!\ell_n$ with the natural Hilbert space structure

$$\langle u,v\rangle := \sum_I u_I v_I, \qquad \text{if} \ \ u=\sum_I u_I e_I,\ v=\sum_I v_I e_I \in \mathcal{C}\!\ell_n. \tag{4.7}$$

It follows directly from definitions that

$$\overline{x} = -x \ \ \text{for each} \ \ x\in\mathbb{R}^n \hookrightarrow \mathcal{C}\!\ell_n, \tag{4.8}$$

and other properties are collected in the lemma below.

Lemma 4.1. *For any $u,v\in\mathcal{C}\!\ell_n$ one has*

$$|u|^2 = (u\odot\overline{u})_0 = (\overline{u}\odot u)_0, \tag{4.9}$$

$$\langle u,v\rangle = (u\odot\overline{v})_0 = (\overline{u}\odot v)_0, \tag{4.10}$$

$$\overline{u\odot v} = \overline{v}\odot\overline{u}, \tag{4.11}$$

$$|\overline{u}| = |u|, \tag{4.12}$$

$$|u\odot v| \le 2^{n/2}|u||v|, \tag{4.13}$$

and

$$|u\odot v| = |u||v| \ \ \textit{if either } u \textit{ or } v \textit{ belongs to } \mathbb{R}^n\hookrightarrow\mathcal{C}\!\ell_n. \tag{4.14}$$

Proof. Properties (4.9)-(4.11) are straightforward consequences of definitions. To justify (4.13), assume $u=\sum_I u_I e_I\in\mathcal{C}\!\ell_n$ and $v=\sum_J v_J e_J\in\mathcal{C}\!\ell_n$ have been given. Then

$$\begin{aligned} |u\odot v| &= \Big|\sum_I\Big(\sum_J u_I v_J e_I\odot e_J\Big)\Big| \le \sum_I\Big|\sum_J u_I v_J e_I\odot e_J\Big| \\ &= \sum_I\Big(\sum_J |u_I v_J|^2\Big)^{1/2} = |v|\sum_I |u_I| \le |v|\Big(\sum_I |u_I|^2\Big)^{1/2}\Big(\sum_I 1\Big)^{1/2} \\ &= 2^{n/2}|u||v|. \end{aligned} \tag{4.15}$$

Above, the triangle inequality has been employed in the second step. The third step relies on (4.5) and the observation that, for each I fixed, the family of Clifford algebra elements $\{e_I\odot e_J\}_J$ coincides modulo signs with the orthonormal basis $\{e_K\}_K$. The penultimate step is the discrete Cauchy-Schwarz inequality.

As regards (4.14), assume that $v\in\mathbb{R}^n\hookrightarrow\mathcal{C}\!\ell_n$ and write

$$\begin{aligned} |u\odot v|^2 &= \big((u\odot v)\odot\overline{u\odot v}\big)_0 = \big(u\odot(v\odot\overline{v})\odot\overline{u}\big)_0 \\ &= |v|^2(u\odot\overline{u})_0 = |u|^2|v|^2, \end{aligned} \tag{4.16}$$

by (4.9), (4.11), (4.8), and (4.1). Finally, the case when $u\in\mathbb{R}^n\hookrightarrow\mathcal{C}\!\ell_n$ follows from what we have just proved with the help of (4.11) and (4.12). □

Next, recall the `Dirac operator`

$$D := \sum_{j=1}^n e_j\partial_j. \tag{4.17}$$

In the sequel, we shall use D_L and D_R to denote the action of D on a $\mathscr{C}^1$ function $u:\Omega\to\mathcal{C}\!\ell_n$ (where Ω is an open subset of $\mathbb{R}^n$) from the left and from the right, respectively. For a sufficiently nice domain Ω with outward unit normal $\nu=(\nu_1,\dots,\nu_n)$ (identified with the

$\mathcal{C}\ell_n$-valued function $\nu = \sum_{j=1}^n \nu_j e_j$) and surface measure σ, and for any two reasonable $\mathcal{C}\ell_n$-valued functions u, v in Ω, the following integration by parts formula holds:

$$\begin{aligned}\int_{\partial\Omega} & u(x) \odot \nu(x) \odot v(x)\, d\sigma(x) \\ & = \int_\Omega \Big\{ (D_R u)(x) \odot v(x) + u(x) \odot (D_L v)(x) \Big\}\, dx. \end{aligned} \tag{4.18}$$

More detailed accounts of these and related matters can be found in [4] and [35]. In general, if $\big(\mathscr{X}, \|\cdot\|_{\mathscr{X}}\big)$ is a Banach space then by $\mathscr{X} \otimes \mathcal{C}\ell_n$ we shall denote the Banach space consisting of elements of the form

$$u = \sum_{l=0}^n {\sum_{|I|=l}}' u_I\, e_I, \quad u_I \in \mathscr{X}, \tag{4.19}$$

equipped with the natural norm

$$\|u\|_{\mathscr{X} \otimes \mathcal{C}\ell_n} := \sum_{l=0}^n {\sum_{|I|=l}}' \|u_I\|_{\mathscr{X}}. \tag{4.20}$$

A simple but useful observation in this context is that

$$\begin{aligned} & \text{if } \Omega \subset \mathbb{R}^n \text{ is a domain of class } \mathscr{C}^{1+\alpha} \text{ for some } \alpha \in (0,1) \text{ then} \\ & \nu\odot : \mathscr{C}^\alpha(\partial\Omega) \otimes \mathcal{C}\ell_n \longrightarrow \mathscr{C}^\alpha(\partial\Omega) \otimes \mathcal{C}\ell_n \quad \text{is an isomorphism} \\ & \text{whose norm and the norm of its inverse are } \leq 2\|\nu\|_{\mathscr{C}^\alpha(\partial\Omega)}. \end{aligned} \tag{4.21}$$

Indeed, by (4.1), its inverse is $-\nu\odot$ and the aforementioned norm estimates are simple consequences of (4.14), bearing in mind that $\|\nu\|_{\mathscr{C}^\alpha(\partial\Omega)} \geq 1$.

For each $s \in \{1, \dots, n\}$ we let $[\,\cdot\,]_s$ denote the projection onto the s-th Euclidean coordinate, i.e., $[x]_s := x_s$ if $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. The following lemma, in the spirit of work of Semmes in [41], will play an important role for us.

Lemma 4.2. *For any odd, harmonic, homogeneous polynomial $P(x)$, $x \in \mathbb{R}^n$ (with $n \geq 2$), of degree $l \geq 3$, there exist a family $P_{rs}(x)$, $1 \leq r, s \leq n$, of harmonic, homogeneous polynomials of degree $l - 2$, as well as a family of odd, $\mathscr{C}^\infty$ functions*

$$k_{rs} : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \hookrightarrow \mathcal{C}\ell_n, \qquad 1 \leq r, s \leq n, \tag{4.22}$$

which are homogeneous of degree $-(n-1)$, and for each $x \in \mathbb{R}^n \setminus \{0\}$ satisfy

$$\frac{P(x)}{|x|^{n-1+l}} = \sum_{r,s=1}^n [k_{rs}(x)]_s \quad \textit{and} \tag{4.23}$$

$$(D_R k_{rs})(x) = \frac{l-1}{n+l-3} \frac{\partial}{\partial x_r} \left(\frac{P_{rs}(x)}{|x|^{n+l-3}} \right), \quad 1 \leq r, s \leq n. \tag{4.24}$$

Moreover, there exists a finite dimensional constant $c_n > 0$ such that

$$\max_{1 \leq r,s \leq n} \|k_{rs}\|_{L^\infty(S^{n-1})} + \max_{1 \leq r,s \leq n} \|\nabla k_{rs}\|_{L^\infty(S^{n-1})} \leq c_n\, 2^l \|P\|_{L^1(S^{n-1})}. \tag{4.25}$$

Proof. Given now an odd, harmonic, homogeneous polynomial $P(x)$ of degree $l \geq 3$ in $\mathbb{R}^n$, for $r, s \in \{1, \dots, n\}$ introduce

$$P_{rs}(x) := \frac{1}{l(l-1)} (\partial_r \partial_s P)(x), \qquad \forall\, x \in \mathbb{R}^n. \tag{4.26}$$

Then each P_{rs} is an odd, harmonic, homogeneous polynomial of degree $l - 2$ in $\mathbb{R}^n$, and Euler's formula for homogeneous functions gives

$$P(x) = \sum_{r,s=1}^n x_r x_s P_{rs}(x), \qquad \forall\, x \in \mathbb{R}^n, \tag{4.27}$$

and, for each $r,s\in\{1,\dots,n\}$,

$$\langle(\nabla P_{rs})(x),x\rangle=(l-2)P_{rs}(x),\qquad\forall\,x\in\mathbb{R}^n. \tag{4.28}$$

To proceed, assume first that $n\geq 3$ and, for each $r,s\in\{1,\dots,n\}$, define the function $k_{rs}:\mathbb{R}^n\setminus\{0\}\longrightarrow\mathbb{R}^n\hookrightarrow\mathcal{C}\ell_n$ by setting

$$k_{rs}(x):=\frac{1}{(n+l-3)(n+l-5)}\sum_{j=1}^{n}\partial_r\partial_j\Big(\frac{P_{rs}(x)}{|x|^{n+l-5}}\Big)e_j,\qquad\forall\,x\in\mathbb{R}^n\setminus\{0\}. \tag{4.29}$$

The fact that $n,l\geq 3$ ensure that both $n+l-3\neq 0$ and $n+l-5\neq 0$ so each k_{rs} is well-defined, odd, $\mathscr{C}^\infty$ and homogeneous of degree $-(n-1)$ in $\mathbb{R}^n\setminus\{0\}$. In addition,

$$k_{rs}(x)=\frac{1}{(n+l-3)(n+l-5)}D_R\Big[\partial_r\Big(\frac{P_{rs}(x)}{|x|^{n+l-5}}\Big)\Big],\qquad\forall\,x\in\mathbb{R}^n\setminus\{0\}, \tag{4.30}$$

hence for all $x\in\mathbb{R}^n\setminus\{0\}$ we may write

$$\begin{aligned}(D_Rk_{rs})(x)&=\frac{1}{(n+l-3)(n+l-5)}D_R^2\Big[\partial_r\Big(\frac{P_{rs}(x)}{|x|^{n+l-5}}\Big)\Big]\\&=\frac{-1}{(n+l-3)(n+l-5)}\Delta\Big[\partial_r\Big(\frac{P_{rs}(x)}{|x|^{n+l-5}}\Big)\Big]\\&=:I+II+III,\end{aligned} \tag{4.31}$$

where

$$\begin{aligned}I&:=\frac{-1}{(n+l-3)(n+l-5)}\partial_r\Big[\frac{(\Delta P_{rs})(x)}{|x|^{n+l-5}}\Big]=0,\\II&:=\frac{-1}{(n+l-3)(n+l-5)}\partial_r\Big[2\big\langle(\nabla P_{rs})(x),\nabla\big[|x|^{-(n+l-5)}\big]\big\rangle\Big]\\&=\frac{2}{n+l-3}\partial_r\Big[\frac{\big\langle(\nabla P_{rs})(x),x\big\rangle}{|x|^{n+l-3}}\Big]\\&=\frac{2(l-2)}{n+l-3}\partial_r\Big[\frac{P_{rs}(x)}{|x|^{n+l-3}}\Big],\\III&:=\frac{-1}{(n+l-3)(n+l-5)}\partial_r\Big[P_{rs}(x)\Delta\big[|x|^{-(n+l-5)}\big]\Big]\\&=\frac{-l+3}{n+l-3}\partial_r\Big[\frac{P_{rs}(x)}{|x|^{n+l-3}}\Big],\end{aligned} \tag{4.32}$$

by the harmonicity of P, (4.28), and straightforward algebra. This proves that (4.23) holds when $n\geq 3$. Going further, from (4.29) and the fact that

$$\sum_{r=1}^{n}(\partial_rP_{rs})(x)=\sum_{s=1}^{n}(\partial_sP_{rs})(x)=0\ \text{ and }\ \sum_{r=1}^{n}P_{rr}(x)=0 \tag{4.33}$$

(as seen from (4.26) and the harmonicity of P), we deduce that for each $x\in\mathbb{R}^n\setminus\{0\}$

$$\begin{aligned}\sum_{r,s=1}^{n}[k_{rs}(x)]_s&=\frac{1}{(n+l-3)(n+l-5)}\sum_{r,s=1}^{n}\partial_r\partial_s\Big(\frac{P_{rs}(x)}{|x|^{n+l-5}}\Big)\\&=\frac{1}{(n+l-3)(n+l-5)}\sum_{r,s=1}^{n}P_{rs}(x)\partial_r\partial_s\big[|x|^{-(n+l-5)}\big]\\&=\frac{-1}{n+l-3}\sum_{r,s=1}^{n}P_{rs}(x)\Big\{\frac{\delta_{rs}}{|x|^{n+l-3}}-(n+l-3)\frac{x_rx_s}{|x|^{n+l-1}}\Big\}\end{aligned}$$

$$= \frac{P(x)}{|x|^{n-1+l}}. \tag{4.34}$$

This establishes (4.24) for $n \geq 3$. Moving on, for each $\gamma \in \mathbb{N}_0^n$, interior estimates for the harmonic function P give

$$\begin{aligned} \|\partial^\gamma P\|_{L^\infty(S^{n-1})} &\leq c_{n,\gamma} \int_{B(0,2)} |P(x)|\,dx = c_{n,\gamma} \int_{S^{n-1}} |P(\omega)| \Big(\int_0^2 r^{n-1+l}\,dr \Big)\,d\omega \\ &= c_{n,\gamma} \frac{2^l}{n+l} \|P\|_{L^1(S^{n-1})}, \end{aligned} \tag{4.35}$$

where we have also used the fact that P is homogeneous of degree l. The estimates in (4.25) now readily follow on account of (4.29), (4.26), and (4.35).

To treat the two-dimensional case, first we observe that if $Q_m(x)$ is an arbitrary homogeneous polynomial of degree $m \in \mathbb{N}_0$ in $\mathbb{R}^n$ with $n \geq 2$ and $\lambda > 0$ then

$$\frac{Q_m(x)}{|x|^{n+m-\lambda}} \text{ is a tempered distribution in } \mathbb{R}^n. \tag{4.36}$$

If, in addition, $Q_m(x)$ is harmonic and $\lambda < n$ then (cf. [42, p. 73]) also

$$\mathcal{F}_{x\to\xi}\Big(\frac{Q_m(x)}{|x|^{n+m-\lambda}} \Big) = \gamma_{n,m,\lambda} \frac{Q_m(\xi)}{|\xi|^{m+\lambda}} \text{ as tempered distributions in } \mathbb{R}^n, \tag{4.37}$$

where $\mathcal{F}_{x\to\xi}$ is an alternative notation for the Fourier transform in $\mathbb{R}^n$ from (3.12) and

$$\gamma_{n,m,\lambda} := (-1)^{3m/2} \pi^{n/2} 2^\lambda \frac{\Gamma(m/2+\lambda/2)}{\Gamma(m/2+n/2-\lambda/2)}. \tag{4.38}$$

Pick now an odd, harmonic, homogeneous polynomial $P(x)$ of degree $l \geq 3$ in $\mathbb{R}^2$ and define P_{rs} for $r, s \in \{1, \dots, n\}$ as in (4.26). Hence, once again each P_{rs} is an odd, harmonic, homogeneous polynomial of degree $l-2$ in $\mathbb{R}^2$, and (4.27) holds. Moreover, (4.37) used for $n = 2$, $m = l-2$, $\lambda = 1$, and $Q_m = P_{rs}$ yields

$$\frac{P_{rs}(x)}{|x|^{l-1}} = -(-1)^{3l/2} 2\pi\, \mathcal{F}^{-1}_{\xi\to x}\Big(\frac{P_{rs}(\xi)}{|\xi|^{l-1}} \Big). \tag{4.39}$$

Now, for each $r, s \in \{1, 2\}$ define the function $k_{rs} : \mathbb{R}^2 \setminus \{0\} \longrightarrow \mathbb{R}^2 \hookrightarrow \mathcal{C}\ell_2$ by setting

$$k_{rs}(x) := (-1)^{3l/2} 2\pi \sum_{j=1}^2 \mathcal{F}^{-1}_{\xi\to x}\Big(\xi_r \xi_j \frac{P_{rs}(\xi)}{|\xi|^{l+1}} \Big) e_j, \qquad \forall\, x \in \mathbb{R}^2 \setminus \{0\}. \tag{4.40}$$

By (4.36) used with $n = 2$, $m = l$, $\lambda = 1$, and $Q_m(\xi) = \xi_r \xi_j P_{rs}(\xi)$, it follows that $\xi_r \xi_j \frac{P_{rs}(\xi)}{|\xi|^{l+1}}$ is a tempered distribution in $\mathbb{R}^2$. Consequently, k_{rs} in (4.40) is meaningfully defined and, from [32, Proposition 4.58, p. 132], we deduce that $k_{rs} \in \mathscr{C}^\infty(\mathbb{R}^2 \setminus \{0\})$. Also, based on standard properties of the Fourier transform (cf., e.g., [32, Chapter 4]) it follows that k_{rs} is odd and homogeneous of degree -1 in $\mathbb{R}^2 \setminus \{0\}$. In addition,

$$\begin{aligned} (D_R k_{rs})(x) &= (-1)^{3l/2} 2\pi \sum_{\ell,j=1}^2 \partial_{x_\ell} \mathcal{F}^{-1}_{\xi\to x}\Big(\xi_r \xi_j \frac{P_{rs}(\xi)}{|\xi|^{l+1}} \Big) e_j \odot e_\ell \\ &= \sqrt{-1}(-1)^{3l/2} 2\pi \sum_{\ell,j=1}^2 \mathcal{F}^{-1}_{\xi\to x}\Big(\xi_r \xi_j \xi_\ell \frac{P_{rs}(\xi)}{|\xi|^{l+1}} \Big) e_j \odot e_\ell =: I + II, \end{aligned} \tag{4.41}$$

where I and II are the pieces produced by summing up over $j = \ell$ and $j \neq \ell$, respectively. Since in the latter scenario $\xi_\ell \xi_j = \xi_j \xi_\ell$ while $e_j \odot e_\ell = -e_\ell \odot e_j$ it follows that $II = 0$. Given that $e_j \odot e_j = -1$ for each $j \in \{1,2\}$, we conclude that

$$(D_R k_{rs})(x) = -\sqrt{-1}(-1)^{3l/2} 2\pi \sum_{j=1}^{2} \mathcal{F}^{-1}_{\xi\to x}\Big(\xi_r \xi_j^2 \frac{P_{rs}(\xi)}{|\xi|^{l+1}}\Big)$$
$$= -\sqrt{-1}(-1)^{3l/2} 2\pi\, \mathcal{F}^{-1}_{\xi\to x}\Big(\xi_r \frac{P_{rs}(\xi)}{|\xi|^{l-1}}\Big)$$
$$= -(-1)^{3l/2} 2\pi\, \partial_{x_r}\Big[\mathcal{F}^{-1}_{\xi\to x}\Big(\frac{P_{rs}(\xi)}{|\xi|^{l-1}}\Big)\Big] = \partial_{x_r}\Big[\frac{P_{rs}(x)}{|x|^{l-1}}\Big], \tag{4.42}$$

where the last step uses (4.39). Hence, (4.23) holds when $n = 2$. Finally, from (4.29), (4.27), and (4.37) (used for P) we deduce that for each $x \in \mathbb{R}^2 \setminus \{0\}$ we have

$$\sum_{r,s=1}^{2} [k_{rs}(x)]_s = (-1)^{3l/2} 2\pi \sum_{r,s=1}^{2} \mathcal{F}^{-1}_{\xi\to x}\Big(\xi_r \xi_s \frac{P_{rs}(\xi)}{|\xi|^{l+1}}\Big)$$
$$= (-1)^{3l/2} 2\pi\, \mathcal{F}^{-1}_{\xi\to x}\Big(\frac{P(\xi)}{|\xi|^{l+1}}\Big) = \frac{P(x)}{|x|^{l+1}}. \tag{4.43}$$

This establishes (4.24) when $n = 2$.

At this stage, there remains to justify (4.25) in the case $n = 2$. To this end, pick $\psi \in \mathscr{C}^\infty_0(\mathbb{R}^2)$ with $0 \le \psi \le 1$, $\psi = 1$ on $B(0,1)$ and $\psi = 0$ on $\mathbb{R}^2 \setminus \overline{B(0,2)}$. Fix $r, s, j \in \{1,2\}$ and abbreviate $u(\xi) := \xi_r \xi_j P_{rs}(\xi)/|\xi|^{l+1}$ for $\xi \in \mathbb{R}^2 \setminus \{0\}$. Then u is locally integrable and defines a tempered distribution in $\mathbb{R}^2$. Hence, for each $\alpha \in \mathbb{N}_0^2$ with $|\alpha| = 2$ and $\xi \in \overline{B(0,1)}$ we may write

$$\big|\mathcal{F}_{x\to\xi}\big(\psi(x)\partial^\alpha u(x)\big)\big| = \big|\big\langle \psi\partial^\alpha u, e^{-i\langle\xi,\cdot\rangle}\big\rangle\big| = \big|\big\langle u, \partial^\alpha\big(\psi e^{-i\langle\xi,\cdot\rangle}\big)\big\rangle\big| \tag{4.44}$$
$$\le C\int_{B(0,2)} |u(x)|\,dx \le C\int_{S^1} |P_{rs}(\omega)|\,d\omega \le C2^l \|P\|_{L^1(S^1)},$$

and

$$\big|\mathcal{F}_{x\to\xi}\big((1-\psi(x))\partial^\alpha u(x)\big)\big| \le \|(1-\psi)\partial^\alpha u\|_{L^1(\mathbb{R}^2)} \le \int_{\mathbb{R}^2\setminus B(0,1)} |\partial^\alpha u(x)|\,dx$$
$$\le C\int_{S^1} |\partial^\alpha u(\omega)|\,d\omega \le C2^l \|P\|_{L^1(S^1)}. \tag{4.45}$$

Collectively, (4.44) and (4.45) give that, for each $\alpha \in \mathbb{N}_0^2$ with $|\alpha| = 2$ and $\xi \in \overline{B(0,1)}$,

$$\big|\mathcal{F}_{x\to\xi}\big(\partial^\alpha u(x)\big)\big| \le \big|\mathcal{F}_{x\to\xi}\big(\psi(x)\partial^\alpha u(x)\big)\big| + \big|\mathcal{F}_{x\to\xi}\big((1-\psi(x))\partial^\alpha u(x)\big)\big|$$
$$\le C2^l \|P\|_{L^1(S^1)}, \tag{4.46}$$

hence for each $\xi \in \overline{B(0,1)}$ we have

$$|\xi|^2 \big|\widehat{u}(\xi)\big| = \sum_{\ell=1}^{2} \big|\xi_\ell^2 \widehat{u}(\xi)\big| = \sum_{\ell=1}^{2} \big|\mathcal{F}_{x\to\xi}\big(\partial_\ell^2 u(x)\big)\big| \le C2^l \|P\|_{L^1(S^1)}. \tag{4.47}$$

In particular, $\|k_{rs}\|_{L^\infty(S^1)} \le C \sup_{|\xi|=1} \big|\widehat{u}(\xi)\big| \le C2^l \|P\|_{L^1(S^1)}$. A similar circle of ideas also yields $\|\nabla k_{rs}\|_{L^\infty(S^1)} \le C2^l \|P\|_{L^1(S^1)}$. This proves (4.25) in the case $n = 2$ and completes the proof of the lemma. □

5. Cauchy-Clifford Operators on Hölder Spaces

Let $\Omega\subset\mathbb{R}^n$ be a set of locally finite perimeter satisfying (2.15). As before, we shall denote by $\nu=(\nu_1,\dots,\nu_n)$ the outward unit normal to Ω and by $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$ the surface measure on $\partial\Omega$. Then the (boundary-to-domain) `Cauchy-Clifford operator` and its principal value (or, boundary-to-boundary) version associated with Ω are, respectively, given by

$$\mathcal{C}f(x):=\frac{1}{\omega_{n-1}}\int_{\partial\Omega}\frac{x-y}{|x-y|^n}\odot\nu(y)\odot f(y)\,d\sigma(y),\qquad x\in\Omega, \tag{5.1}$$

and

$$\mathbf{C}f(x):=\lim_{\varepsilon\to 0^+}\frac{1}{\omega_{n-1}}\int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}}\frac{x-y}{|x-y|^n}\odot\nu(y)\odot f(y)\,d\sigma(y),\qquad x\in\partial\Omega, \tag{5.2}$$

where f is a $\mathcal{C}\!\ell_n$-valued function defined on $\partial\Omega$. At the present time, these definitions are informal as more conditions need to be imposed on the function f and the underlying domain Ω in order to ensure that these operators are well-defined and enjoy desirable properties in various settings of interest. We start by recording the following result, in the context of uniformly rectifiable domains.

Proposition 5.1. *Let $\Omega\subset\mathbb{R}^n$ be a* UR *domain. Then for every $f\in L^p(\partial\Omega,\sigma)\otimes\mathcal{C}\!\ell_n$ with $p\in[1,\infty)$, the function $\mathbf{C}f$ is meaningfully defined σ-a.e. on $\partial\Omega$, and the actions of the two Cauchy-Clifford operators on f are related via the boundary behavior*

$$\Big(\mathcal{C}f\Big|_{\partial\Omega}^{\rm n.t.}\Big)(x):=\lim_{\Gamma_\kappa(x)\ni z\to x}\mathcal{C}f(z)=\big(\tfrac12 I+\mathbf{C}\big)f(x),\quad \sigma\text{-a.e. } x\in\partial\Omega, \tag{5.3}$$

where I is the identity operator. Moreover, for each $p\in(1,\infty)$, there exists a finite constant $M=M(n,p,\Omega)>0$ such that

$$\|\mathcal{N}(\mathcal{C}f)\|_{L^p(\partial\Omega,\sigma)}\le M\,\|f\|_{L^p(\partial\Omega,\sigma)\otimes\mathcal{C}\!\ell_n}, \tag{5.4}$$

the operator $\mathbf{C}$ is well-defined and bounded on $L^p(\partial\Omega,\sigma)\otimes\mathcal{C}\!\ell_n$, and the Poincaré-Bertrand formula[5] on Lebesgue spaces

$$\mathbf{C}^2=\tfrac14 I\quad\text{on}\quad L^p(\partial\Omega,\sigma)\otimes\mathcal{C}\!\ell_n \tag{5.5}$$

holds.

Proof. With the exception of (5.5) (which has been proved in [20]; cf. also [34] for very general results of this type), all claims follow from Theorem 3.1. □

The goal in this section is to prove similar results when the Lebesgue scale is replaced by Hölder spaces, in a class of domains considerably more general than the category of uniformly rectifiable domains. We begin by proving the following result.

Lemma 5.2. *Let $\Omega\subseteq\mathbb{R}^n$ be a Lebesgue measurable set whose boundary is compact and upper Ahlfors regular (hence, in particular, Ω is of locally finite perimeter by (2.21)). Denote by ν the geometric measure theoretic outward unit normal to Ω and define $\sigma:=\mathcal{H}^{n-1}\lfloor\partial_*\Omega$. Then there exists a number $N=N(n,c)\in(0,\infty)$, depending only on the dimension n and the upper Ahlfors regularity constant c of $\partial\Omega$, with the property that*

$$\Big|\int_{\partial_*\Omega\setminus B(x,r)}\frac{x-y}{|x-y|^n}\odot\nu(y)\,d\sigma(y)\Big|\le N,\qquad\forall\,x\in\mathbb{R}^n,\quad\forall\,r\in(0,\infty). \tag{5.6}$$

[5] concerning the superposition of singular integrals

Proof. We shall first show that, whenever $\Omega \subseteq \mathbb{R}^n$ is a bounded set of locally finite perimeter, having fixed an arbitrary $x \in \mathbb{R}^n$, for $\mathscr{L}^1$-a.e. $\varepsilon > 0$ we have

$$\int_{\partial_*\Omega\setminus B(x,\varepsilon)} \frac{x-y}{|x-y|^n} \odot \nu(y)\, d\sigma(y) = \int_{\Omega\cap\partial B(x,\varepsilon)} \frac{x-y}{|x-y|^n} \odot \nu(y)\, d\mathcal{H}^{n-1}(y)$$
$$= \frac{\mathcal{H}^{n-1}\big(\Omega\cap\partial B(x,\varepsilon)\big)}{\varepsilon^{n-1}}. \tag{5.7}$$

To justify this claim, we start by noting that the second equality (which holds for any measurable set $\Omega \subset \mathbb{R}^n$) is an immediate consequence of the fact that

$$y \in \partial B(x,\varepsilon) \ \text{ implies } \ (x-y)\odot\nu(y) = (x-y)\odot(y-x)/\varepsilon = \varepsilon. \tag{5.8}$$

As regards the first equality in (5.7), for each $j,k \in \{1,\dots,n\}$ consider the vector field

$$\vec{F}_{jk}(y) := \Big(0,\dots,0,\frac{x_j-y_j}{|x-y|^n},0,\dots,0\Big), \qquad \forall\, y\in\mathbb{R}^n\setminus\{x\}, \tag{5.9}$$

with the non-zero component on the k-th slot. Thus, we have $\vec{F}_{jk} \in \mathscr{C}^1(\mathbb{R}^n\setminus\{x\},\mathbb{R}^n)$ and, if E_Δ stands for the standard fundamental solution for the Laplacian $\Delta = \partial_1^2 + \cdots \partial_n^2$ in $\mathbb{R}^n$, given by

$$E_\Delta(x) := \begin{cases} \dfrac{1}{\omega_{n-1}(2-n)}\dfrac{1}{|x|^{n-2}} & \text{if } n\geq 3,\\[2ex] \dfrac{1}{2\pi}\ln|x| & \text{if } n=2,\end{cases} \qquad \forall\, x\in\mathbb{R}^n\setminus\{0\}, \tag{5.10}$$

then

$$(\operatorname{div}\vec{F}_{jk})(y) = \omega_{n-1}(\partial_j\partial_k E_\Delta)(x-y) \qquad \forall\, y \in \mathbb{R}^n\setminus\{x\}. \tag{5.11}$$

As a consequence, in $\mathbb{R}^n\setminus\{x\}$ we have

$$\sum_{j,k=1}^n (\operatorname{div}\vec{F}_{jk})e_j\odot e_k = \sum_{1\leq j\neq k\leq n}(\operatorname{div}\vec{F}_{jk})e_j\odot e_k - \sum_{j=1}^n \operatorname{div}\vec{F}_{jj}$$
$$= \omega_{n-1}\sum_{1\leq j\neq k\leq n}(\partial_j\partial_k E_\Delta)(x-\cdot)e_j\odot e_k - \omega_{n-1}(\Delta E_\Delta)(x-\cdot)$$
$$= 0, \tag{5.12}$$

using the fact that $e_j\odot e_k = -e_k\odot e_j$ for $j\neq k$ and the harmonicity of $E_\Delta(x-\cdot)$ in $\mathbb{R}^n\setminus\{x\}$.

At this stage, fix an arbitrary $\varepsilon_o \in (0,\infty)$ and alter each $\vec{F}_{jk}$ both inside $B(x,\varepsilon_o)$ and outside an open neighborhood of $\overline{\Omega}$ to a vector field $\vec{G}_{jk}\in\mathscr{C}^1_0(\mathbb{R}^n,\mathbb{R}^n)$ (this is possible given the working assumption that Ω is bounded). Then for $\mathscr{L}^1$-a.e. $\varepsilon\in(\varepsilon_o,\infty)$ based on the formula (2.58) used for $\vec{F} := \vec{G}_{jk}$, $D := \Omega$, and $r := \varepsilon$ we may write

$$0 = \sum_{j,k=1}^n\Big(\int_{\Omega\setminus B(x,\varepsilon)}\operatorname{div}\vec{F}_{jk}\,d\mathscr{L}^n\Big)e_j\odot e_k = \sum_{j,k=1}^n\Big(\int_{\Omega\setminus B(x,\varepsilon)}\operatorname{div}\vec{G}_{jk}\,d\mathscr{L}^n\Big)e_j\odot e_k$$
$$= \sum_{j,k=1}^n\Big(\int_{\partial_*\Omega\setminus B(x,\varepsilon)}\langle\vec{G}_{jk},\nu\rangle\,d\sigma\Big)e_j\odot e_k - \sum_{j,k=1}^n\Big(\int_{\Omega\cap\partial B(x,\varepsilon)}\langle\vec{G}_{jk},\nu\rangle\,d\mathcal{H}^{n-1}\Big)e_j\odot e_k$$
$$= \sum_{j,k=1}^n\Big(\int_{\partial_*\Omega\setminus B(x,\varepsilon)}\langle\vec{F}_{jk},\nu\rangle\,d\sigma\Big)e_j\odot e_k - \sum_{j,k=1}^n\Big(\int_{\Omega\cap\partial B(x,\varepsilon)}\langle\vec{F}_{jk},\nu\rangle\,d\mathcal{H}^{n-1}\Big)e_j\odot e_k$$
$$= \sum_{j,k=1}^n\Big(\int_{\partial_*\Omega\setminus B(x,\varepsilon)}\frac{(x_j-y_j)\nu_k(y)}{|x-y|^n}\,d\sigma(y)\Big)e_j\odot e_k$$

$$
\begin{aligned}
&-\sum_{j,k=1}^{n}\Big(\int_{\Omega\cap\partial B(x,\varepsilon)}\frac{(x_j-y_j)\nu_k(y)}{|x-y|^n}\,d\mathcal{H}^{n-1}(y)\Big)e_j\odot e_k\\
&=\int_{\partial_*\Omega\setminus B(x,\varepsilon)}\frac{x-y}{|x-y|^n}\odot\nu(y)\,d\sigma(y)-\int_{\Omega\cap\partial B(x,\varepsilon)}\frac{x-y}{|x-y|^n}\odot\nu(y)\,d\mathcal{H}^{n-1}(y). \qquad (5.13)
\end{aligned}
$$

With this in hand, the first equality in (5.7) readily follows. Thus, (5.7) is fully proved.

To proceed, assume that $\Omega\subseteq\mathbb{R}^n$ is a bounded Lebesgue measurable set whose boundary is upper Ahlfors regular. Then (5.7) implies that for each $x\in\mathbb{R}^n$

$$
\Big|\int_{\partial_*\Omega\setminus B(x,\varepsilon)}\frac{x-y}{|x-y|^n}\odot\nu(y)\,d\sigma(y)\Big|\leq\frac{\mathcal{H}^{n-1}\big(\partial B(x,\,\varepsilon)\big)}{\varepsilon^{n-1}}=\omega_{n-1}, \qquad (5.14)
$$

for $\mathscr{L}^1$-a.e. $\varepsilon>0$. Fix now $x\in\mathbb{R}^n$ and pick an arbitrary $r\in(0,\infty)$. Based on (5.14) we conclude that there exists $\varepsilon\in(r/2,r)$ such that

$$
\Big|\int_{\partial_*\Omega\setminus B(x,\varepsilon)}\frac{x-y}{|x-y|^n}\odot\nu(y)\,d\sigma(y)\Big|\leq\omega_{n-1}. \qquad (5.15)
$$

For this choice of ε we may then estimate

$$
\begin{aligned}
&\Big|\int_{\partial_*\Omega\setminus B(x,r)}\frac{x-y}{|x-y|^n}\odot\nu(y)\,d\sigma(y)\Big|\\
&\qquad\leq\Big|\int_{\partial_*\Omega\setminus B(x,\varepsilon)}\frac{x-y}{|x-y|^n}\odot\nu(y)\,d\sigma(y)\Big|\\
&\qquad\quad+\Big|\int_{[B(x,r)\setminus B(x,\varepsilon)]\cap\partial_*\Omega}\frac{x-y}{|x-y|^n}\odot\nu(y)\,d\sigma(y)\Big|\\
&\qquad\leq\omega_{n-1}+\int_{[B(x,r)\setminus B(x,\varepsilon)]\cap\partial\Omega}\frac{d\mathcal{H}^{n-1}(y)}{|x-y|^{n-1}}\\
&\qquad\leq\omega_{n-1}+\int_{[B(x,2\varepsilon)\setminus B(x,\varepsilon)]\cap\partial\Omega}\frac{d\mathcal{H}^{n-1}(y)}{|x-y|^{n-1}}\\
&\qquad\leq\omega_{n-1}+\varepsilon^{-(n-1)}\mathcal{H}^{n-1}\big(B(x,\,2\varepsilon)\cap\partial\Omega\big). \qquad (5.16)
\end{aligned}
$$

If $\operatorname{dist}(x,\partial\Omega)\leq 2\varepsilon$, pick a point $x_0\in\partial\Omega$ such that $\operatorname{dist}(x,\partial\Omega)=|x-x_0|$. In particular, $|x-x_0|\leq 2\varepsilon$ which forces $B(x,2\varepsilon)\subseteq B(x_0,4\varepsilon)$. As such,

$$
\mathcal{H}^{n-1}\big(B(x,2\varepsilon)\cap\partial\Omega\big)\leq\mathcal{H}^{n-1}\big(B(x_0,4\varepsilon)\cap\partial\Omega\big)\leq c(4\varepsilon)^{n-1}, \qquad (5.17)
$$

with $c\in(0,\infty)$ standing for the upper Ahlfors regularity constant of $\partial\Omega$. On the other hand, if $\operatorname{dist}(x,\partial\Omega)>2\varepsilon$ then $\mathcal{H}^{n-1}\big(B(x,2\varepsilon)\cap\partial\Omega\big)=0$. Thus, taking $N:=\omega_{n-1}+c\,4^{n-1}$ the desired conclusion follows from (5.16) and (5.17), in the case when Ω is as in the statement of the lemma and also bounded.

Finally, when Ω is as in the statement of the lemma but unbounded, consider $\Omega^c:=\mathbb{R}^n\setminus\Omega$. Then $\Omega^c\subseteq\mathbb{R}^n$ is a bounded, Lebesgue measurable set, with the property that $\partial(\Omega^c)=\partial\Omega$ and $\partial_*(\Omega^c)=\partial_*\Omega$. Moreover, the geometric measure theoretic outward unit normal to Ω^c is $-\nu$. Then (5.6) follows from what we have proved so far applied to Ω^c. □

It is clear from (5.1) that, the boundary-to-domain Cauchy-Clifford operator is well-defined on $L^1(\partial\Omega,\sigma)$. To state our next lemma, recall that $\rho(\cdot)$ has been introduced in (2.52).

Lemma 5.3. *Let $\Omega\subset\mathbb{R}^n$ be a Lebesgue measurable set whose boundary is compact, upper Ahlfors regular, and satisfies (2.15). Then the Cauchy-Clifford operator (5.1) has the* *property that*

$$\mathcal{C}1 = \begin{cases} 1 & \textit{in } \Omega \quad \textit{if } \Omega \textit{ bounded,} \\ 0 & \textit{in } \Omega \quad \textit{if } \Omega \textit{ unbounded,} \end{cases} \tag{5.18}$$

and for each $\alpha \in (0,1)$ *there exists a finite* $M > 0$*, depending only on* n*,* α*,* $\operatorname{diam}(\partial\Omega)$*, and the upper Ahlfors regularity constant of* $\partial\Omega$*, such that for every* $f \in \mathscr{C}^\alpha(\partial\Omega) \otimes \mathcal{C}\!\ell_n$ *one has*

$$\sup_{x\in\Omega} \big|(\mathcal{C}f)(x)\big| + \sup_{x\in\Omega} \Big\{\rho(x)^{1-\alpha}\big|\nabla(\mathcal{C}f)(x)\big|\Big\} \le M\,\|f\|_{\mathscr{C}^\alpha(\partial\Omega)\otimes\mathcal{C}\!\ell_n}. \tag{5.19}$$

Proof. The fact that $\mathcal{C}1 = 1$ in Ω when Ω is bounded follows from (5.7), written for $x \in \Omega$ and suitably small $\varepsilon > 0$. That $(\mathcal{C}1)(x) = 0$ for each $x \in \Omega$ when Ω is unbounded also follows from formula (5.7), this time considered for the bounded set $\Omega^c := \mathbb{R}^n \setminus \Omega$ (since in this case $\Omega^c \cap \partial B(x,\varepsilon) = \emptyset$ if $\varepsilon > 0$ is sufficiently small). Having proved (5.18), then (5.19) follows with the help of Lemma 3.4. □

In contrast to Lemma 5.3 (cf. also Lemma 5.4 below), we note that there exists a bounded open set $\Omega \subset \mathbb{R}^2 \equiv \mathbb{C}$ whose boundary is a rectifiable Jordan curve, and there exists a complex-valued function $f \in \mathscr{C}^{1/2}(\partial\Omega)$ with the property that the boundary-to-domain Cauchy operator naturally associated with Ω acting on f is actually an unbounded function in Ω. See the discussion in [9], [10].

Lemma 5.4. *Let* $\Omega \subset \mathbb{R}^n$ *be a uniform domain whose boundary is compact, upper Ahlfors regular, and satisfies* (2.15)*. Then the boundary-to-domain Cauchy-Clifford operator has the property that for each* $\alpha \in (0,1)$ *is well-defined, linear, and bounded in the context*

$$\mathcal{C} : \mathscr{C}^\alpha(\partial\Omega) \otimes \mathcal{C}\!\ell_n \longrightarrow \mathscr{C}^\alpha\big(\overline{\Omega}\big) \otimes \mathcal{C}\!\ell_n, \tag{5.20}$$

with operator norm controlled in terms of n*,* α*,* $\operatorname{diam}(\partial\Omega)$*, and the upper Ahlfors regularity constant of* $\partial\Omega$*.*

Proof. This is a direct consequence of Lemma 5.3 and Lemma 2.9. □

In the class of UR domains with compact boundaries that are also uniform domains, it follows from Lemma 5.4 and the jump-formula (5.3) that the principal value Cauchy-Clifford operator $\mathbf{C}$ defines a bounded mapping from $\mathscr{C}^\alpha(\partial\Omega) \otimes \mathcal{C}\!\ell_n$ into itself for each $\alpha \in (0,1)$. The goal is to prove that this boundedness result actually holds under much more relaxed background assumptions on the underlying domain. In this regard, a key aspect has to do with the action of $\mathbf{C}$ on constants. Note that when $\Omega \subset \mathbb{R}^n$ is a UR domain with compact boundary, it follows from (5.18) and (5.3) that the principal value Cauchy-Clifford operator satisfies

$$\mathbf{C}1 = \begin{cases} +\frac{1}{2} & \text{on } \partial\Omega \quad \text{if } \Omega \text{ is bounded,} \\ -\frac{1}{2} & \text{on } \partial\Omega \quad \text{if } \Omega \text{ is unbounded.} \end{cases} \tag{5.21}$$

The lemma below establishes a formula similar in spirit in a much larger class of domains.

Lemma 5.5. *Let* $\Omega \subset \mathbb{R}^n$ *be a Lebesgue measurable set whose boundary is compact, Ahlfors regular, and such that* (2.15) *is satisfied (hence, in particular,* Ω *has locally finite perimeter). As in the past, consider* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ *and let* ν *denote the outward unit normal to* Ω*. Then for* σ*-a.e.* $x \in \partial\Omega$ *there holds*

$$\lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \frac{x-y}{|x-y|^n} \odot \nu(y)\, d\sigma(y) = \begin{cases} +\frac{1}{2} & \textit{if } \Omega \textit{ is bounded,} \\ -\frac{1}{2} & \textit{if } \Omega \textit{ is unbounded.} \end{cases} \tag{5.22}$$

Proof. Consider first the case when Ω is bounded. Fix $x \in \partial^*\Omega$ and pick an arbitrary $\delta > 0$. From Lemma 2.4 we know that there exist $\mathcal{O}_x \subset (0,1)$ of density 1 at 0 (i.e., satisfying

(2.30)) and some $r_\delta > 0$ with the property that

$$\left|\frac{\mathcal{H}^{n-1}\big(\Omega\cap\partial B(x,r)\big)}{\omega_{n-1}r^{n-1}}-\frac{1}{2}\right|<\delta,\qquad \forall\, r\in\mathcal{O}_x\cap(0,r_\delta). \tag{5.23}$$

Since (2.30) entails

$$\begin{aligned}\lim_{\varepsilon\to 0^+}&\frac{\mathscr{L}^1\big(\mathcal{O}_x\cap(\varepsilon/2,\varepsilon)\big)}{\varepsilon}\\ &=\lim_{\varepsilon\to 0^+}\frac{\mathscr{L}^1\big(\mathcal{O}_x\cap(0,\varepsilon)\big)}{\varepsilon}-\lim_{\varepsilon\to 0^+}\frac{\mathscr{L}^1\big(\mathcal{O}_x\cap(0,\varepsilon/2)\big)}{\varepsilon}\\ &=1-\frac{1}{2}=\frac{1}{2},\end{aligned} \tag{5.24}$$

it follows that there exists $\varepsilon_\delta\in(0,r_\delta)$ with the property that

$$\mathscr{L}^1\big(\mathcal{O}_x\cap(\varepsilon/2,\varepsilon)\big)>0,\qquad \forall\,\varepsilon\in(0,\varepsilon_\delta). \tag{5.25}$$

From our assumptions on Ω and (5.7) we also know that

$$\exists\, N_x\subset(0,\infty)\text{ with }\mathscr{L}^1(N_x)=0\text{ such that }\forall\, r\in(0,\infty)\setminus N_x\text{ we have}$$

$$\frac{1}{\omega_{n-1}}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>r}}\frac{x-y}{|x-y|^n}\odot\nu(y)\,d\sigma(y)=\frac{\mathcal{H}^{n-1}\big(\Omega\cap\partial B(x,r)\big)}{\omega_{n-1}r^{n-1}}. \tag{5.26}$$

Consider next $\varepsilon\in(0,\varepsilon_\delta)$ and note that $\big[\mathcal{O}_x\cap(\varepsilon/2,\varepsilon)\big]\setminus N_x\neq\emptyset$, thanks to (5.25). As such, it is possible to select $r\in\big[\mathcal{O}_x\cap(\varepsilon/2,\varepsilon)\big]\setminus N_x$ for which we then write

$$\begin{aligned}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon/4}}\frac{x-y}{|x-y|^n}\odot\nu(y)\,d\sigma(y)=&\int\limits_{\substack{y\in\partial\Omega\\ r\geq|x-y|>\varepsilon/4}}\frac{x-y}{|x-y|^n}\odot\nu(y)\,d\sigma(y)\\ &+\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>r}}\frac{x-y}{|x-y|^n}\odot\nu(y)\,d\sigma(y).\end{aligned} \tag{5.27}$$

In turn, (5.27), (5.26), and (5.23) permit us to estimate

$$\begin{aligned}&\left|\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon/4}}\frac{x-y}{|x-y|^n}\odot\nu(y)\,d\sigma(y)-\frac{1}{2}\right|\\ &\qquad\leq\left|\int\limits_{\substack{y\in\partial\Omega\\ r\geq|x-y|>\varepsilon/4}}\frac{x-y}{|x-y|^n}\odot\nu(y)\,d\sigma(y)\right|+\left|\frac{\mathcal{H}^{n-1}\big(\Omega\cap\partial B(x,r)\big)}{\omega_{n-1}r^{n-1}}-\frac{1}{2}\right|\\ &\qquad\leq\sup_{r\in(\varepsilon/2,\,\varepsilon)}\left|\int\limits_{\substack{y\in\partial\Omega\\ r\geq|x-y|>\varepsilon/4}}\frac{x-y}{|x-y|^n}\odot\nu(y)\,d\sigma(y)\right|+\delta\end{aligned} \tag{5.28}$$

which, in light of Proposition 2.3 (whose applicability in the current setting is ensured by (2.18)), then yields (bearing in mind (2.13))

$$\limsup_{\varepsilon\to 0^+}\left|\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon/4}}\frac{x-y}{|x-y|^n}\odot\nu(y)\,d\sigma(y)-\frac{1}{2}\right|\leq\delta\ \text{ for }\sigma\text{-a.e. }x\in\partial\Omega. \tag{5.29}$$

Given that $\delta > 0$ has been arbitrarily chosen, the version of (5.22) for Ω bounded readily follows from this. Finally, the version of (5.22) corresponding to Ω unbounded is a consequence of what we have proved so far, applied to the bounded set $\Omega^c := \mathbb{R}^n \setminus \Omega$ (whose geometric measure theoretic outward unit normal is $-\nu$). $\square$

The stage has been set to show that, under much less restrictive conditions on the underlying set Ω (than the class of UR domains with compact boundaries that are also uniform domains), the principal value Cauchy-Clifford operator $\mathbf{C}$ continues to be a bounded mapping from $\mathscr{C}^\alpha(\partial\Omega) \otimes \mathcal{C}\!\ell_n$ into itself for each $\alpha \in (0,1)$. In this regard, our result can be thought of as the higher-dimensional generalization of the classical Plemelj-Privalov theorem according to which the Cauchy integral operator on a piecewise smooth Jordan curve without cusps in the plane is bounded on Hölder spaces (cf. [38], [39], [40], as well as the discussion in [36, §19, pp. 45-49]). In addition, we also establish a natural jump formula and prove that $2\mathbf{C}$ is idempotent on $\mathscr{C}^\alpha(\partial\Omega) \otimes \mathcal{C}\!\ell_n$ with $\alpha \in (0,1)$. We wish to stress that, even in the more general geometric measure theoretic setting considered below, we retain (5.2) as the definition of the Cauchy-Clifford operator $\mathbf{C}$.

Theorem 5.6. *Let $\Omega \subset \mathbb{R}^n$ be a Lebesgue measurable set whose boundary is compact, upper Ahlfors regular, and satisfies (2.15). As in the past, define $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, and fix an arbitrary $\alpha \in (0,1)$. Then for each $f \in \mathscr{C}^\alpha(\partial\Omega) \otimes \mathcal{C}\!\ell_n$ the limit defining $\mathbf{C}f(x)$ as in (5.2) exists for σ-a.e. $x \in \partial\Omega$, and the operator $\mathbf{C}$ induces a well-defined, linear, and bounded mapping*

$$\mathbf{C} : \mathscr{C}^\alpha(\partial\Omega) \otimes \mathcal{C}\!\ell_n \longrightarrow \mathscr{C}^\alpha(\partial\Omega) \otimes \mathcal{C}\!\ell_n. \tag{5.30}$$

Furthermore, the jump formula (5.3) holds for every $f \in \mathscr{C}^\alpha(\partial\Omega) \otimes \mathcal{C}\!\ell_n$.

Finally, under the additional assumption that the set Ω is open, one also has the Poincaré-Bertrand formula on Hölder spaces

$$\mathbf{C}^2 = \tfrac{1}{4} I \quad \text{on} \quad \mathscr{C}^\alpha(\partial\Omega) \otimes \mathcal{C}\!\ell_n. \tag{5.31}$$

Incidentally, given an open set Ω in the plane, the fact that its boundary is a piecewise smooth Jordan curve implies that $\partial\Omega$ is compact and upper Ahlfors regular, while the additional property that $\partial\Omega$ lacks cusps implies that (2.15) holds. Hence, our demands on the underlying domain Ω are weaker versions of the hypotheses in the formulation of the classical Plemelj-Privalov theorem mentioned earlier.

Proof of Theorem 5.6. Fix $\alpha \in (0,1)$ and pick an arbitrary function $f \in \mathscr{C}^\alpha(\partial\Omega) \otimes \mathcal{C}\!\ell_n$. Then for σ-a.e. $x \in \partial\Omega$, Lemma 5.5 allows us to write

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \frac{x-y}{|x-y|^n} \odot \nu(y) \odot f(y)\, d\sigma(y) \\
&\quad = \lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \frac{x-y}{|x-y|^n} \odot \nu(y) \odot \big(f(y) - f(x)\big)\, d\sigma(y) \pm \frac{1}{2} f(x) \\
&\quad = \frac{1}{\omega_{n-1}} \int\limits_{\partial\Omega} \frac{x-y}{|x-y|^n} \odot \nu(y) \odot \big(f(y) - f(x)\big)\, d\sigma(y) \pm \frac{1}{2} f(x),
\end{aligned} \tag{5.32}$$

where the sign of $\frac{1}{2}f(x)$ is plus if Ω is bounded and minus if Ω is unbounded. For the last equality, we have used Lebesgue's Dominated Convergence Theorem. Indeed, given that $f(y) - f(x) = O(|x-y|^\alpha)$, an estimate based on the upper Ahlfors regularity of $\partial\Omega$ in the spirit of (3.38) shows that the last integrand above is absolutely integrable for each fixed $x \in \partial\Omega$. In turn, (5.32) allows us to conclude that the limit defining $\mathbf{C}f(x)$ as in (5.2) exists for σ-a.e. $x \in \partial\Omega$. Furthermore, by redefining $\mathbf{C}f$ on a set of zero σ-measure, there is no loss of generality in assuming that, for each $f \in \mathscr{C}^\alpha(\partial\Omega) \otimes \mathcal{C}\ell_n$ with $\alpha \in (0,1)$,

$$\mathbf{C}f(x) = \pm\tfrac{1}{2}f(x) + \frac{1}{\omega_{n-1}} \int\limits_{\partial\Omega} \frac{x-y}{|x-y|^n} \odot \nu(y) \odot \big(f(y) - f(x)\big)\, d\sigma(y), \qquad \forall\, x \in \partial\Omega, \quad (5.33)$$

with the sign dictated by whether Ω is bounded (plus), or Ω is unbounded (minus).

We now proceed to showing that, in the context of (5.30), the operator (5.33) is well-defined and bounded. To this end, fix distinct points $x_1, x_2 \in \partial\Omega$ and starting from (5.33) write

$$\mathbf{C}f(x_1) - \mathbf{C}f(x_2) = I + II \quad (5.34)$$

where

$$I := \pm\tfrac{1}{2}\big(f(x_1) - f(x_2)\big) \quad (5.35)$$

and

$$II := \frac{1}{\omega_{n-1}} \int_{\partial\Omega} \Big\{ \frac{x_1-y}{|x_1-y|^n} \odot \nu(y) \odot \big(f(y) - f(x_1)\big) - \frac{x_2-y}{|x_2-y|^n} \odot \nu(y) \odot \big(f(y) - f(x_2)\big) \Big\}\, d\sigma(y). \quad (5.36)$$

Next, introduce $r := |x_1 - x_2| > 0$ and estimate

$$\big|II\big| \leq II_1 + II_2 + II_3, \quad (5.37)$$

where

$$II_1 := \frac{1}{\omega_{n-1}} \Bigg| \int\limits_{\substack{y\in\partial\Omega \\ |x_1-y|>2r}} \frac{x_1-y}{|x_1-y|^n} \odot \nu(y) \odot \big(f(y) - f(x_1)\big) - \frac{x_2-y}{|x_2-y|^n} \odot \nu(y) \odot \big(f(y) - f(x_2)\big)\, d\sigma(y) \Bigg|, \quad (5.38)$$

while

$$II_2 := \frac{1}{\omega_{n-1}} \int\limits_{\substack{y\in\partial\Omega \\ |x_1-y|\leq 2r}} \left| \frac{x_1-y}{|x_1-y|^n} \odot \nu(y) \odot \big(f(y) - f(x_1)\big) \right| d\sigma(y), \quad (5.39)$$

$$II_3 := \frac{1}{\omega_{n-1}} \int\limits_{\substack{y\in\partial\Omega \\ |x_1-y|\leq 2r}} \left| \frac{x_2-y}{|x_2-y|^n} \odot \nu(y) \odot \big(f(y) - f(x_2)\big) \right| d\sigma(y). \quad (5.40)$$

Note that

$$II_2 \leq c_n [f]_{\dot{\mathscr{C}}^\alpha(\partial\Omega)\otimes\mathcal{C}\ell_n} \int\limits_{\substack{y\in\partial\Omega \\ |x_1-y|\leq 2r}} \frac{d\sigma(y)}{|x_1-y|^{n-1-\alpha}}, \quad (5.41)$$

and, given that $|x_1 - y| \leq 2r$ forces $|x_2 - y| \leq |x_1 - x_2| + |x_1 - y| \leq 3r$,

$$II_3 \leq \frac{1}{\omega_{n-1}} \int\limits_{\substack{y\in\partial\Omega \\ |x_2-y|\leq 3r}} \left| \frac{x_2-y}{|x_2-y|^n} \odot \nu(y) \odot \big(f(y) - f(x_2)\big) \right| d\sigma(y)$$

$$\leq c_n[f]_{\dot{\mathscr{C}}^\alpha(\partial\Omega)\otimes\mathcal{C}\ell_n}\int\limits_{\substack{y\in\partial\Omega\\|x_2-y|\leq 3r}}\frac{d\sigma(y)}{|x_2-y|^{n-1-\alpha}}. \tag{5.42}$$

On the other hand, with $c\in(0,\infty)$ denoting the upper Ahlfors regularity constant of $\partial\Omega$, for every $z\in\partial\Omega$ and $R\in(0,\infty)$ we may estimate

$$\begin{aligned}\int\limits_{\substack{y\in\partial\Omega\\|z-y|<R}}\frac{d\sigma(y)}{|z-y|^{n-1-\alpha}}&=\sum_{j=1}^{\infty}\int_{[B(z,2^{1-j}R)\setminus B(z,2^{-j}R)]\cap\partial\Omega}\frac{d\sigma(y)}{|z-y|^{n-1-\alpha}}\\&\leq\sum_{j=1}^{\infty}(2^{-j}R)^{-(n-1-\alpha)}\sigma\big(B(z,2^{1-j}R)\cap\partial\Omega\big)\\&\leq c\,2^{n-1}\sum_{j=1}^{\infty}(2^{-j}R)^{\alpha}=MR^{\alpha},\end{aligned} \tag{5.43}$$

for some constant $M=M(n,\alpha,c)\in(0,\infty)$. In light of this, we obtain from (5.41) and (5.42) (keeping in mind the significance of the number r) that there exists some constant $M=M(n,\alpha,c)\in(0,\infty)$ with the property that

$$II_2+II_3\leq M[f]_{\dot{\mathscr{C}}^\alpha(\partial\Omega)}|x_1-x_2|^{\alpha}. \tag{5.44}$$

Going further, bound

$$II_1\leq II_1^a+II_1^b, \tag{5.45}$$

with

$$\begin{aligned}II_1^a&:=\frac{1}{\omega_{n-1}}\Bigg|\int\limits_{\substack{y\in\partial\Omega\\|x_1-y|>2r}}\frac{x_1-y}{|x_1-y|^n}\odot\nu(y)\odot\big(f(x_2)-f(x_1)\big)\,d\sigma(y)\Bigg|\\&=\frac{1}{\omega_{n-1}}\Bigg|\Bigg(\int\limits_{\substack{y\in\partial\Omega\\|x_1-y|>2r}}\frac{x_1-y}{|x_1-y|^n}\odot\nu(y)\,d\sigma(y)\Bigg)\odot\big(f(x_2)-f(x_1)\big)\Bigg|\\&\leq\frac{2^{n/2}}{\omega_{n-1}}\Bigg|\int\limits_{\substack{y\in\partial\Omega\\|x_1-y|>2r}}\frac{x_1-y}{|x_1-y|^n}\odot\nu(y)\,d\sigma(y)\Bigg|\big|f(x_2)-f(x_1)\big|\\&\leq M(n,c)\,r^{\alpha}\,[f]_{\dot{\mathscr{C}}^\alpha(\partial\Omega)\otimes\mathcal{C}\ell_n},\end{aligned} \tag{5.46}$$

where the penultimate inequality uses (4.13) while the last inequality is based on (5.6), and

$$\begin{aligned}II_1^b&:=\frac{1}{\omega_{n-1}}\Bigg|\int\limits_{\substack{y\in\partial\Omega\\|x_1-y|>2r}}\Big(\frac{x_1-y}{|x_1-y|^n}-\frac{x_2-y}{|x_2-y|^n}\Big)\odot\nu(y)\odot\big(f(y)-f(x_2)\big)\,d\sigma(y)\Bigg|\\&\leq\frac{2^{n/2}}{\omega_{n-1}}\int\limits_{\substack{y\in\partial\Omega\\|x_1-y|>2r}}\Big|\frac{x_1-y}{|x_1-y|^n}-\frac{x_2-y}{|x_2-y|^n}\Big||f(y)-f(x_2)|\,d\sigma(y)\\&\leq c_n\,r\,[f]_{\dot{\mathscr{C}}^\alpha(\partial\Omega)\otimes\mathcal{C}\ell_n}\int\limits_{\substack{y\in\partial\Omega\\|x_1-y|>2r}}\frac{d\sigma(y)}{|x_1-y|^{n-\alpha}},\end{aligned} \tag{5.47}$$

using the Mean Value Theorem and the fact that f is Hölder of order α. Here it helps to note that if $y\in\partial\Omega$ and $|x_1-y|>2r$ then $|\xi-y|\approx|x_1-y|$ for all $\xi\in[x_1,x_2]$, and also $|y-x_2|<|y-x_1|/2$. To continue, with $c\in(0,\infty)$ denoting the upper Ahlfors regularity constant of $\partial\Omega$ we observe that

$$\begin{aligned}\int\limits_{\substack{y\in\partial\Omega\\|x_1-y|>2r}}\frac{d\sigma(y)}{|x_1-y|^{n-\alpha}}&=\sum_{j=1}^{\infty}\int_{[B(x_1,2^{j+1}r)\setminus B(x_1,2^jr)]\cap\partial\Omega}\frac{d\sigma(y)}{|x_1-y|^{n-\alpha}}\\&\leq\sum_{j=1}^{\infty}(2^jr)^{-(n-\alpha)}\sigma\big(B(x_1,2^{j+1}r)\cap\partial\Omega\big)\\&\leq c\,2^{n-1}\sum_{j=1}^{\infty}(2^jr)^{-1+\alpha}=Mr^{-1+\alpha},\end{aligned}\tag{5.48}$$

for some constant $M=M(n,\alpha,c)\in(0,\infty)$. Combining (5.45), (5.46), (5.47), and (5.48) we conclude that there exists a constant $M=M(n,\alpha,c)\in(0,\infty)$ with the property that

$$II_1\leq M[f]_{\dot{\mathscr{C}}^\alpha(\partial\Omega)\otimes\mathcal{C}\!\ell_n}|x_1-x_2|^\alpha.\tag{5.49}$$

From (5.34)-(5.35), (5.37), (5.44), and (5.49) we may then conclude that

$$\big|\mathbf{C}f(x_1)-\mathbf{C}f(x_2)\big|\leq M[f]_{\dot{\mathscr{C}}^\alpha(\partial\Omega)\otimes\mathcal{C}\!\ell_n}|x_1-x_2|^\alpha,\qquad\forall\,x_1,x_2\in\partial\Omega,\tag{5.50}$$

for some constant $M=M(n,\alpha,c)\in(0,\infty)$. The argument so far gives that the Cauchy-Clifford singular integral operator $\mathbf{C}$ maps $\dot{\mathscr{C}}^\alpha(\partial\Omega)\otimes\mathcal{C}\!\ell_n$ boundedly into itself. Having established this, Lemma 3.2 may be invoked (bearing in mind that (5.33) forces $\mathbf{C}1=\pm\frac12$) in order to finish the proof of the theorem.

Turning our attention to jump-formula (5.3), it has been already noted that the action of the boundary-to-domain Cauchy-Clifford operator (5.1) is meaningful on Hölder functions. Also, observe that Proposition 2.7 ensures that it is meaningful to consider the limit in (5.3) whenever $\Omega\subseteq\mathbb{R}^n$ is an open set with an Ahlfors regular boundary, satisfying (2.15). Assume now that some $f\in\mathscr{C}^\alpha(\partial\Omega)\otimes\mathcal{C}\!\ell_n$ with $\alpha\in(0,1)$ has been given and fix $x\in\partial^*\Omega$. Let $\mathcal{O}_x$ be the set given by Lemma 2.4 applied with Ω replaced by $\mathbb{R}^n\setminus\Omega$. In particular,

$$\lim_{\mathcal{O}_x\ni\varepsilon\to0^+}\frac{\mathcal{H}^{n-1}\big(\partial B(x,\varepsilon)\setminus\Omega\big)}{\omega_{n-1}\varepsilon^{n-1}}=\frac12.\tag{5.51}$$

For some $\kappa>0$ fixed, write

$$\begin{aligned}\lim_{\substack{z\in\Gamma_\kappa(x)\\z\to x}}\mathcal{C}f(z)=&\lim_{\mathcal{O}_x\ni\varepsilon\to0^+}\lim_{\substack{z\in\Gamma_\kappa(x)\\z\to x}}\frac{1}{\omega_{n-1}}\int\limits_{\substack{|x-y|>\varepsilon\\y\in\partial\Omega}}\frac{z-y}{|z-y|^n}\odot\nu(y)\odot f(y)\,d\sigma(y)\\&+\lim_{\mathcal{O}_x\ni\varepsilon\to0^+}\lim_{\substack{z\in\Gamma_\kappa(x)\\z\to x}}\frac{1}{\omega_{n-1}}\int\limits_{\substack{|x-y|<\varepsilon\\y\in\partial\Omega}}\frac{z-y}{|z-y|^n}\odot\nu(y)\odot(f(y)-f(x))\,d\sigma(y)\\&+\left(\lim_{\mathcal{O}_x\ni\varepsilon\to0^+}\lim_{\substack{z\in\Gamma_\kappa(x)\\z\to x}}\frac{1}{\omega_{n-1}}\int\limits_{\substack{|x-y|<\varepsilon\\y\in\partial\Omega}}\frac{z-y}{|z-y|^n}\odot\nu(y)\,d\sigma(y)\right)\odot f(x)\\&=:I_1+I_2+I_3.\end{aligned}\tag{5.52}$$

For each fixed $\varepsilon > 0$, Lebesgue's Dominated Convergence Theorem applies to the limit as $\Gamma_\kappa(x) \ni z \to x$ in I_1 and yields

$$I_1 = \lim_{\varepsilon \to 0^+} \frac{1}{\omega_{n-1}} \int\limits_{\substack{|x-y|>\varepsilon \\ y \in \partial\Omega}} \frac{x-y}{|x-y|^n} \odot \nu(y) \odot f(y)\, d\sigma(y) = \mathbf{C}f(x). \tag{5.53}$$

To handle I_2, we first observe that for every $x, y \in \partial\Omega$ and $z \in \Gamma_\kappa(x)$,

$$\begin{aligned} |x-y| &\leq |z-y| + |z-x| \leq |z-y| + (1+\kappa)\,\mathrm{dist}(z,\partial\Omega) \\ &\leq |z-y| + (1+\kappa)|z-y| = (2+\kappa)|z-y|. \end{aligned} \tag{5.54}$$

Hence, since f is Hölder of order α,

$$\left| \frac{z-y}{|z-y|^n} \odot \nu(y) \right| |f(y) - f(x)| \leq [f]_{\dot{\mathscr{C}}^\alpha(\partial\Omega)\otimes \mathcal{C}\ell_n} \frac{(2+\kappa)^{n-1}}{|x-y|^{n-1-\alpha}}, \tag{5.55}$$

so that, based on the upper Ahlfors regularity of $\partial\Omega$ and once again Lebesgue's Dominated Convergence Theorem, we obtain that

$$I_2 = 0. \tag{5.56}$$

To treat I_3 in (5.52), we first claim that, having fixed $z \in \Omega$, for $\mathscr{L}^1$-a.e $\varepsilon > 0$ we have

$$\int\limits_{\substack{|x-y|<\varepsilon \\ y \in \partial\Omega}} \frac{z-y}{|z-y|^n} \odot \nu(y)\, d\sigma(y) = \int\limits_{\substack{|x-y|=\varepsilon \\ y \in \mathbb{R}^n \setminus \Omega}} \frac{z-y}{|z-y|^n} \odot \nu(y)\, d\sigma(y). \tag{5.57}$$

To justify this, pick a large $R > 0$ and apply (2.57) to $D := B(0,R) \setminus \Omega$ and, for each $j, k \in \{1, \dots, n\}$, to the vector field

$$\vec{F}_{jk}(y) := \Big(0, \dots, 0, \frac{z_j - y_j}{|z-y|^n}, 0, \dots, 0\Big), \qquad \forall\, y \in \mathbb{R}^n \setminus \{z\}, \tag{5.58}$$

with the non-zero component on the k-th slot. Agree to alter each $\vec{F}_{jk}$ outside a compact neighborhood of $\overline{D}$ to a vector field $\vec{G}_{jk} \in \mathscr{C}^1_0(\mathbb{R}^n \setminus \{z\}, \mathbb{R}^n)$. Then by reasoning as in (5.11)-(5.13), formula (5.57) follows. Consequently, starting with (5.57), then using (5.8), and then (5.51), we obtain

$$\begin{aligned} &\lim_{\mathcal{O}_x \ni \varepsilon \to 0^+} \lim_{\substack{z \in \Gamma_\kappa(x) \\ z \to x}} \frac{1}{\omega_{n-1}} \int\limits_{\substack{|x-y|<\varepsilon \\ y \in \partial\Omega}} \frac{z-y}{|z-y|^n} \odot \nu(y)\, d\sigma(y) \\ &\qquad = \lim_{\mathcal{O}_x \ni \varepsilon \to 0^+} \frac{1}{\omega_{n-1}} \int\limits_{\substack{|x-y|=\varepsilon \\ y \in \mathbb{R}^n \setminus \Omega}} \frac{x-y}{|x-y|^n} \odot \nu(y)\, d\sigma(y) \\ &\qquad = \lim_{\mathcal{O}_x \ni \varepsilon \to 0^+} \frac{\mathcal{H}^{n-1}\big(\partial B(x,\varepsilon) \setminus \Omega\big)}{\omega_{n-1}\varepsilon^{n-1}} = \frac{1}{2}. \end{aligned} \tag{5.59}$$

A combination of (5.52), (5.53), (5.56), and (5.59) shows that the limit in the left hand-side of (5.52) exists and matches $(\frac{1}{2}I + \mathbf{C})f(x)$. This proves that formula (5.3) holds for each $f \in \mathscr{C}^\alpha(\partial\Omega) \otimes \mathcal{C}\ell_n$ at every $x \in \partial^*\Omega$ (hence at σ-a.e. point in $\partial\Omega$, by (2.13) and the assumption (2.15)).

To finish the proof of the theorem, there remains to establish the Poincaré-Bertrand formula (5.31) under the additional assumption that the set Ω is open. Assume that this is the case and bring in the version of the Cauchy reproducing formula from [34] to the effect that, under the current assumptions on the set Ω,

$$\left.\begin{array}{r} u:\Omega\to \mathcal{C}\!\ell_n \ \text{ continuous}, \ \ D_L u=0 \ \text{ in } \ \Omega \\ \mathcal{N}u\in L^1(\partial\Omega,\sigma), \ \ u\big|^{\rm n.t.}_{\partial\Omega} \ \text{ exists } \sigma\text{-a.e. on } \ \partial\Omega \end{array}\right\} \Rightarrow \ u=\mathcal{C}\big(u\big|^{\rm n.t.}_{\partial\Omega}\big) \ \text{ in } \ \Omega. \tag{5.60}$$

Now, given any $f\in\mathscr{C}^\alpha(\partial\Omega)\otimes\mathcal{C}\!\ell_n$, define $u:=\mathcal{C}f$ in Ω. Then, by design, $u\in\mathscr{C}^\infty(\Omega)$ and $D_L u=0$ in Ω. Also, (5.19) gives that $\sup_{x\in\Omega}\big|u(x)\big|\leq M\|f\|_{\mathscr{C}^\alpha(\partial\Omega)\otimes\mathcal{C}\!\ell_n}$ which, in turn, forces $\mathcal{N}u\in L^\infty(\partial\Omega,\sigma)\subset L^1(\partial\Omega,\sigma)$ given that $\partial\Omega$ has finite measure. Finally, the jump formula (5.3) for Hölder functions, established earlier in the proof, yields

$$\big(u\big|^{\rm n.t.}_{\partial\Omega}\big)(x)=\big(\tfrac12 I+\mathbf{C}\big)f(x) \quad \text{ for } \ \sigma\text{-a.e. } \ x\in\partial\Omega. \tag{5.61}$$

Granted these, (5.60) applies and gives that $u=\mathcal{C}\big(u\big|^{\rm n.t.}_{\partial\Omega}\big)$ in Ω. Going to the boundary nontangentially and relying on (5.61) then allow us to conclude that

$$\big(\tfrac12 I+\mathbf{C}\big)f=\big(\tfrac12 I+\mathbf{C}\big)\big(\tfrac12 I+\mathbf{C}\big)f \quad \sigma\text{-a.e. on } \ \partial\Omega, \tag{5.62}$$

from which (5.31) now readily follows. □

In this last part of this section we briefly consider harmonic layer potentials. Recall the standard fundamental solution E_Δ for the Laplacian in $\mathbb{R}^n$ from (5.10). Given a nonempty open proper subset Ω of $\mathbb{R}^n$, let $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$. Then the `harmonic single layer operator` associated with Ω acts on a function f defined on $\partial\Omega$ according to

$$\mathscr{S}f(x):=\int_{\partial\Omega}E_\Delta(x-y)f(y)\,d\sigma(y), \qquad x\in\Omega. \tag{5.63}$$

Assume that Ω is a set of locally finite perimeter for which (2.15) holds and denote by ν its (geometric measure theoretic) outward unit normal. In this context, it follows from (4.17), (5.63), (5.1), and the fact that $\nu\odot\nu=-1$ (cf. (4.1)), that the harmonic single layer operator and the Cauchy-Clifford operator are related via

$$D_L\mathscr{S}f=-\mathcal{C}(\nu\odot f) \ \text{ in } \ \Omega. \tag{5.64}$$

Parenthetically, we wish to note that, in the same setting, the `harmonic double layer operator` associated with Ω is defined as

$$\mathscr{D}f(x):=\frac{1}{\omega_{n-1}}\int_{\partial\Omega}\frac{\langle\nu(y),y-x\rangle}{|x-y|^n}f(y)\,d\sigma(y), \qquad x\in\Omega, \tag{5.65}$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product of vectors in $\mathbb{R}^n$. In particular, from (5.1), (4.10), (4.8), and (5.65), it follows that

$$\text{if } f \text{ is scalar-valued then } \ \mathscr{D}f=\big(\mathcal{C}f\big)_0 \ \text{ in } \ \Omega. \tag{5.66}$$

As a consequence of this and (5.20), we see that if $\Omega\subset\mathbb{R}^n$ is a uniform domain whose boundary is compact, upper Ahlfors regular, and satisfies (2.15) then for each $\alpha\in(0,1)$ the harmonic double layer operator induces a well-defined, linear, and bounded mapping

$$\mathscr{D}:\mathscr{C}^\alpha(\partial\Omega)\longrightarrow\mathscr{C}^\alpha\big(\overline{\Omega}\big). \tag{5.67}$$

Returning to the mainstream discussion, make the convention that ∇^2 is the vector of all second order partial derivatives in $\mathbb{R}^n$. Also, once again, recall (2.52).

Lemma 5.7. *Let Ω be a domain of class $\mathscr{C}^{1+\alpha}$ for some $\alpha\in(0,1)$ with compact boundary. Then*

$$A:=\sup_{x\in\Omega}\big|\nabla(\mathscr{S}1)(x)\big|+\sup_{x\in\Omega}\Big\{\rho(x)^{1-\alpha}\big|\nabla^2(\mathscr{S}1)(x)\big|\Big\}<+\infty \tag{5.68}$$

and, in fact, this quantity may be estimated in terms of n, α, $\operatorname{diam}(\partial\Omega)$, $\|\nu\|_{\mathscr{C}^\alpha(\partial\Omega)}$, and the upper Ahlfors regularity constant of $\partial\Omega$.

Proof. Via the identification (4.3) we obtain from (5.64) that

$$\nabla(\mathscr{S}1) \equiv D_L \mathscr{S}1 = -\mathcal{C}\nu \quad \text{in} \quad \Omega. \tag{5.69}$$

Then, keeping in mind that $\nu \in \mathscr{C}^\alpha(\partial\Omega) \otimes \mathcal{C}\ell_n$ under the present assumption on Ω, the claim in (5.68) readily follows by combining (5.69) with (5.19). □

6. The Proofs of Theorem 1.1 and Theorem 1.2

We start by presenting the proof of Theorem 1.1.

Proof of (a) ⇒ (e) *in Theorem 1.1.* Let Ω be a domain of class $\mathscr{C}^{1+\alpha}$, $\alpha \in (0,1)$, with compact boundary (hence, in particular, Ω is a UR domain). Also, assume $P(x)$ is an odd, homogeneous, harmonic polynomial of degree $l \geq 1$ in $\mathbb{R}^n$ and, with it, associate the singular integral operator

$$\mathbb{T}f(x) := \int_{\partial\Omega} \frac{P(x-y)}{|x-y|^{n-1+l}} f(y)\, d\sigma(y), \qquad x \in \Omega. \tag{6.1}$$

In a first stage, the goal is to prove that there exists a constant $C \in (1,\infty)$, depending only on n, α, diam $(\partial\Omega)$, $\|\nu\|_{\mathscr{C}^\alpha(\partial\Omega)}$, and the upper Ahlfors regularity constant of $\partial\Omega$ (something we shall indicate by writing $C = C(n,\alpha,\Omega)$) such that for every $f \in \mathscr{C}^\alpha(\partial\Omega)$ we have

$$\sup_{x\in\Omega} |\mathbb{T}f(x)| + \sup_{x\in\Omega} \Big\{ \rho(x)^{1-\alpha} \big|\nabla(\mathbb{T}f)(x)\big| \Big\} \leq C^l 2^{l^2} \|P\|_{L^1(S^{n-1})} \|f\|_{\mathscr{C}^\alpha(\partial\Omega)}. \tag{6.2}$$

We shall do so by induction on $l \in 2\mathbb{N}-1$, the degree of the homogeneous harmonic polynomial P. When $l = 1$ we have $P(x) = \sum_{j=1}^n a_j x_j$ for each $x = (x_1,\dots,x_n) \in \mathbb{R}^n$, where the a_j's are some fixed constants. Hence, in this case,

$$\max_{1\leq j\leq n} |a_j| \leq \|P\|_{L^\infty(S^{n-1})} \leq c_n \|P\|_{L^1(S^{n-1})} \tag{6.3}$$

where the last inequality is a consequence of (4.35) (with $c_n \in (0,\infty)$ denoting a dimensional constant), and

$$\mathbb{T} = \omega_{n-1} \sum_{j=1}^n a_j \partial_j \mathscr{S}. \tag{6.4}$$

Then (6.2) follows from (6.3), (6.4), Lemma 5.7, and Lemma 3.4. To proceed, fix some odd integer $l \geq 3$ and assume that there exists $C = C(n,\alpha,\Omega) \in (1,\infty)$ such that

$$\begin{array}{c}\text{the estimate in (6.2) holds whenever } \mathbb{T} \text{ is associated as in (6.1) with} \\ \text{an odd harmonic homogeneous polynomial of degree} \leq l-2 \text{ in } \mathbb{R}^n.\end{array} \tag{6.5}$$

Also, pick an arbitrary odd harmonic homogeneous polynomial $P(x)$ of degree l in $\mathbb{R}^n$ and let $\mathbb{T}$ be as in (6.1) for this choice of P. Consider the family $P_{rs}(x)$, $1 \leq r,s \leq n$, of odd harmonic homogeneous polynomials of degree $l-2$, as well as the family of odd, $\mathscr{C}^\infty$ functions $k_{rs} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \hookrightarrow \mathcal{C}\ell_n$, associated with P as in Lemma 4.2. For each $1 \leq i,j \leq n$ set

$$k^{rs}(x) := P_{rs}(x)/|x|^{n+l-3} \quad \text{for} \quad x \in \mathbb{R}^n \setminus \{0\}, \tag{6.6}$$

and introduce the integral operator acting on Clifford algebra-valued functions, $f = \sum_I f_I e_I$ with Hölder scalar components f_I defined on $\partial\Omega$, according to

$$\begin{aligned}\mathbb{T}^{rs} f(x) &:= \int_{\partial\Omega} k^{rs}(x-y) f(y)\, d\sigma(y) \\ &= \sum_I \Big(\int_{\partial\Omega} k^{rs}(x-y) f_I(y)\, d\sigma(y) \Big) e_I, \qquad x \in \Omega.\end{aligned} \tag{6.7}$$

Fix such an arbitrary $f \in \mathscr{C}^\alpha(\partial\Omega) \otimes \mathcal{C}\ell_n$. Then from the properties of the P_{rs}'s and the induction hypothesis (6.5) (used component-wise, keeping in mind that the sum in (6.7) is performed over a set of cardinality 2^n) we conclude that for each $1 \leq r, s \leq n$ we have

$$\begin{aligned}\sup_{x\in\Omega} |(\mathbb{T}^{rs} f)(x)| + \sup_{x\in\Omega} \Big\{\rho(x)^{1-\alpha}\big|\nabla(\mathbb{T}^{rs} f)(x)\big|\Big\} \\ \leq 2^{n/2} C^{l-2} 2^{(l-2)^2} \|P_{rs}\|_{L^1(S^{n-1})} \|f\|_{\mathscr{C}^\alpha(\partial\Omega)\otimes\mathcal{C}\ell_n} \\ \leq c_n\, C^{l-2} 2^{(l-2)^2} 2^l \|P\|_{L^1(S^{n-1})} \|f\|_{\mathscr{C}^\alpha(\partial\Omega)\otimes\mathcal{C}\ell_n}. \end{aligned} \tag{6.8}$$

Moving on, for every $r, s \in \{1, \ldots, n\}$ and $f : \partial\Omega \to \mathcal{C}\ell_n$ with Hölder scalar components we set

$$\mathbb{T}_{rs} f(x) := \int_{\partial\Omega} k_{rs}(x-y) \odot f(y)\, d\sigma(y), \qquad x \in \Omega. \tag{6.9}$$

Then, thanks to formula (4.23), whenever the function f is actually scalar-valued (i.e., $f : \partial\Omega \to \mathbb{R} \hookrightarrow \mathcal{C}\ell_n$) the original operator $\mathbb{T}$ from (6.1) may be recovered from the above $\mathbb{T}_{rs}$'s by means of the identity

$$\mathbb{T} f(x) = \sum_{r,s=1}^{n} \big[\mathbb{T}_{rs} f(x)\big]_s \quad \text{for all } \ x \in \Omega. \tag{6.10}$$

To proceed, consider first the case when Ω is unbounded. In this scenario, fix some $x \in \Omega$ and select

$$R_1 \in \big(0, \operatorname{dist}(x, \partial\Omega)\big) \ \text{ along with } \ R_2 > \operatorname{dist}(x, \partial\Omega) + \operatorname{diam}(\partial\Omega). \tag{6.11}$$

Set $\Omega_{R_1,R_2} := \big(B(x, R_2) \setminus \overline{B(x, R_1)}\,\big) \cap \Omega$ which is a bounded $\mathscr{C}^{1+\alpha}$ domain in $\mathbb{R}^n$ with the property that

$$\partial\Omega_{R_1,R_2} = \partial B(x, R_2) \cup \partial B(x, R_1) \cup \partial\Omega. \tag{6.12}$$

We continue to denote by ν and σ the outward unit normal and surface measure for Ω_{R_1,R_2}. As a consequence of (4.18) (used with Ω_{R_1,R_2} in place of Ω, $u = k_{rs}(x - \cdot) \in \mathscr{C}^\infty\big(\overline{\Omega_{R_1,R_2}}\,\big)$, and $v \equiv 1$) and (4.24), we then obtain that for each $r, s \in \{1, \ldots, n\}$

$$\begin{aligned}\int_{\partial\Omega_{R_1,R_2}} k_{rs}(x-y) \odot \nu(y)\, d\sigma(y) &= -\int_{\Omega_{R_1,R_2}} (D_R k_{rs})(x-y)\, dy \\ &= \frac{l-1}{n+l-3} \int_{\Omega_{R_1,R_2}} \frac{\partial}{\partial y_r} \left(\frac{P_{rs}(x-y)}{|x-y|^{n+l-3}}\right) dy \\ &= \frac{l-1}{n+l-3} \int_{\partial\Omega_{R_1,R_2}} k^{rs}(x-y)\nu_r(y)\, d\sigma(y). \end{aligned} \tag{6.13}$$

Hence,

$$\begin{aligned}(\mathbb{T}_{rs}\nu)(x) &= \int_{\partial\Omega} k_{rs}(x-y) \odot \nu(y)\, d\sigma(y) \\ &= \int_{\partial\Omega_{R_1,R_2}} k_{rs}(x-y) \odot \nu(y)\, d\sigma(y) - \int_{\partial B(x,R_1)} k_{rs}(x-y) \odot \frac{x-y}{|x-y|}\, d\sigma(y) \\ &\quad + \int_{\partial B(x,R_2)} k_{rs}(x-y) \odot \frac{x-y}{|x-y|}\, d\sigma(y) \\ &= \frac{l-1}{n+l-3} \int_{\partial\Omega_{R_1,R_2}} k^{rs}(x-y)\nu_r(y)\, d\sigma(y) \\ &\quad - \int_{S^{n-1}} k_{rs}(\omega) \odot \omega\, d\omega + \int_{S^{n-1}} k_{rs}(\omega) \odot \omega\, d\omega \end{aligned}$$

$$
\begin{aligned}
&= \frac{l-1}{n+l-3}\int_{\partial\Omega} k^{rs}(x-y)\nu_r(y)\,d\sigma(y)\\
&\quad - \frac{l-1}{n+l-3}\int_{\partial B(x,R_1)} k^{rs}(x-y)\frac{x_r-y_r}{|x-y|}\,d\sigma(y)\\
&\quad + \frac{l-1}{n+l-3}\int_{\partial B(x,R_2)} k^{rs}(x-y)\frac{x_r-y_r}{|x-y|}\,d\sigma(y)\\
&= \frac{l-1}{n+l-3}(\mathbb{T}^{rs}\nu_r)(x)\\
&\quad - \frac{l-1}{n+l-3}\int_{S^{n-1}} k^{rs}(\omega)\omega_r\,d\omega + \frac{l-1}{n+l-3}\int_{S^{n-1}} k^{rs}(\omega)\omega_r\,d\omega\\
&= \frac{l-1}{n+l-3}(\mathbb{T}^{rs}\nu_r)(x). \qquad (6.14)
\end{aligned}
$$

From (6.14) and (6.8) used with $f=\nu_r\in\mathscr{C}^\alpha(\partial\Omega)$, for $1\le r,s\le n$ we obtain

$$
\begin{aligned}
&\sup_{x\in\Omega}|(\mathbb{T}_{rs}\nu)(x)| + \sup_{x\in\Omega}\Big\{\rho(x)^{1-\alpha}\big|\nabla(\mathbb{T}_{rs}\nu)(x)\big|\Big\}\\
&\qquad\qquad \le \sup_{x\in\Omega}|(\mathbb{T}^{rs}\nu_r)(x)| + \sup_{x\in\Omega}\Big\{\rho(x)^{1-\alpha}\big|\nabla(\mathbb{T}^{rs}\nu_r)(x)\big|\Big\}\\
&\qquad\qquad \le c_n\, C^{l-2}2^{(l-2)^2}2^l\|P\|_{L^1(S^{n-1})}\|\nu\|_{\mathscr{C}^\alpha(\partial\Omega)}, \qquad (6.15)
\end{aligned}
$$

in the case when Ω is an unbounded domain.

When Ω is a bounded domain, we once again consider Ω_{R_1,R_2} as before and carry out a computation similar in spirit to what we have just done above. This time, however, $\Omega_{R_1,R_2}=\Omega\setminus\overline{B(x,R_1)}$ and in place of (6.12) we have $\partial\Omega_{R_1,R_2}=\partial B(x,R_1)\cup\partial\Omega$. Consequently, in place of (6.14) we now obtain

$$
\begin{aligned}
(\mathbb{T}_{rs}\nu)(x) &= \frac{l-1}{n+l-3}(\mathbb{T}^{rs}\nu_r)(x)\\
&\qquad - \frac{l-1}{n+l-3}\int_{S^{n-1}} k^{rs}(\omega)\omega_r\,d\omega - \int_{S^{n-1}} k_{rs}(\omega)\odot\omega\,d\omega. \qquad (6.16)
\end{aligned}
$$

To estimate the integrals on the unit sphere we note that, in view of (6.6), (4.26), (4.35), and (4.25), we have

$$
\|k^{rs}\|_{L^\infty(S^{n-1})} + \|k_{rs}\|_{L^\infty(S^{n-1})} \le c_n\,2^l\|P\|_{L^1(S^{n-1})}. \qquad (6.17)
$$

Upon observing that $\|\nu\|_{\mathscr{C}^\alpha(\partial\Omega)}\ge 1$, from (6.16) and (6.17) we deduce that an estimate similar to (6.15) also holds in the case when Ω is a bounded domain (this time, replacing the constant c_n appearing in (6.15) by $2c_n$, which is inconsequential for our purposes). In summary, (6.16) may be assumed to hold irrespective of whether Ω is bounded or not.

Going further, let $\widetilde{\mathbb{T}}_{rs}$ be the version of $\mathbb{T}_{rs}$ from (6.9) in which $\nu(y)$ has been absorbed in the integral kernel. That is, for $f:\partial\Omega\to\mathcal{C}\ell_n$ with Hölder scalar components set

$$
\widetilde{\mathbb{T}}_{rs}f(x) := \int_{\partial\Omega}\big(k_{rs}(x-y)\odot\nu(y)\big)\odot f(y)\,d\sigma(y), \qquad x\in\Omega, \qquad (6.18)
$$

for each $r,s\in\{1,\dots,n\}$. Since $\widetilde{\mathbb{T}}_{rs}1=\mathbb{T}_{rs}\nu$, from (6.15) we conclude that for each $r,s\in\{1,\dots,n\}$

$$
\begin{aligned}
&\sup_{x\in\Omega}|(\widetilde{\mathbb{T}}_{rs}1)(x)| + \sup_{x\in\Omega}\Big\{\rho(x)^{1-\alpha}\big|\nabla(\widetilde{\mathbb{T}}_{rs}1)(x)\big|\Big\}\\
&\qquad\qquad \le c_n\, C^{l-2}2^{(l-2)^2}2^l\|P\|_{L^1(S^{n-1})}\|\nu\|_{\mathscr{C}^\alpha(\partial\Omega)}. \qquad (6.19)
\end{aligned}
$$

Given that the integral kernel of $\widetilde{\mathbb{T}}_{rs}$ satisfies

$$|k_{rs}(x-y)\odot\nu(y)| \leq \frac{\|k_{rs}\|_{L^\infty(S^{n-1})}}{|x-y|^{n-1}} \leq \frac{c_n 2^l \|P\|_{L^1(S^{n-1})}}{|x-y|^{n-1}}, \tag{6.20}$$

and

$$\big|\nabla_x\big[k_{rs}(x-y)\odot\nu(y)\big]\big| \leq \frac{\|\nabla k_{rs}\|_{L^\infty(S^{n-1})}}{|x-y|^{n}} \leq \frac{c_n 2^l \|P\|_{L^1(S^{n-1})}}{|x-y|^{n}}, \tag{6.21}$$

we may invoke Lemma 3.4 with

$$A := c_n\, C^{l-2} 2^{(l-2)^2} 2^l \|P\|_{L^1(S^{n-1})}\|\nu\|_{\mathscr{C}^\alpha(\partial\Omega)} \ \text{ and } \ B := c_n 2^l \|P\|_{L^1(S^{n-1})} \tag{6.22}$$

in order to conclude that if $1 \leq r, s \leq n$ then

$$\begin{aligned} &\sup_{x\in\Omega}|\widetilde{\mathbb{T}}_{rs}f(x)| + \sup_{x\in\Omega}\Big\{\rho(x)^{1-\alpha}\big|\nabla(\widetilde{\mathbb{T}}_{rs}f)(x)\big|\Big\} \\ &\qquad \leq C_{n,\alpha,\Omega}\Big\{C^{l-2}2^{(l-2)^2}2^l\|\nu\|_{\mathscr{C}^\alpha(\partial\Omega)} + 2^l\Big\}\|P\|_{L^1(S^{n-1})}\|f\|_{\mathscr{C}^\alpha(\partial\Omega)\otimes\mathcal{C}\ell_n} \end{aligned} \tag{6.23}$$

for every $f \in \mathscr{C}^\alpha(\partial\Omega)\otimes\mathcal{C}\ell_n$. Writing (6.23) for f replaced by $\nu\odot f$ then yields, in light of (6.18), (6.9), and (4.21) (bearing in mind that $\nu\odot\nu = -1$), that for $1 \leq r, s \leq n$ we have

$$\begin{aligned} &\sup_{x\in\Omega}|\mathbb{T}_{rs}f(x)| + \sup_{x\in\Omega}\Big\{\rho(x)^{1-\alpha}\big|\nabla(\mathbb{T}_{rs}f)(x)\big|\Big\} \\ &\qquad \leq C_{n,\alpha,\Omega}\Big\{C^{l-2}2^{(l-2)^2}2^l\|\nu\|_{\mathscr{C}^\alpha(\partial\Omega)} + 2^l\Big\}\times \\ &\qquad\qquad \times 2\|\nu\|_{\mathscr{C}^\alpha(\partial\Omega)}\|P\|_{L^1(S^{n-1})}\|f\|_{\mathscr{C}^\alpha(\partial\Omega)\otimes\mathcal{C}\ell_n} \end{aligned} \tag{6.24}$$

for every $f \in \mathscr{C}^\alpha(\partial\Omega)\otimes\mathcal{C}\ell_n$. In turn, from this and (6.10) we finally conclude that

$$\begin{aligned} &\sup_{x\in\Omega}|\mathbb{T}f(x)| + \sup_{x\in\Omega}\Big\{\rho(x)^{1-\alpha}\big|\nabla(\mathbb{T}f)(x)\big|\Big\} \\ &\qquad \leq n^2\, C_{n,\alpha,\Omega}\Big\{C^{l-2}2^{(l-2)^2}2^l\|\nu\|_{\mathscr{C}^\alpha(\partial\Omega)} + 2^l\Big\}\times \\ &\qquad\qquad \times 2\|\nu\|_{\mathscr{C}^\alpha(\partial\Omega)}\|P\|_{L^1(S^{n-1})}\|f\|_{\mathscr{C}^\alpha(\partial\Omega)} \end{aligned} \tag{6.25}$$

for every $f \in \mathscr{C}^\alpha(\partial\Omega)$. Having established (6.25), we now see that (6.2) holds provided the constant $C \in (1,\infty)$ is chosen in such a way that

$$n^2\, C_{n,\alpha,\Omega}\Big\{C^{l-2}2^{(l-2)^2}2^l\|\nu\|_{\mathscr{C}^\alpha(\partial\Omega)} + 2^l\Big\}2\|\nu\|_{\mathscr{C}^\alpha(\partial\Omega)} \leq C^l 2^{l^2} \tag{6.26}$$

for each odd number $l \in \mathbb{N}$, $l \geq 3$. Since $2^{(l-2)^2}2^l \leq 2\cdot 2^{l^2}$ and $2^l \leq C^{l-2}2^{l^2}$, it follows that the left-hand side of (6.26) is $\leq C(n,\alpha,\Omega)C^{l-2}2^{l^2}$. This, in turn, is majorized by the right-hand side of (6.26) granted that $C \geq \max\big\{1, \sqrt{C(n,\alpha,\Omega)}\,\big\}$. In summary, choosing C in the manner just described, to begin with, ensures that (6.2) holds.

Next, we aim to show that (6.2) continues to be valid if the harmonicity condition on P is dropped, that is, when

$$P(x) \text{ is a homogeneous polynomial in } \mathbb{R}^n \text{ of degree } l \in 2\mathbb{N}-1. \tag{6.27}$$

Indeed, a standard fact about arbitrary homogeneous polynomials $P(x)$ is the decomposition (cf. [42, § 3.1.2, p. 69])

$$\begin{gathered} P(x) = P_1(x) + |x|^2 Q_1(x) \ \text{ in } \ \mathbb{R}^n, \quad \text{where} \\ P_1, Q_1 \text{ are homogeneous polynomials and } P_1 \text{ is harmonic.} \end{gathered} \tag{6.28}$$

Hence, if $P(x)$ is a homogeneous polynomial of degree $l = 2N+1$ in $\mathbb{R}^n$, for some $N \in \mathbb{N}_0$, not necessarily harmonic, then by iterating (6.28) we obtain

$$P(x) = \sum_{j=1}^{N+1} |x|^{2(j-1)} P_j(x) \ \text{ in } \ \mathbb{R}^n, \quad \text{where each } P_j \text{ is} \\ \text{a harmonic homogeneous polynomial of degree } l-2(j-1). \tag{6.29}$$

Since the restrictions to the unit sphere of any two homogeneous harmonic polynomials of different degrees are orthogonal in $L^2(S^{n-1})$ (cf. [42, § 3.1.1, p. 69]), it follows from (6.29) that

$$\|P\|^2_{L^2(S^{n-1})} = \sum_{j=1}^{N+1} \|P_j\|^2_{L^2(S^{n-1})}. \tag{6.30}$$

In particular, for each j, Hölder's inequality and (6.30) permit us to estimate

$$\|P_j\|_{L^1(S^{n-1})} \leq c_n \|P_j\|_{L^2(S^{n-1})} \leq c_n \|P\|_{L^2(S^{n-1})}. \tag{6.31}$$

Combining (6.1) and (6.29), for any $x \in \Omega$ and $f \in \mathscr{C}^\alpha(\partial\Omega)$ we obtain

$$\mathbb{T}f(x) = \sum_{j=1}^{N+1} \int_{\partial\Omega} \frac{P_j(x-y)}{|x-y|^{n-1+(l-2(j-1))}} f(y)\, d\sigma(y), \tag{6.32}$$

and each integral operator appearing in the sum above is constructed according to the same blue-print as the original $\mathbb{T}$ in (6.1), including the property that the intervening homogeneous polynomial is harmonic. As such, repeated applications of (6.2) yield

$$\sup_{x\in\Omega} |\mathbb{T}f(x)| + \sup_{x\in\Omega} \Big\{ \rho(x)^{1-\alpha} \big|\nabla(\mathbb{T}f)(x)\big| \Big\} \leq c_n l C^l 2^{l^2} \|P\|_{L^2(S^{n-1})} \|f\|_{\mathscr{C}^\alpha(\partial\Omega)}, \tag{6.33}$$

for each $f \in \mathscr{C}^\alpha(\partial\Omega)$. Since if C is bigger than a suitable dimensional constant we have $c_n l \leq C^l$ for all l's, by eventually replacing C by C^2 in (6.33). Ultimately, with the help of Lemma 2.9 (while keeping (2.51) in mind), we deduce that (1.9) holds for $\mathbb{T}_+$ in Ω_+. That $\mathbb{T}_-$ also satisfies similar properties follows in a similar manner, working in Ω_- (in place of Ω_+), which continues to be a domain of class $\mathscr{C}^{1+\alpha}$ with compact boundary. ☐

Proof of (e) ⇒ (d) *in Theorem 1.1.* This is obvious, since the operators $\mathscr{R}_j^\pm$ from (1.7) are particular cases of those considered in (1.8). ☐

Proof of (d) ⇒ (a) *in Theorem 1.1.* Since we are presently assuming that Ω is a UR domain, Theorem 3.1 applies in $\Omega_\pm$ and yields (bearing (2.45) in mind) the following jump-formulas

$$\Big(\mathscr{R}_j^\pm f\Big|^{\text{n.t.}}_{\partial\Omega_\pm}\Big)(x) = \mp\frac{1}{2}\nu_j(x) f(x) + \lim_{\varepsilon\to 0^+} \int_{\partial\Omega\setminus B(x,\varepsilon)} (\partial_j E_\Delta)(x-y) f(y)\, d\sigma(y), \tag{6.34}$$

for each $f \in L^p(\partial\Omega, \sigma)$ with $p \in [1,\infty)$, each $j \in \{1,\dots,n\}$, and σ-a.e. $x \in \partial\Omega$. Hence, by (6.34) and (1.6), we have

$$\nu_j = \mathscr{R}_j^- 1\Big|_{\partial\Omega_-} - \mathscr{R}_j^+ 1\Big|_{\partial\Omega_+} \in \mathscr{C}^\alpha(\partial\Omega), \qquad \forall\, j \in \{1,\dots,n\}. \tag{6.35}$$

Given the present background assumptions on Ω, Theorem 2.2 then gives that Ω is a $\mathscr{C}^{1+\alpha}$ domain. ☐

Proof of (a) ⇒ (c) *in Theorem 1.1.* Assume that Ω is a domain of class $\mathscr{C}^{1+\alpha}$, $\alpha \in (0,1)$, with compact boundary. Here, the task is to prove that the principal value singular integral operator T, originally defined in (1.5), is a well-defined, linear and bounded mapping from $\mathscr{C}^\alpha(\partial\Omega)$ into itself. In the process, we shall also show that (1.10) holds. Since (a) ⇒ (e) has already been established, we know that the singular integral operator (6.1) maps $\mathscr{C}^\alpha(\partial\Omega)$ boundedly into $\mathscr{C}^\alpha\big(\overline{\Omega}\big)$ with

$$\big\|\mathbb{T}f\big\|_{\mathscr{C}^\alpha(\overline{\Omega})} \leq C^l 2^{l^2} \|P\|_{L^2(S^{n-1})} \|f\|_{\mathscr{C}^\alpha(\partial\Omega)}, \qquad \forall\, f \in \mathscr{C}^\alpha(\partial\Omega). \tag{6.36}$$

For starters, let us operate under the additional assumption that the homogeneous polynomial P is also harmonic, and abbreviate

$$k(x) := \frac{P(x)}{|x|^{n-1+l}}, \qquad \forall\, x \in \mathbb{R}^n \setminus \{0\}. \tag{6.37}$$

In such a scenario, (4.37) gives that

$$\widehat{k}(\xi) = \mathcal{F}_{x\to\xi}\Big(\frac{P(x)}{|x|^{n+l-1}}\Big) = \gamma_{n,l,1}\,\frac{P(\xi)}{|\xi|^{l+1}}, \qquad \forall\, \xi \in \mathbb{R}^n \setminus \{0\}. \tag{6.38}$$

Moreover, a direct computation using Stirling's approximation formula

$$\sqrt{2\pi}\, m^{m+1/2}e^{-m} \leq m! \leq e\, m^{m+1/2}e^{-m}, \qquad \forall\, m \in \mathbb{N}, \tag{6.39}$$

shows that

$$\gamma_{n,l,1} = \begin{cases} O(l^{-(n-2)/2}) & \text{if } n \text{ even,} \\ O(l^{-(n-4)/2}) & \text{if } n \text{ odd,} \end{cases} \qquad \text{as } \ l \to \infty. \tag{6.40}$$

We continue by observing that, thanks to (4.35),

$$\sup_{x\in\partial\Omega} |P(\nu(x))| \leq \|P\|_{L^\infty(S^{n-1})} \leq c_n 2^l l^{-1} \|P\|_{L^1(S^{n-1})}. \tag{6.41}$$

Next we note that $|\nu(x)-\nu(y)| \geq 1/2$ forces $|x-y|^\alpha \geq 1/(2\|\nu\|_{\mathscr{C}^\alpha(\partial\Omega)})$ which further implies

$$\begin{aligned} \frac{|P(\nu(x)) - P(\nu(y))|}{|x-y|^\alpha} &\leq 4\|\nu\|_{\mathscr{C}^\alpha(\partial\Omega)}\|P\|_{L^\infty(S^{n-1})} \\ &\leq c_n 2^l l^{-1} \|\nu\|_{\mathscr{C}^\alpha(\partial\Omega)}\|P\|_{L^1(S^{n-1})}, \end{aligned} \tag{6.42}$$

by virtue of (4.35), while if $|\nu(x)-\nu(y)| \leq 1/2$ the Mean Value Theorem and (4.35) permit us to once again estimate

$$\begin{aligned} \frac{|P(\nu(x)) - P(\nu(y))|}{|x-y|^\alpha} &\leq \Big(\sup_{z\in[\nu(x),\nu(y)]} |(\nabla P)(z)|\Big)\|\nu\|_{\mathscr{C}^\alpha(\partial\Omega)} \\ &\leq \|\nabla P\|_{L^\infty(S^{n-1})}\|\nu\|_{\mathscr{C}^\alpha(\partial\Omega)} \\ &\leq c_n 2^l l^{-1} \|\nu\|_{\mathscr{C}^\alpha(\partial\Omega)}\|P\|_{L^1(S^{n-1})}. \end{aligned} \tag{6.43}$$

By combining (6.38) and (6.40)-(6.43) we therefore arrive at the conclusion that

$$\begin{gathered} \text{the mapping } \partial\Omega \ni x \mapsto \widehat{k}(\nu(x)) \in \mathbb{C} \text{ belongs to } \mathscr{C}^\alpha(\partial\Omega) \\ \text{and } \big\|\widehat{k}(\nu(\cdot))\big\|_{\mathscr{C}^\alpha(\partial\Omega)} \leq c_n 2^l \|\nu\|_{\mathscr{C}^\alpha(\partial\Omega)}\|P\|_{L^1(S^{n-1})}. \end{gathered} \tag{6.44}$$

Next, the assumptions on Ω imply (cf. the discussion in §2) that this is both a UR domain and a uniform domain. As such, Theorem 3.1 applies. Since $\mathbb{T}$ from (6.1) corresponds to the operator $\mathcal{T}$ defined as in (3.2) with k as in (6.37), for each $f \in \mathscr{C}^\alpha(\partial\Omega)$ we obtain from (3.11), (6.44), and (6.36) that

$$\begin{aligned} \|Tf\|_{\mathscr{C}^\alpha(\partial\Omega)} &\leq \big\|\tfrac{1}{2i}\widehat{k}(\nu(\cdot))f + Tf\big\|_{\mathscr{C}^\alpha(\partial\Omega)} + \big\|\tfrac{1}{2i}\widehat{k}(\nu(\cdot))f\big\|_{\mathscr{C}^\alpha(\partial\Omega)} \\ &\leq \big\|\mathbb{T}f\big|^{\text{n.t.}}_{\partial\Omega}\big\|_{\mathscr{C}^\alpha(\partial\Omega)} + 2^{-1}\big\|\widehat{k}(\nu(\cdot))\big\|_{\mathscr{C}^\alpha(\partial\Omega)}\|f\|_{\mathscr{C}^\alpha(\partial\Omega)} \\ &= \big\|\mathbb{T}f\big|_{\partial\Omega}\big\|_{\mathscr{C}^\alpha(\partial\Omega)} + c_n 2^l \|\nu\|_{\mathscr{C}^\alpha(\partial\Omega)}\|P\|_{L^1(S^{n-1})}\|f\|_{\mathscr{C}^\alpha(\partial\Omega)} \\ &\leq \big\|\mathbb{T}f\big\|_{\mathscr{C}^\alpha(\overline{\Omega})} + c_n 2^l \|\nu\|_{\mathscr{C}^\alpha(\partial\Omega)}\|P\|_{L^2(S^{n-1})}\|f\|_{\mathscr{C}^\alpha(\partial\Omega)} \\ &\leq \Big\{C^l 2^{l^2} + c_n 2^l \|\nu\|_{\mathscr{C}^\alpha(\partial\Omega)}\Big\}\|P\|_{L^2(S^{n-1})}\|f\|_{\mathscr{C}^\alpha(\partial\Omega)} \\ &\leq (C^2)^l 2^{l^2}\|P\|_{L^2(S^{n-1})}\|f\|_{\mathscr{C}^\alpha(\partial\Omega)}, \end{aligned} \tag{6.45}$$

assuming, without loss of generality, that $C \geq 2 + c_n\|\nu\|_{\mathscr{C}^\alpha(\partial\Omega)}$ to begin with. Note that the estimate just derived has the format demanded in (1.10).

To treat the general case when P is merely as in (6.27), consider the decomposition (6.29) and, for each $f \in \mathscr{C}^\alpha(\partial\Omega)$, write

$$Tf(x) = \sum_{j=1}^{N+1} \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \frac{P_j(x-y)}{|x-y|^{n-1+(l-2(j-1))}} f(y)\,d\sigma(y), \quad x \in \partial\Omega. \tag{6.46}$$

Since every integral operator appearing in the right-hand side of (6.46) is of the same type as the original T in (1.5), with the additional property that the intervening homogeneous polynomial is harmonic, repeated applications of (6.45) give

$$\|Tf\|_{\mathscr{C}^\alpha(\partial\Omega)} \leq l(C^2)^l 2^{l^2}\|P\|_{L^2(S^{n-1})}\|f\|_{\mathscr{C}^\alpha(\partial\Omega)}, \qquad \forall\, f \in \mathscr{C}^\alpha(\partial\Omega). \tag{6.47}$$

Using $l \leq (C^2)^l$ for all l's if C is sufficiently large and re-denoting C^4 simply as C, estimate (1.10) finally follows. $\square$

Proof of (c) $\Rightarrow$ (b) *in Theorem 1.1.* This is trivial, since the Riesz transforms from (1.1) are special cases of the principal value singular integral operators defined in (1.5). $\square$

Proof of (b) $\Rightarrow$ (a) *in Theorem 1.1.* Given the assumption made in (1.4) and the background hypotheses on Ω, equivalence (1.3) may be used to conclude that Ω is a UR domain. Next, observe that since $\nu \odot \nu = -1$ and $x - y = \sum_{j=1}^n (x_j - y_j)e_j$, from (5.2) and (1.1) we obtain

$$\mathbf{C}\nu = -\sum_{j=1}^n (R_j 1)e_j \quad \sigma\text{-a.e. on } \partial\Omega \tag{6.48}$$

which, on account of (5.5), further yields

$$\tfrac{1}{4}\nu = \mathbf{C}(\mathbf{C}\nu) = -\mathbf{C}\Big(\sum_{j=1}^n (R_j 1)e_j\Big) \quad \sigma\text{-a.e. on } \partial\Omega. \tag{6.49}$$

With this in hand, it readily follows from Theorem 5.6 that if condition (1.4) holds then $\nu \in \mathscr{C}^\alpha(\partial\Omega, \mathbb{R}^n)$. Having established this, Theorem 2.2 applies and gives that Ω is a domain of class $\mathscr{C}^{1+\alpha}$. $\square$

This concludes the proof of Theorem 1.1, and we now turn to the proof of Theorem 1.2.

Proof of Theorem 1.2. This is a direct consequence of Proposition 2.11 and Theorem 1.1 upon observing that $\mathfrak{C} = iR_1 + R_2$. $\square$

We finally present the proof of Theorem 1.5.

Proof of Theorem 1.5. Let

$$k\big|_{S^{n-1}} = \sum_{l=0}^\infty Y_l \tag{6.50}$$

be the decomposition of $k\big|_{S^{n-1}} \in L^2(S^{n-1})$ in surface spherical harmonics. That is, $\{Y_l\}_{l\in\mathbb{N}_0}$ are mutually orthogonal functions in $L^2(S^{n-1})$ with the property that for each $l \in \mathbb{N}_0$ the function

$$P_l(x) := \begin{cases} |x|^l Y_l(x/|x|) & \text{if } x \in \mathbb{R}^n \setminus \{0\}, \\ 0 & \text{if } x = 0, \end{cases} \tag{6.51}$$

is a homogeneous harmonic polynomial of degree l in $\mathbb{R}^n$. In particular,

$$\Delta_{S^{n-1}} Y_l = -l(l+n-2)Y_l \quad \text{on } S^{n-1}, \qquad \forall\, l \in \mathbb{N}_0. \tag{6.52}$$

See, for example, [42, pp. 68-70] for a discussion. Then for each $l\in\mathbb{N}_0$ we may write

$$\begin{aligned}\big[-l(l+n-2)\big]^{m_l}&\|Y_l\|^2_{L^2(S^{n-1})}\\ &=\big[-l(l+n-2)\big]^{m_l}\int_{S^{n-1}}k\overline{Y_l}\,d\omega\\ &=\int_{S^{n-1}}k\Delta^{m_l}_{S^{n-1}}\overline{Y_l}\,d\omega=\int_{S^{n-1}}\big(\Delta^{m_l}_{S^{n-1}}k\big)\overline{Y_l}\,d\omega,\end{aligned}\tag{6.53}$$

where the first equality uses (6.50), the second one is based on (6.52), and the third one follows via repeated integrations by parts. In turn, from (6.53) and the Cauchy-Schwarz inequality we obtain

$$\|Y_l\|_{L^2(S^{n-1})}\leq l^{-2m_l}\big\|\Delta^{m_l}_{S^{n-1}}k\big\|_{L^2(S^{n-1})},\qquad\forall\,l\in\mathbb{N}_0.\tag{6.54}$$

We continue by noting that the homogeneity of k together with (6.50) and (6.51) permit us to express

$$k(x)=\frac{k(x/|x|)}{|x|^{n-1}}=\sum_{l=0}^{\infty}\frac{Y_l(x/|x|)}{|x|^{n-1}}=\sum_{l=0}^{\infty}\frac{P_l(x/|x|)}{|x|^{n-1}}=\sum_{l=0}^{\infty}\frac{P_l(x)}{|x|^{n-1+l}},\tag{6.55}$$

for each $x\in\mathbb{R}^n\backslash\{0\}$. For each $l\in\mathbb{N}_0$, let $\mathbb{T}_l$, T_l be the integral operators defined analogously to (1.18) and (1.19) in which the kernel $k(x-y)$ has been replaced by $P_l(x-y)|x-y|^{-(n-1+l)}$. Then for each $f\in\mathscr{C}^\alpha(\partial\Omega)$ we may estimate

$$\begin{aligned}\sum_{l=0}^{\infty}\|\mathbb{T}_lf\|_{\mathscr{C}^\alpha(\overline{\Omega})}&\leq\sum_{l=0}^{\infty}C^l2^{l^2}\|P_l\|_{L^2(S^{n-1})}\|f\|_{\mathscr{C}^\alpha(\partial\Omega)}\\ &=\sum_{l=0}^{\infty}C^l2^{l^2}\|Y_l\|_{L^2(S^{n-1})}\|f\|_{\mathscr{C}^\alpha(\partial\Omega)}\\ &\leq\Big\{\sum_{l=0}^{\infty}C^l2^{l^2}l^{-2m_l}\big\|\Delta^{m_l}_{S^{n-1}}k\big\|_{L^2(S^{n-1})}\Big\}\|f\|_{\mathscr{C}^\alpha(\partial\Omega)},\end{aligned}\tag{6.56}$$

by invoking (1.9), (6.54), and keeping in mind that $P\big|_{S^{n-1}}=Y_l$ (cf. (6.51)). Since for l large we have $C^l2^{l^2}\leq 4^{l^2}$, it follows from (1.17) that the series in the curly bracket in (6.56) is convergent to some finite constant M. Based on this and (6.55) we may then conclude that $\|\mathbb{T}f\|_{\mathscr{C}^\alpha(\overline{\Omega})}\leq\sum_{l=0}^{\infty}\|\mathbb{T}_lf\|_{\mathscr{C}^\alpha(\overline{\Omega})}\leq M\|f\|_{\mathscr{C}^\alpha(\partial\Omega)}$. This proves the boundedness of the first operator in (1.20), and the second operator in (1.20) is treated similarly (making use of (1.10)). □

Remark 6.1. *We claim that condition (1.17) is satisfied whenever the kernel k is of the form $P(x)/|x|^{n-1+l_o}$ for some homogeneous polynomial P of degree $l_o\in 2\mathbb{N}-1$ in $\mathbb{R}^n$. Indeed, writing $P(x)/|x|^{n-1+l_o}=P(x/|x|)/|x|^{n-1}$ and invoking (6.29), there is no loss of generality in assuming that P is also harmonic to begin with. Granted this, it follows that $k\big|_{S^{n-1}}=P\big|_{S^{n-1}}$ is a surface spherical harmonic of degree l_o, hence (cf. [42, § 3.1.4, p. 70]) $\Delta_{S^{n-1}}\big(k\big|_{S^{n-1}}\big)=-l_o(l_o+n-2)\big(k\big|_{S^{n-1}}\big)$. Choosing $m_l:=l^2$ for each $l\in\mathbb{N}_0$ and iterating this formula then shows that the series in (1.17) is dominated by*

$$\sum_{l=0}^{\infty}4^{l^2}l^{-2l^2}\big[l_o(l_o+n-2)\big]^{l^2}\|k\|_{L^2(S^{n-1})}<+\infty.\tag{6.57}$$

7. Further Results

We start by recalling some definitions. First, given a compact Ahlfors regular set $\Sigma \subset \mathbb{R}^n$ introduce $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$ and define the John-Nirenberg space of functions of bounded mean oscillations on Σ as

$$\mathrm{BMO}(\Sigma) := \big\{ f \in L^1(\Sigma, \sigma) : \, f^{\#,p} \in L^\infty(\Sigma, \sigma) \big\}, \tag{7.1}$$

where $p \in [1, \infty)$ is a fixed parameter and

$$f^{\#,p}(x) := \sup_{r>0} \left(\frac{1}{\sigma(\Sigma \cap B(x,r))} \int\limits_{\Sigma \cap B(x,r)} |f(y) - f_{\Delta(x,r)}|^p \, d\sigma(y) \right)^{\frac{1}{p}}, \tag{7.2}$$

with $f_{\Delta(x,r)}$ the mean value of f on $\Sigma \cap B(x,r)$. As is well known, various choices of p give the same space. Keeping this in mind, we define the seminorm

$$[f]_{\mathrm{BMO}(\Sigma)} := \|f^{\#,p}\|_{L^\infty(\Sigma,\sigma)}. \tag{7.3}$$

We then define the Sarason space $\mathrm{VMO}(\Sigma)$ of functions of vanishing mean oscillations on Σ as the closure in $\mathrm{BMO}(\Sigma)$ of $\mathscr{C}^0(\Sigma)$, the space of continuous functions on Σ. Alternatively, given any $\alpha \in (0,1)$, the space $\mathrm{VMO}(\Sigma)$ may be described (cf. [20, Proposition 2.15, p. 2602]) as the closure in $\mathrm{BMO}(\Sigma)$ of $\mathscr{C}^\alpha(\Sigma)$. Hence, in the present context,

$$\bigcup_{0 \le \alpha < 1} \mathscr{C}^\alpha(\Sigma) \hookrightarrow \mathrm{VMO}(\Sigma) \hookrightarrow \mathrm{BMO}(\Sigma) \hookrightarrow \bigcap_{0<p<\infty} L^p(\Sigma, \sigma). \tag{7.4}$$

Proposition 7.1. *If $\Omega \subseteq \mathbb{R}^n$ is a* UR *domain with compact boundary then the principal value Cauchy-Clifford operator $\mathbf{C}$ from (5.2) is bounded both on $\mathrm{BMO}(\partial\Omega) \otimes \mathcal{C}\ell_n$, as well as on $\mathrm{VMO}(\partial\Omega) \otimes \mathcal{C}\ell_n$. Moreover, $\mathbf{C}^2 = \frac{1}{4}I$ both on $\mathrm{BMO}(\partial\Omega) \otimes \mathcal{C}\ell_n$ and on $\mathrm{VMO}(\partial\Omega) \otimes \mathcal{C}\ell_n$. Hence, in particular, $\mathbf{C}$ is an isomorphism when acting on either of these spaces.*

Proof. To begin with, observe that in the present setting (5.21) ensures that $\mathbf{C}$ is well-defined on $\mathrm{BMO}(\partial\Omega) \otimes \mathcal{C}\ell_n$. Fix now $f \in \mathrm{BMO}(\partial\Omega) \otimes \mathcal{C}\ell_n$ and pick some $x_0 \in \partial\Omega$ and $r > 0$. For each $R > 0$, let us agree to abbreviate $\Delta_R := \partial\Omega \cap B(x_0, R)$. Denote by ν the geometric measure theoretic outward unit normal to Ω and, with $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, introduce

$$A(x_0, r) := \frac{1}{\omega_{n-1}} \int\limits_{\substack{y \in \partial\Omega \\ |x_0 - y| \ge 2r}} \frac{x_0 - y}{|x_0 - y|^n} \odot \nu(y) \odot \big(f(y) - f_{\Delta_{2r}}\big) \, d\sigma(y) \pm \frac{1}{2} f_{\Delta_{2r}} \tag{7.5}$$

where the sign is chosen to be plus if Ω is bounded and minus if Ω is unbounded, and where $f_{\Delta_{2r}}$ stands for the integral average of f over Δ_{2r}. For $x \in \Delta_r$ use (5.21) to split

$$\begin{aligned} \mathbf{C}f(x) = \lim_{\varepsilon \to 0^+} \frac{1}{\omega_{n-1}} &\int\limits_{\substack{y \in \partial\Omega \setminus B(x,\varepsilon) \\ |x_0 - y| < 2r}} \frac{x - y}{|x - y|^n} \odot \nu(y) \odot \big(f(y) - f_{\Delta_{2r}}\big) \, d\sigma(y) \\ &+ \frac{1}{\omega_{n-1}} \int\limits_{\substack{y \in \partial\Omega \\ |x_0 - y| \ge 2r}} \Big(\frac{x - y}{|x - y|^n} - \frac{x_0 - y}{|x_0 - y|^n} \Big) \odot \nu(y) \odot \big(f(y) - f_{\Delta_{2r}}\big) \, d\sigma(y) \\ &+ A(x_0, r), \end{aligned} \tag{7.6}$$

then employ this representation (and Minkowski's inequality) in order to estimate

$$\left(\frac{1}{\sigma(\Delta_r)} \int_{\Delta_r} \big| \mathbf{C}f(x) - A(x_0, r) \big|^2 \, d\sigma(x) \right)^{\frac{1}{2}} \le c\big(I + II\big), \tag{7.7}$$

where $c \in (0,\infty)$ depends only on Ω and

$$I := \left(\frac{1}{\sigma(\Delta_r)} \int_{\partial\Omega} \big|\mathbf{C}\big((f - f_{\Delta_{2r}})\mathbf{1}_{\Delta_{2r}}\big)\big|^2 \, d\sigma \right)^{\frac{1}{2}}$$

and

$$II := r^{-\frac{n-1}{2}} \int\limits_{\substack{y\in\partial\Omega \\ |x_0-y|\geq 2r}} \left(\int_{\Delta_r} \left| \frac{x-y}{|x-y|^n} - \frac{x_0-y}{|x_0-y|^n} \right|^2 d\sigma(x) \right)^{\frac{1}{2}} \big|f(y) - f_{\Delta_{2r}}\big| \, d\sigma(y).$$

Now, the boundedness of $\mathbf{C}$ on $L^2(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n$ from Proposition 5.1 gives (bearing in mind that σ is doubling)

$$I \leq c \left(\frac{1}{\sigma(\Delta_{2r})} \int_{\Delta_{2r}} \big|f - f_{\Delta_{2r}}\big|^2 \, d\sigma \right)^{\frac{1}{2}} \leq c[f]_{\mathrm{BMO}(\partial\Omega)\otimes\mathcal{C}\ell_n}, \tag{7.8}$$

which suits our purposes. Next, we write

$$\begin{aligned} II &\leq c \int\limits_{\substack{y\in\partial\Omega \\ |x_0-y|\geq 2r}} \frac{r}{|x_0-y|^n} \big|f(y) - f_{\Delta_{2r}}\big| \, d\sigma(y) \\ &\leq c \sum_{j=1}^{\infty} \int_{\Delta_{2^{j+1}r}\setminus\Delta_{2^j r}} \frac{r}{(2^j r)^n} |f(y) - f_{\Delta_{2r}}| \, d\sigma(y) \\ &\leq c \sum_{j=1}^{\infty} \frac{1}{2^j} \fint_{\Delta_{2^{j+1}r}} |f - f_{\Delta_{2r}}| \, d\sigma \\ &\leq c \sum_{j=1}^{\infty} \frac{1}{2^j} \fint_{\Delta_{2^{j+1}r}} \Big[|f - f_{\Delta_{2^{j+1}r}}| + \sum_{k=1}^{j} |f_{\Delta_{2^{k+1}r}} - f_{\Delta_{2^k r}}| \Big] \, d\sigma \\ &\leq c \sum_{j=1}^{\infty} \frac{1}{2^j} (1+j) f^{\#,1}(x_0) \leq c f^{\#,1}(x_0) \leq c[f]_{\mathrm{BMO}(\partial\Omega)\otimes\mathcal{C}\ell_n}. \end{aligned} \tag{7.9}$$

Above, the first inequality follows from the Mean Value Theorem, while the second inequality is a consequence of writing the integral over $\partial\Omega \setminus \Delta_{2r}$ as the telescopic sum over $\Delta_{2^{j+1}r} \setminus \Delta_{2^j r}$, $j \in \mathbb{N}$ and the fact that $|x_0 - y| \geq 2^j r$ for $y \in \Delta_{2^{j+1}r} \setminus \Delta_{2^j r}$. The third inequality is a result of enlarging the domain of integration from $\Delta_{2^{j+1}r} \setminus \Delta_{2^j r}$ to $\Delta_{2^{j+1}r}$ and using $\sigma\big(\Delta_{2^{j+1}r}\big) \approx (2^j r)^{n-1}$. The fourth inequality follows from the triangle inequality after writing

$$f - f_{\Delta_{2r}} = f - f_{\Delta_{2^{j+1}r}} + \sum_{k=1}^{j} (f_{\Delta_{2^{k+1}r}} - f_{\Delta_{2^k r}}). \tag{7.10}$$

The fifth inequality is a consequence of the fact that, for each k, we have

$$\begin{aligned} \big|f_{\Delta_{2^{k+1}r}} - f_{\Delta_{2^k r}}\big| &= \Big| \fint_{\Delta_{2^k r}} (f - f_{\Delta_{2^{k+1}r}}) \, d\sigma \Big| \\ &\leq c \fint_{\Delta_{2^{k+1}r}} \big|f - f_{\Delta_{2^{k+1}r}}\big| \, d\sigma \leq c\, f^{\#,1}(x_0). \end{aligned} \tag{7.11}$$

The sixth inequality is a consequence of $\sum\limits_{j=1}^{\infty} 2^{-j}(1+j) < +\infty$ and, finally, the last inequality is seen from (7.3).

From (7.7)-(7.9) we eventually obtain $\|(\mathbf{C}f)^{\#,2}\|_{L^\infty(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n} \leq c[f]_{\mathrm{BMO}(\partial\Omega)\otimes\mathcal{C}\ell_n}$, hence

$$[\mathbf{C}f]_{\mathrm{BMO}(\partial\Omega)\otimes\mathcal{C}\ell_n} \leq c[f]_{\mathrm{BMO}(\partial\Omega)\otimes\mathcal{C}\ell_n} \tag{7.12}$$

from which we conclude that the operator

$$\mathbf{C} : \mathrm{BMO}(\partial\Omega) \otimes \mathcal{C}\ell_n \longrightarrow \mathrm{BMO}(\partial\Omega) \otimes \mathcal{C}\ell_n \tag{7.13}$$

is well-defined and bounded. Next, that

$$\mathbf{C} : \mathrm{VMO}(\partial\Omega) \otimes \mathcal{C}\ell_n \longrightarrow \mathrm{VMO}(\partial\Omega) \otimes \mathcal{C}\ell_n \tag{7.14}$$

is also well-defined and bounded follows from (7.13), the characterization of $\mathrm{VMO}(\partial\Omega)\otimes\mathcal{C}\ell_n$ as the closure in $\mathrm{BMO}(\partial\Omega) \otimes \mathcal{C}\ell_n$ of $\mathscr{C}^\alpha(\partial\Omega) \otimes \mathcal{C}\ell_n$ for each $\alpha \in (0, 1)$, and Theorem 5.6.

Finally, the claims in the last part of the statement of the proposition are direct consequences of what we have proved so far, (7.4), and (5.5). □

When $\Omega \subseteq \mathbb{R}^n$ is a UR domain with compact boundary (a scenario in which the Riesz transforms are bounded on $L^2(\partial\Omega, \sigma)$), standard Calderón-Zygmund theory implies that $R_j : L^\infty(\partial\Omega, \sigma) \to \mathrm{BMO}(\partial\Omega)$ is bounded for each $j \in \{1, \dots, n\}$. Hence, in this case, we have $R_j 1 \in \mathrm{BMO}(\partial\Omega)$ for each $j \in \{1, \dots, n\}$. Remarkably, the proximity of the BMO functions $R_j 1$, $1 \leq j \leq n$, to the space $\mathrm{VMO}(\partial\Omega)$ controls how close the outward unit normal ν to Ω is to being in $\mathrm{VMO}(\partial\Omega, \mathbb{R}^n)$. Specifically, we have the following result.

Theorem 7.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a* UR *domain with compact boundary and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, let $\|\mathbf{C}\|_*$ stand for the operator norm of the Cauchy-Clifford singular integral operator acting on the space* $\mathrm{BMO}(\partial\Omega) \otimes \mathcal{C}\ell_n$*. Then, with distances considered in* $\mathrm{BMO}(\partial\Omega, \mathbb{R}^n)$ *or simply* $\mathrm{BMO}(\partial\Omega)$*, as appropriate, one has*

$$\mathrm{dist}\big(\nu\,,\, \mathrm{VMO}(\partial\Omega, \mathbb{R}^n)\big) \leq 4\|\mathbf{C}\|_* \Big(\sum_{j=1}^n \mathrm{dist}\big(R_j 1\,,\, \mathrm{VMO}(\partial\Omega)\big)^2 \Big)^{1/2}, \tag{7.15}$$

$$\Big(\sum_{j=1}^n \mathrm{dist}\big(R_j 1\,,\, \mathrm{VMO}(\partial\Omega)\big)^2 \Big)^{1/2} \leq \|\mathbf{C}\|_* \,\mathrm{dist}\big(\nu\,,\, \mathrm{VMO}(\partial\Omega, \mathbb{R}^n)\big). \tag{7.16}$$

Proof. On the one had, based on (6.49) and Proposition 7.1 we may estimate

$$\begin{aligned}
\mathrm{dist}\big(\nu\,,\, \mathrm{VMO}(\partial\Omega, \mathbb{R}^n)\big) &= \inf_{\eta \in \mathrm{VMO}(\partial\Omega, \mathbb{R}^n)} [\nu - \eta]_{\mathrm{BMO}(\partial\Omega, \mathbb{R}^n)} \\
&= \inf_{\eta \in \mathrm{VMO}(\partial\Omega)\otimes\mathcal{C}\ell_n} [\nu - \eta]_{\mathrm{BMO}(\partial\Omega)\otimes\mathcal{C}\ell_n} \\
&= \inf_{\eta \in \mathrm{VMO}(\partial\Omega)\otimes\mathcal{C}\ell_n} 4\Big[\mathbf{C}\Big(\sum_{j=1}^n (R_j 1)e_j + \mathbf{C}\eta \Big)\Big]_{\mathrm{BMO}(\partial\Omega)\otimes\mathcal{C}\ell_n} \\
&\leq 4\|\mathbf{C}\|_* \inf_{\eta \in \mathrm{VMO}(\partial\Omega)\otimes\mathcal{C}\ell_n} \Big[\sum_{j=1}^n (R_j 1)e_j + \mathbf{C}\eta \Big]_{\mathrm{BMO}(\partial\Omega)\otimes\mathcal{C}\ell_n} \\
&= 4\|\mathbf{C}\|_* \inf_{\xi \in \mathrm{VMO}(\partial\Omega)\otimes\mathcal{C}\ell_n} \Big[\sum_{j=1}^n (R_j 1)e_j - \xi \Big]_{\mathrm{BMO}(\partial\Omega)\otimes\mathcal{C}\ell_n} \\
&= 4\|\mathbf{C}\|_* \inf_{\xi \in \mathrm{VMO}(\partial\Omega, \mathbb{R}^n)} \Big[\sum_{j=1}^n (R_j 1)e_j - \xi \Big]_{\mathrm{BMO}(\partial\Omega, \mathbb{R}^n)} \\
&= 4\|\mathbf{C}\|_* \Big(\sum_{j=1}^n \mathrm{dist}\big(R_j 1\,,\, \mathrm{VMO}(\partial\Omega)\big)^2 \Big)^{1/2},
\end{aligned} \tag{7.17}$$

yielding (7.15). On the other hand, from (6.48) and Proposition 7.1 we deduce

$$\begin{aligned}\Big(\sum_{j=1}^{n}\mathrm{dist}\big(R_j 1\,,\,\mathrm{VMO}(\partial\Omega)\big)^2\Big)^{1/2} &= \inf_{\xi\in\mathrm{VMO}(\partial\Omega)\otimes\mathcal{C}\!\ell_n}\Big[\sum_{j=1}^{n}(R_j 1)e_j-\xi\Big]_{\mathrm{BMO}(\partial\Omega)\otimes\mathcal{C}\!\ell_n}\\ &= \inf_{\xi\in\mathrm{VMO}(\partial\Omega)\otimes\mathcal{C}\!\ell_n}\Big[\mathbf{C}\nu-\xi\Big]_{\mathrm{BMO}(\partial\Omega)\otimes\mathcal{C}\!\ell_n}\\ &= \inf_{\eta\in\mathrm{VMO}(\partial\Omega)\otimes\mathcal{C}\!\ell_n}\Big[\mathbf{C}(\nu-\eta)\Big]_{\mathrm{BMO}(\partial\Omega)\otimes\mathcal{C}\!\ell_n}\\ &\leq \|\mathbf{C}\|_*\inf_{\eta\in\mathrm{VMO}(\partial\Omega)\otimes\mathcal{C}\!\ell_n}\big[\nu-\eta\big]_{\mathrm{BMO}(\partial\Omega)\otimes\mathcal{C}\!\ell_n}\\ &= \|\mathbf{C}\|_*\inf_{\eta\in\mathrm{VMO}(\partial\Omega,\mathbb{R}^n)}[\nu-\eta]_{\mathrm{BMO}(\partial\Omega,\mathbb{R}^n)}\\ &= \|\mathbf{C}\|_*\,\mathrm{dist}\big(\nu\,,\,\mathrm{VMO}(\partial\Omega,\mathbb{R}^n)\big), \end{aligned}\tag{7.18}$$

establishing (7.16). □

Having established Theorem 7.2, we are now in a position to present the proof of Theorem 1.3.

Proof of Theorem 1.3. For the left-to-right implication in (1.14), start by observing that Ω is a UR domain (cf. Definition 2.6). As such, Theorem 7.2 applies and (7.16) gives that $R_j 1\in\mathrm{VMO}(\partial\Omega)$ for each $j\in\{1,\dots,n\}$. For the right-to-left implication in (1.14), use (1.3) and the background assumptions on Ω to conclude that Ω is a UR domain, then invoke (7.15) from Theorem 7.2 to conclude that $\nu\in\mathrm{VMO}(\partial\Omega,\mathbb{R}^n)$. □

Moving on, we record the following definition.

Definition 7.3. *Let $\Omega\subset\mathbb{R}^n$ be an open set with compact boundary. Then Ω is said to satisfy a* `John condition` *if there exist $\theta\in(0,1)$ and $R\in(0,\infty)$, called the John constants of Ω, with the following significance. For every $p\in\partial\Omega$ and $r\in(0,R)$ one can find $p_r\in B(p,r)\cap\Omega$ such that $B(p_r,\theta r)\subset\Omega$ and with the property that for each $x\in B(p,r)\cap\partial\Omega$ there exists a rectifiable path $\gamma_x:[0,1]\to\overline{\Omega}$, whose length is $\leq\theta^{-1}r$ and*

$$\gamma_x(0)=x,\quad \gamma_x(1)=p_r,\quad\text{and}\;\; \mathrm{dist}\,(\gamma_x(t),\partial\Omega)>\theta\,|\gamma_x(t)-x|\quad\forall\,t\in(0,1]. \tag{7.19}$$

Furthermore, Ω is said to satisfy a `two-sided John condition` *if both Ω and $\mathbb{R}^n\setminus\overline{\Omega}$ satisfy a John condition.*

The above definition appears in [20], where it has been noted that any NTA domain (in the sense of D. Jerison and C. Kenig; [23]) with compact boundary satisfies a John condition.

Next, we recall the concept of δ-Reifenberg flat domain, following [25], [26]. As a preamble, the reader is reminded that the Pompeiu-Hausdorff distance between two sets $A,B\subseteq\mathbb{R}^n$ is given by

$$D[A,B]:=\max\Big\{\sup\{\mathrm{dist}(a,B):\,a\in A\}\,,\,\sup\{\mathrm{dist}(b,A):\,b\in B\}\Big\}. \tag{7.20}$$

Definition 7.4. *Let $\Sigma\subset\mathbb{R}^n$ be a compact set and let $\delta\in(0,\frac{1}{4\sqrt{2}})$. Call Σ a δ-*`Reifenberg flat set` *if there exists $R>0$ such that for every $x\in\Sigma$ and every $r\in(0,R]$ there exists an $(n-1)$-dimensional plane $L(x,r)$ which contains x and such that*

$$D\big[\Sigma\cap B(x,r)\,,\,L(x,r)\cap B(x,r)\big]\leq\delta r. \tag{7.21}$$

Definition 7.5. *Say that a bounded open set $\Omega\subset\mathbb{R}^n$ has the* `separation property` *if there exists $R>0$ such that for every $x\in\partial\Omega$ and $r\in(0,R]$ there exists an $(n-1)$-dimensional plane $\mathcal{L}(x,r)$ containing x and a choice of unit normal vector to $\mathcal{L}(x,r)$, call* *it $\vec{n}_{x,r}$, satisfying*

$$\begin{aligned} &\{y + t\,\vec{n}_{x,r} \in B(x,r) : \, y \in \mathcal{L}(x,r),\ t < -\tfrac{r}{4}\} \subset \Omega, \\ &\{y + t\,\vec{n}_{x,r} \in B(x,r) : \, y \in \mathcal{L}(x,r),\ t > \tfrac{r}{4}\} \subset \mathbb{R}^n \setminus \Omega. \end{aligned} \tag{7.22}$$

Moreover, if Ω is unbounded, it is also required that $\partial\Omega$ divides $\mathbb{R}^n$ into two distinct connected components and that $\mathbb{R}^n \setminus \Omega$ has a non-empty interior.

Definition 7.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $\delta \in (0, \delta_n)$. Call Ω a δ-*`Reifenberg flat domain` *if Ω has the separation property and $\partial\Omega$ is a δ-Reifenberg flat set.*

The notion of `Reifenberg flat domain with vanishing constant` *is introduced in a similar fashion, this time allowing the constant δ appearing in (7.21) to depend on r, say $\delta = \delta(r)$, and demanding that $\lim\limits_{r\to 0^+} \delta(r) = 0$.*

As our next result shows, under appropriate background assumptions (of a "large" geometry nature) the proximity of the vector-valued function $(R_1 1, R_2 1, \dots, R_n 1)$ to the space $\mathrm{VMO}(\partial\Omega, \mathbb{R}^n)$, measured in $\mathrm{BMO}(\partial\Omega, \mathbb{R}^n)$, can be used to quantify Reifenberg flatness.

Theorem 7.7. *Assume $\Omega \subseteq \mathbb{R}^n$ is an open set with a compact Ahlfors regular boundary, satisfying a two-sided John condition. If, with distances considered in $\mathrm{BMO}(\partial\Omega)$,*

$$\sum_{j=1}^{n} \mathrm{dist}\big(R_j 1\,,\, \mathrm{VMO}(\partial\Omega)\big) < \varepsilon \tag{7.23}$$

then Ω is a δ-Reifenberg flat domain for $\delta = C_o \cdot \varepsilon$, where $C_o \in (0,\infty)$ depends only on the Ahlfors regularity and John constants of Ω.

As a consequence, if $R_j 1 \in \mathrm{VMO}(\partial\Omega)$ for every $j \in \{1, \dots, n\}$ then actually Ω is a Reifenberg flat domain with vanishing constant.

Proof. It is known that if $\Omega \subseteq \mathbb{R}^n$ is an open set with a compact Ahlfors regular boundary, satisfying a two-sided John condition, and such that

$$\mathrm{dist}\big(\nu\,,\, \mathrm{VMO}(\partial\Omega, \mathbb{R}^n)\big) < \varepsilon \tag{7.24}$$

(with the distance considered in $\mathrm{BMO}(\partial\Omega, \mathbb{R}^n)$), then Ω is a δ-Reifenberg flat domain for $\delta = C_o \cdot \varepsilon$, where the constant $C_o \in (0,\infty)$ is as in the statement of the theorem. See [20, Definition 4.7 p. 2690 and Corollary 4.20 p. 2710] in this regard. Granted this, the desired conclusion follows by invoking Theorem 7.2, since our assumptions on Ω guarantee that this is a UR domain (cf. (1.15)). □

In this last part of this section we discuss a (partial) extension of Theorem 1.1 in the context of Besov spaces. We begin by defining this scale, and recalling some of its most basic properties.

Definition 7.8. *Assume that $\Sigma \subset \mathbb{R}^n$ is an Ahlfors regular set and let $\sigma := \mathcal{H}^{n-1} \lfloor \Sigma$. Then, given $1 \leq p \leq \infty$ and $0 < s < 1$, define the Besov space*

$$B^{p,p}_s(\Sigma) := \big\{ f \in L^p(\Sigma, \sigma) : \, \|f\|_{B^{p,p}_s(\Sigma)} < +\infty \big\} \tag{7.25}$$

where

$$\|f\|_{B^{p,p}_s(\Sigma)} := \|f\|_{L^p(\Sigma,\sigma)} + \Big(\int_\Sigma \int_\Sigma \frac{|f(x) - f(y)|^p}{|x-y|^{n-1+sp}} \, d\sigma(x)\, d\sigma(y) \Big)^{1/p}, \tag{7.26}$$

with the convention that

$$B^{\infty,\infty}_s(\Sigma) := \mathscr{C}^s(\Sigma) \quad \text{and} \quad \|f\|_{B^{\infty,\infty}_s(\Sigma)} := \|f\|_{\mathscr{C}^s(\Sigma)}. \tag{7.27}$$

Finally, denote by $B^{p,p}_{s,\mathrm{loc}}(\Sigma)$ the space of functions whose truncations by smooth and compactly supported functions belong to $B^{p,p}_s(\Sigma)$.

Consider Σ as in Definition 7.8 and suppose $1 \leq p_0, p_1 \leq \infty$ and $s_0, s_1 \in (0,1)$ are such that

$$\frac{1}{p_1} - \frac{s_1}{n-1} = \frac{1}{p_0} - \frac{s_0}{n-1} \quad \text{and} \quad s_0 \geq s_1. \tag{7.28}$$

Then [24, Proposition 5, p. 213] gives that

$$B^{p_0,p_0}_{s_0}(\Sigma) \hookrightarrow B^{p_1,p_1}_{s_1}(\Sigma) \ \text{ continuously.} \tag{7.29}$$

In particular,

$$B^{p,p}_s(\Sigma) \hookrightarrow \mathscr{C}^\alpha(\Sigma) \ \text{ if } \ p \in [1,\infty],\ s \in (0,1),\ sp > n-1,\ \alpha := s - \tfrac{n-1}{p}. \tag{7.30}$$

In turn, from (7.25)-(7.26) and (7.30) one may easily deduce that

$$\text{if } \ p \in [1,\infty],\ s \in (0,1) \ \text{ satisfy } \ sp > n-1, \text{ then } \ B^{p,p}_s(\Sigma) \ \text{ is an algebra,} \tag{7.31}$$

and

$$\begin{array}{c} f/g \in B^{p,p}_s(\Sigma) \ \text{ whenever } f, g \in B^{p,p}_s(\Sigma) \\ \text{and } |g| \geq c > 0 \ \sigma\text{-a.e. on } \Sigma. \end{array} \tag{7.32}$$

Another useful simple property is that, given any $p \in [1,\infty]$ and $s \in (0,1)$, if $F : \mathbb{R} \to \mathbb{R}$ is a bounded Lipschitz function then

$$F \circ f \in B^{p,p}_{s,\mathrm{loc}}(\Sigma) \ \text{ for every } \ f \in B^{p,p}_s(\Sigma). \tag{7.33}$$

Finally, we note that in the case when Σ is the graph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$, from [33, Proposition 2.9, p. 33] and real interpolation we obtain that, for each $p \in (1,\infty)$ and $s \in (0,1)$,

$$f \in B^{p,p}_s(\Sigma) \Longleftrightarrow f(\cdot,\varphi(\cdot)) \in B^{p,p}_s(\mathbb{R}^{n-1}). \tag{7.34}$$

Proposition 7.9. *Assume $\Omega \subset \mathbb{R}^n$ is a Lebesgue measurable set whose boundary is compact, Ahlfors regular, and satisfies* (2.15). *Then*

$$\mathbf{C} : B^{p,p}_s(\partial\Omega) \otimes \mathcal{C}\ell_n \longrightarrow B^{p,p}_s(\partial\Omega) \otimes \mathcal{C}\ell_n \tag{7.35}$$

is well-defined and bounded for each $p \in [1,\infty]$ and $s \in (0,1)$.

Proof. One way to see this is via real interpolation (cf. [17, S 8.1] for a version suiting the current setting) between the boundedness result proved in Theorem 5.6 (corresponding to (7.35) when $p = \infty$; cf. (7.27)), and the fact that the operator $\mathbf{C}$ in (7.35) with $p = 1$ is also bounded (which follows from the atomic/molecular theory for the Besov scale on spaces of homogeneous type from [18]). □

In order to present the extension of Theorem 1.1 mentioned earlier to the scale of Besov spaces, we make the following definition.

Definition 7.10. *Given $p \in [1,\infty]$ and $s \in (0,1)$, call a nonempty, open, proper subset Ω of $\mathbb{R}^n$ a $B^{p,p}_{s+1}$-*`domain` *provided it may be locally identified*[6] *near boundary points with the upper-graph of a real-valued function φ defined in $\mathbb{R}^{n-1}$ with the property that $\partial_j\varphi \in B^{p,p}_s(\mathbb{R}^{n-1})$ for each $j \in \{1,\dots,n-1\}$.*

The stage has been set for stating and proving the following result.

Theorem 7.11. *Assume $\Omega \subseteq \mathbb{R}^n$ is an open set with a compact Ahlfors regular boundary, satisfying $\partial\Omega = \partial(\overline{\Omega})$ and $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$. Then for each $s \in (0,1)$ and $p \in [1,\infty]$ with the property that $sp > n-1$ the following claims are equivalent:*

(a) *Ω is a $B^{p,p}_{s+1}$-domain;*
(b) *the Riesz transforms, defined as in* (1.1), *satisfy*

$$R_j 1 \in B^{p,p}_s(\partial\Omega) \quad \textit{for each} \ \ j \in \{1,\dots,n\}; \tag{7.36}$$

[6] in the sense described in Definition 2.1

Proof. Consider the implication (b) $\Rightarrow$ (a). The starting point is the observation that (7.36) and (7.30) imply (1.4) for $\alpha := s - \frac{n-1}{p} \in (0,1)$. As such, Theorem 1.1 applies and gives that Ω is a domain of class $\mathscr{C}^{1+\alpha}$. Hence, locally, the outward unit normal ν to Ω has components $(\nu_j)_{1\leq j\leq n}$ of the form

$$\nu_j(x',\varphi(x')) = \begin{cases} \frac{\partial_j\varphi(x')}{\sqrt{1+|\nabla\varphi(x')|^2}} & \text{if } 1\leq j\leq n-1, \\ -\frac{1}{\sqrt{1+|\nabla\varphi(x')|^2}} & \text{if } j=n, \end{cases} \tag{7.37}$$

where $\varphi \in \mathscr{C}^{1+\alpha}(\mathbb{R}^{n-1})$ is a real-valued function whose upper-graph locally describes Ω. Without loss of generality it may be assumed that φ has compact support.

On the other hand, from the assumption (7.36), Proposition 7.9, and (6.49) we may conclude that

$$\nu \in B^{p,p}_s(\partial\Omega,\mathbb{R}^n). \tag{7.38}$$

On account of this membership and (7.34) we obtain

$$\nu_j(\cdot,\varphi(\cdot)) \in B^{p,p}_s(\mathbb{R}^{n-1}) \ \text{ for each } \ j\in\{1,\dots,n\}. \tag{7.39}$$

Upon recalling (7.31)-(7.32), this further yields

$$\partial_j\varphi = \frac{\nu_j(\cdot,\varphi(\cdot))}{\nu_n(\cdot,\varphi(\cdot))} \in B^{p,p}_s(\mathbb{R}^{n-1}) \ \text{ for each } \ j\in\{1,\dots,n-1\}, \tag{7.40}$$

proving that Ω is a $B^{p,p}_{s+1}$-domain.

Concerning the implication (a) $\Rightarrow$ (b), assume that Ω is a $B^{p,p}_{s+1}$-domain with s,p as before. From definitions and (7.30) (used with $\Sigma := \mathbb{R}^{n-1}$) it follows that Ω is a domain of class $\mathscr{C}^{1+\alpha}$ with $\alpha := s - \frac{n-1}{p}$. Hence, in particular, Ω is a Lipschitz domain. We claim that (7.38) holds. Thanks to (7.34), justifying this claim comes down to proving that (7.39) holds, where φ is a real-valued function defined in $\mathbb{R}^{n-1}$ satisfying $\partial_j\varphi \in B^{p,p}_s(\mathbb{R}^{n-1})$ for each $j\in\{1,\dots,n-1\}$, and whose upper-graph locally describes Ω (again, without loss of generality it may be assumed that φ has compact support). To this end, consider the function $F:\mathbb{R}\to\mathbb{R}$ given by $F(t) := \frac{1}{\sqrt{1+|t|}}$ for each $t\in\mathbb{R}$, and note that F is both bounded and Lipschitz. Since by (7.31)

$$|\nabla\varphi|^2 = \sum_{j=1}^{n-1}(\partial_j\varphi)(\partial_j\varphi) \in B^{p,p}_s(\mathbb{R}^{n-1}), \tag{7.41}$$

it follows from (7.33) that

$$\nu_n(\cdot,\varphi(\cdot)) = -F\circ|\nabla\varphi|^2 \in B^{p,p}_{s,\,\mathrm{loc}}(\mathbb{R}^{n-1}). \tag{7.42}$$

Granted this, another reference to (7.31) gives that for each $j\in\{1,\dots,n-1\}$

$$\nu_j(\cdot,\varphi(\cdot)) = \frac{\partial_j\varphi}{\sqrt{1+|\nabla\varphi|^2}} = -\partial_j\varphi\cdot\nu_n(\cdot,\varphi(\cdot)) \in B^{p,p}_s(\mathbb{R}^{n-1}). \tag{7.43}$$

This finishes the proof of (7.39), hence completing the justification of (7.38). Having established this, bring in identity (6.48) in order to conclude on account of Proposition 7.9 that

$$\sum_{j=1}^{n}(R_j 1)e_j = -\mathbf{C}\nu \in B^{p,p}_s(\partial\Omega)\otimes\mathcal{C}\ell_n. \tag{7.44}$$

Since this readily implies (7.36), the implication (a) $\Rightarrow$ (b) is established. □

Lastly, we remark that the limiting case $s=1$ of Theorem 7.11 also holds provided $p\in(n-1,\infty)$ and the Besov space intervening in (7.36) is replaced by $L^p_1(\partial\Omega)$, the L^p-based Sobolev space of order one on $\partial\Omega$ considered in [20] (in which scenario Ω is an L^p_2-domain, in a natural sense). The proof follows the same blue-print, and makes use of the fact that $\mathbf{C}$ is a bounded operator from $L^p_1(\partial\Omega)\otimes\mathcal{C}\ell_n$ into itself (cf. [34] in this regard).

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Dorina Mitrea
Department of Mathematics
University of Missouri at Columbia
Columbia, MO 65211, USA

e-mail: *mitread@missouri.edu*

Marius Mitrea
Department of Mathematics
University of Missouri at Columbia
Columbia, MO 65211, USA

e-mail: *mitream@missouri.edu*

Joan Verdera
Department de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra, Barcelona, Catalonia

e-mail: *jvm@mat.uab.cat*