Explicit minimisers of some nonlocal anisotropic energies: a short proof

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Abstract

In this paper we consider nonlocal energies defined on probability measures in the plane, given by a convolution interaction term plus a quadratic confinement. The interaction kernel is $-\log|z| + \alpha\, x^2/|z|^2$, $z = x + iy$, with $-1 < \alpha < 1$. This kernel is anisotropic except for the Coulombic case $\alpha = 0$. We present a short compact proof of the known surprising fact that the unique minimiser of the energy is the normalised characteristic function of the domain enclosed by an ellipse with horizontal semi-axis $\sqrt{1-\alpha}$ and vertical semi-axis $\sqrt{1+\alpha}$. Letting $\alpha \to 1^-$ we find that the semicircle law on the vertical axis is the unique minimiser of the corresponding energy, a result related to interacting dislocations, and previously obtained by some of the authors. We devote the first sections of this paper to presenting some well-known background material in the simplest way possible, so that readers unfamiliar with the subject find the proofs accessible.

1   Introduction

Consider the energy functional defined on a probability measure μ in the plane by

$$(1) \qquad I_\alpha(\mu) = \iint W_\alpha(z - w)\, d\mu(z)\, d\mu(w) + \int |z|^2\, d\mu(z),$$

where the interaction kernel is

$$(2) \qquad W_\alpha(z) = -\log|z| + \alpha \frac{x^2}{|z|^2}, \quad z = x + iy \in \mathbb{C}, \quad z \neq 0, \quad \alpha \in \mathbb{R}.$$

The result we discuss here was proved in [CMM+] and reads as follows.

Theorem. *If* $-1 < \alpha < 1$, *then the unique minimiser of the energy functional* I_α *is the normalised characteristic function of the domain enclosed by the ellipse (centred at zero) with horizontal semi-axis* $\sqrt{1-\alpha}$ *and vertical semi-axis* $\sqrt{1+\alpha}$.

For $\alpha = 0$ this result is already proved in Frostman's thesis [Fro]. A simple argument shows that as $\alpha \to 1^-$ the minimiser in the Theorem tends in the weak $\star$ topology of finite Radon measures to the semi-circle law on the vertical axis, namely, to the probability measure

$$\frac{1}{\pi}\sqrt{2-y^2}\,\chi_{[-\sqrt{2}i,\sqrt{2}i]}(y)\,dy.$$

One can show that indeed the semi-circle law on the vertical axis is the unique minimiser of the energy (1) with $\alpha = 1$. This was proved in [MRS], and solved a long standing conjecture on the behaviour of interacting dislocations in metals, which predicted the formation of walls. Swapping variables one gets a similar result for $\alpha = -1$ involving the semicircle law on the horizontal axis. An argument based on energy comparison then leads to the conclusion that for $|\alpha| > 1$ the unique minimiser of (1) is one of the semicircle laws, which completes the picture.

Energies of the form (1), with various types of interaction kernels, arise frequently in models in which individuals repel each other if they get too close and are attracted if they get far from the centre of mass. The work done so far has concentrated mostly on radial interaction kernels and only very recently non-isotropy has entered the scene. See the introductions of [MRS], [CMM+], [CMM+2] and [MMR+] for more information about that and for references to previous work on the subject.

The main goal of this paper is to present a short compact proof of the Theorem. Along the way we also present some well-known background results, for the sake of the reader unfamiliar with the subject. The original proof in [CMM+] relies on computing explicitly, at every point in the plane, the potential P of the normalised characteristic function of the compact set E enclosed by a generic ellipse (see (5) below for the definition of the potential of a finite Radon measure μ). With the full potential at hand, it is then shown that there exists a unique ellipse such that the corresponding potential satisfies the so-called Euler-Lagrange conditions (the first condition on E, the second outside E), which are necessary conditions for minimality. This ellipse is shown to provide a minimiser of I_α, which is unique by the strict convexity of the energy. This computational approach is very powerful, but it does not provide much insight about the deep nature of the problem. In the argument we present here one needs instead to know explicitly the potential P only inside ellipses, which is much easier. The explicit formula for the potential inside the ellipse yields appropriate values for the semi-axes so that the potential is constant on E. In other words, it leads to finding a solution to the first Euler–Lagrange condition. It remains to show that P is everywhere larger than or equal to the constant value it takes on E, which is the second Euler–Lagrange condition. We achieve that in two steps. The first one consists in showing that the Laplacian of P on the exterior of E has positive boundary

limits. This we do via the classical Plemelj jump formula for the Cauchy Integral. The second step is an application of the minimum principle to a suitably constructed function, which exploits the fact that P is biharmonic outside E.

The short compact proof we present here, however, does not generalise to dimensions higher than two, unlike the original computational approach, which was exploited in [CMM+2] to prove the natural higher-dimensional version of the result in [CMM+]. Moreover, other energies in the plane involving other natural interaction kernels do not seem to be covered by what we do in this paper.

In conclusion, the problem of understanding deeply why ellipses appear in minimising the energy (1) for interaction kernels with a similar structure to (2) turns out to be challenging, and remains at present rather obscure. Further work seems to be needed to unveil its real nature.

The paper is organised as follows. Sections 2, 3 and 4 are expository and aimed at readers unfamiliar with the subject. We discuss some properties of the potentials we will be dealing with, the Euler–Lagrange conditions for the energy minimisers, and existence and uniqueness of minimisers. Sections 5 and 6 contain the proof of the Theorem. In section 5 we find a candidate ellipse such that the potential of the normalised characteristic function of the interior domain satisfies the first Euler–Lagrange condition. In section 6 we prove that such potential satisfies the second Euler–Lagrange condition. Section 7 is an appendix devoted to the Plemelj jump formula.

2   The potential

Given a mass distribution μ, one can define a potential of μ associated with the energy (1). This potential arises in computing the directional derivative of I_α at μ along a measure δ in the space of finite Radon measures (not necessarily positive, not necessarily with total mass 1) with finite energy

$$(3) \qquad\qquad\qquad I_\alpha(|\delta|) < \infty.$$

First of all, since the interaction kernel W_α in (2) is even,

$$(4) \qquad \frac{d}{dt}I_\alpha(\mu + t\delta)\big|_{t=0} = 2\int\left(\int W_\alpha(z-w)\,d\mu(w) + \frac{|z|^2}{2}\right)\,d\delta(z),$$

and so the following expression, which one calls the potential of μ, arises naturally:

$$(5) \qquad\qquad P(\mu)(z) = \left(W_\alpha \star \mu\right)(z) + \frac{1}{2}|z|^2, \quad z \in \mathbb{C}.$$

We claim that if μ is a probability measure minimising I_α then

$$(6) \qquad\qquad P(\mu)(z) = C_0, \quad , \quad \mu - a.e.\ \text{on}\ \text{spt}\,\mu,$$

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where $\operatorname{spt}\mu$ stands for the support of μ and C_0 is a constant. Set $C_0 = \int P(\mu)\,d\mu$ and define

$$F_1 = \{z \in \operatorname{spt}\mu : P(\mu)(z) \le C_0\} \qquad \text{and} \qquad F_2 = \{z \in \operatorname{spt}\mu : P(\mu)(z) > C_0\}.$$

Assume that $\mu(F_2) > 0$. If one has $\mu(F_1) = 0$ then $C_0 = \int P(\mu)\,d\mu > C_0$. Thus $\mu(F_1) > 0$. The measure

$$\delta = \frac{1}{\mu(F_1)}\chi_{F_1}\mu - \frac{1}{\mu(F_2)}\chi_{F_2}\mu$$

has total integral 0 and finite energy (3). Furthermore $\mu + t\delta$ is a positive measure with total mass 1, provided

$$-\mu(F_1) < t < \mu(F_2).$$

Note that the derivative in (4) vanishes, because $t = 0$ is a minimum of $I_\alpha(\mu + t\delta)$. The right-hand side of (4), however, is

$$2\left(\frac{1}{\mu(F_1)}\int_{F_1} P(\mu)\,d\mu - \frac{1}{\mu(F_2)}\int_{F_2} P(\mu)\,d\mu\right) < 0,$$

which is a contradiction. Therefore $\mu(F_2) = 0$ and so $P(\mu)(z) \le C_0$, μ-a.e. on $\operatorname{spt}\mu$. Since $\int P(\mu)\,d\mu = C_0$ we obtain (6).

The argument above holds for kernels much more general than W_α.

Moreover, for the kernel W_α in (2), one has that

$$P(\mu)(z) \le C_0, \quad z \in \operatorname{spt}\mu.$$

This can be proved readily as follows. Since μ has finite I_α energy, its logarithmic energy is finite too. Hence μ has no atoms and $(x^2/|z|^2) * \mu$ is a continuous function on the plane. Therefore $P(\mu)$ is lower semicontinuous. The set $\{z \in \mathbb{C} : P(\mu)(z) > C_0\}$ is open and has zero μ measure and so does not intersect the support of μ.

As a final remark note that the definition (5) yields

$$\int P(\mu)\,d\mu = I_\alpha(\mu) - \frac{1}{2}\int |z|^2\,d\mu(z).$$

If one wants that, in analogy to what happens in electrostatics, $I_\alpha(\mu) = \int P(\mu)\,d\mu$, then one has to add the constant $\frac{1}{2}\int |z|^2\,d\mu(z)$ to the right hand side of (5). The resulting potential will also be μ-a.e. constant on the support of μ if μ is a minimiser of I_α.

3   The Euler–Lagrange conditions

The potential P of a minimiser μ satisfies two properties, called the Euler–Lagrange conditions. Let Cap stand for the logarithmic capacity. The Euler–Lagrange conditions **EL1** and **EL2** are the following.

EL1: *There exists a constant C_0 such that $P(\mu)(z) = C_0$, Cap-a.e. on spt μ.*

Before proceeding to the proof of the statement above we make a remark. Let ν be a probability measure with finite energy and set $\delta = \nu - \mu$. Then $\mu + t\delta$ is a probability measure with finite energy for $0 \leq t \leq 1$. Since the function of $t \to I_\alpha(\mu + t\delta)$ has a minimum at $t = 0$ we conclude from (4) that

$$(7) \qquad\qquad \int P(\mu)\, d\nu \geq \int P(\mu)\, d\mu.$$

We now prove **EL1**. If the set $\{z \in \operatorname{spt} \mu : P(\mu) < C_0\}$ has positive capacity, then there exists a probability measure ν supported on that set with finite logarithmic energy and thus with finite I_α energy. Now the right-hand side in (7) is C_0 and the left-hand side is strictly less than C_0, which is a contradiction.

EL2: $P(\mu)(z) \geq C_0, \quad z \notin \operatorname{spt} \mu$, C_0 *the constant in* **EL1**.

The argument for proving **EL2** is rather simple. Given a point $z \notin \operatorname{spt} \mu$, we set

$$\nu = \frac{1}{|B(z,r)|} \chi_{B(z,r)}(w)\, dA(w),$$

where dA is Lebesgue measure in the plane. By (7)

$$C_0 = \int P(\mu)\, d\mu \leq \frac{1}{|B(z,r)|} \int_{B(z,r)} P(\mu)(w)\, dA(w) \xrightarrow{r \to 0} P(\mu)(z),$$

since $P(\mu)$ is continuous on the complement of the support of μ. Hence **EL2** holds.

Remark 1 (Sufficiency of the Euler-Lagrange conditions)**.** *In the case of the functional I_α in (1) the Euler-Lagrange conditions are not only necessary conditions for minimisers, but they are also sufficient. In other words, they characterise minimisers of I_α. This follows from the strict convexity of I_α, which is proved in the next section.*

4 Existence and uniqueness of minimisers

The existence of a minimiser for I_α follows by a standard compactness argument based on the lower semicontinuity and the coercivity of the interaction kernel. In addition a minimiser has compact support since the quadratic confinement beats the interaction potential at infinity. Uniqueness follows from the fact that the Fourier transform of the interaction kernel W_α in (2) is non-negative on test functions with zero integral. All this can be found in [MRS]. Here we discuss some of the steps in the computation of the Fourier transform of W_α and the proof of the uniqueness of minimisers from positivity of the Fourier transform of W_α.

The definition of the Fourier transform we use is

$$\widehat{\phi}(\xi) = \int \phi(z) e^{-i\xi \cdot z} \, dA(z), \quad \xi \in \mathbb{C},$$

where ϕ is a function in the Schwartz class $\mathcal{S}$. The Fourier transform of the logarithmic term is

$$-\log|z| \xrightarrow{\text{Fourier}} \frac{2\pi}{|\xi|^2} + c_0\delta_0,$$

where c_0 is a constant, and the tempered distribution $1/|\xi|^2$ acts on $\varphi \in \mathcal{S}$ as

$$(8) \qquad \int_{|\xi|<1} \frac{\varphi(\xi) - \varphi(0)}{|\xi|^2} \, dA(\xi) + \int_{|\xi|>1} \frac{\varphi(\xi)}{|\xi|^2} \, dA(\xi).$$

To compute the Fourier transform of the anisotropic term in W_α we write

$$\frac{x^2}{|z|^2} = \frac{1}{2}\left(\frac{x^2 - y^2}{|z|^2} + 1 \right).$$

Since the homogeneous polynomial $x^2 - y^2$ is harmonic we may resort to the well-known formula [Ste, Chapter 3, Section 3, p.73]

$$\frac{x^2 - y^2}{|z|^2} \xrightarrow{\text{Fourier}} -4\pi \text{ p. v. } \frac{\xi_1^2 - \xi_2^2}{|\xi|^4},$$

where p. v. stands for principal value. Hence, for a constant c_α depending only on α,

$$(9) \qquad \begin{aligned} W_\alpha &\xrightarrow{\text{Fourier}} \frac{2\pi}{|\xi|^2} - \frac{\alpha}{2}\left(4\pi \text{ p. v. } \frac{\xi_1^2 - \xi_2^2}{|\xi|^4} \right) + c_\alpha\delta_0 \\ &= 2\pi \frac{(1-\alpha)\xi_1^2 + (1+\alpha)\xi_2^2}{|\xi|^4} + c_\alpha\delta_0, \end{aligned}$$

and the tempered distribution of homogeneity -2 in the last line acts on $\mathcal{S}$ in a way analogous to (8). From (9) and by Plancherel's identity one gets

$$(2\pi)^2 \iint W_\alpha(z-w)\phi(z)\phi(w) \, dA(z)dA(w) = \int \widehat{W_\alpha}(\xi)|\widehat{\phi}(\xi)|^2 \, dA(\xi) \geq 0,$$

provided ϕ is a function in $\mathcal{S}$ with vanishing integral.

We now extend the above formula to a more general context.

Lemma. *If μ_1 and μ_2 are compactly supported probability measures with finite energy, then*

$$(10) \quad (2\pi)^2 \iint W_\alpha(z-w) \, d(\mu_1 - \mu_2)(z) \, d(\mu_1 - \mu_2)(w) = \int \widehat{W_\alpha}(\xi)|\widehat{\mu_1 - \mu_2}(\xi)|^2 \, dA(\xi).$$

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Remark 2. *The case $\alpha = 0$ in (10) corresponds to a purely logarithmic potential and is well-known. See, for example, [ST, Lemma 1.8, p.29] and [L, Theorem 1.16, p.80], where it is shown that the left-hand side is non-negative. Indeed, in both references one proves that*

$$\iint \log \frac{1}{|z-w|} \, d\nu(z) \, d\nu(w) = \int \left(\int \frac{1}{|z-w|} \, d\nu(w) \right)^2 dA(z),$$

provided ν is a signed compactly supported measure with finite logarithmic energy and $\nu(1) = 0$. See [ST, formula (1.30), p.34] and [L, p.80]. Applying Plancherel's identity to the right hand side above one gets (10) for $\alpha = 0$.

Before proving the Lemma we show that (10) implies uniqueness of the minimisers of I_α. We recall that at the beginning of this section we observed that a minimiser has compact support. Hence we can assume that we are miminising over compactly supported probability measures. First of all, since the right-hand side in (10) is non-negative, the Lemma yields

(11)
$$2 \iint W_\alpha(z-w) \, d\mu_1(z) \, d\mu_2(w) \le \iint W_\alpha(z-w) \, d\mu_1(z) \, d\mu_1(w)$$
$$+ \iint W_\alpha(z-w) \, d\mu_2(z) \, d\mu_2(w),$$

with strict inequality unless $\mu_1 = \mu_2$.

Now the strict convexity of the energy functional I_α follows immediately. Note that, since the confinement term is linear one just needs to look at the energy associated with the interaction kernel, namely,

$$J_\alpha(\mu) = \iint W_\alpha(z-w) \, d\mu(z) \, d\mu(w).$$

Now, if $0 \le t \le 1$ and μ_1 and μ_2 are probability measures with $J_\alpha(\mu_j) < \infty$, $j = 1, 2$, then

$$J_\alpha((1-t)\mu_1 + t\mu_2) = (1-t)^2 J_\alpha(\mu_1) + t^2 J_\alpha(\mu_2) + 2t(1-t) \iint W_\alpha(z-w) \, d\mu_1(z) \, d\mu_2(w)$$

$$\overset{(11)}{\le} (1-t)^2 J_\alpha(\mu_1) + t^2 J_\alpha(\mu_2) + t(1-t) \left(J_\alpha(\mu_1) + J_\alpha(\mu_2) \right)$$

$$= (1-t) J_\alpha(\mu_1) + t J_\alpha(\mu_2)$$

and the inequality is strict unless $\mu_1 = \mu_2$.

Of course strict convexity implies uniqueness of minimisers.

Proof of the Lemma. We follow closely the argument in [CMM+2].

Let φ be a C^∞ function, supported on the unit disc $B_1(0) \subset \mathbb{R}^2$, non-negative, radial, and with $\int_{\mathbb{R}^2} \varphi(z)\, dz = 1$. Set $\nu = \mu_1 - \mu_2$. For $\varepsilon > 0$ we define

$$\varphi_\varepsilon(z) = \frac{1}{\varepsilon^2} \varphi\left(\frac{z}{\varepsilon}\right) \qquad \text{and} \qquad \nu_\varepsilon = \nu \star \varphi_\varepsilon.$$

We claim that

$$(12) \qquad (2\pi)^2 \int_{\mathbb{R}^2} (W_\alpha \star \nu_\varepsilon)(z) \nu_\varepsilon(z)\, dA(z) = \int_{\mathbb{R}^2} \widehat{W_\alpha}(\xi) |\widehat{\nu_\varepsilon}(\xi)|^2\, dA(\xi).$$

To show this, we set $f := W_\alpha \star \nu_\varepsilon$ and $g := \nu_\varepsilon$, and note that $g \in C_c^\infty(\mathbb{R}^2)$ and $f \in C^\infty(\mathbb{R}^2)$. Moreover, since $\widehat{\nu_\varepsilon} \in \mathcal{S}$, $\widehat{\nu_\varepsilon}(0) = 0$ and $\widehat{W_\alpha}$ behaves as $1/|\xi|^2$ at infinity in view of (9), we have that $\widehat{f} = \widehat{W_\alpha} \widehat{\nu_\varepsilon} \in L^1(\mathbb{R}^2)$. Let $\psi \in C_c^\infty(\mathbb{R}^2)$ be such that $\psi = 1$ on $B_1(0)$ and let $R > 0$ be such that the support of g is contained in $B_R(0)$. If $\tau > 0$ is such that $\tau R < 1$, then, by Parseval's formula,

$$(13) \qquad \begin{aligned} (2\pi)^2 \int_{\mathbb{R}^2} f(z)g(z)\, dA(z) &= (2\pi)^2 \int_{\mathbb{R}^2} \psi(\tau z) f(z) g(z)\, dA(z) \\ &= \int_{\mathbb{R}^2} (\widehat{\psi(\tau \cdot)\, f})(\xi)\, \overline{\widehat{g}(\xi)}\, dA(\xi) \\ &= \int_{\mathbb{R}^2} (\widehat{\psi}_\tau \star \widehat{f})(\xi)\, \overline{\widehat{g}(\xi)}\, dA(\xi), \end{aligned}$$

where $\widehat{\psi}_\tau(z) := (2\pi\tau)^{-2} \widehat{\psi}(z/\tau)$. We have $\widehat{\psi} \in \mathcal{S} \subset L^1(\mathbb{R}^2)$ and $(2\pi)^{-2} \int_{\mathbb{R}^2} \widehat{\psi}(\xi)\, d\xi = \psi(0) = 1$. Hence the family $(\widehat{\psi}_\tau)_\tau$ is an approximate identity. Being $\widehat{f} \in L^1(\mathbb{R}^2)$, we conclude that $\widehat{\psi}_\tau \star \widehat{f}$ converges to $\widehat{f}$ in $L^1(\mathbb{R}^2)$, as $\tau \to 0$.

Since $\widehat{g} \in L^\infty(\mathbb{R}^2)$, we deduce that

$$\lim_{\tau \to 0} \int_{\mathbb{R}^2} (\widehat{\psi}_\tau \star \widehat{f})(\xi)\, \overline{\widehat{g}(\xi)}\, dA(\xi) = \int_{\mathbb{R}^2} \widehat{f}(\xi)\, \overline{\widehat{g}(\xi)}\, dA(\xi),$$

which, together with (13), proves (12).

We now let $\varepsilon \to 0$ in (12). For the right-hand side we remark that for every $\xi \in \mathbb{R}^2$

$$\widehat{\varphi_\varepsilon}(\xi) = \widehat{\varphi}(\varepsilon\xi) \to \widehat{\varphi}(0) = 1,$$

as $\varepsilon \to 0$, and that $\|\widehat{\varphi_\varepsilon}\|_{L^\infty} \leq \|\varphi\|_{L^1} = 1$ for each $\varepsilon > 0$. Therefore, by the Dominated Convergence Theorem, we have

$$\int_{\mathbb{R}^2} \widehat{W_\alpha}(\xi) |\widehat{\nu_\varepsilon}(\xi)|^2\, dA(\xi) = \int_{\mathbb{R}^2} \widehat{W_\alpha}(\xi) |\widehat{\nu}(\xi)|^2 |\widehat{\varphi_\varepsilon}(\xi)|^2\, dA(\xi) \to \int_{\mathbb{R}^2} \widehat{W_\alpha}(\xi) |\widehat{\nu}(\xi)|^2\, dA(\xi),$$

as $\varepsilon \to 0$, even if the right-hand side is infinite.

To deal with the left-hand side of (12), we take $R > 1$ such that the support of ν is contained in the disc centred at the origin and with radius $R/4$. Then

$$|W_\alpha(z)| \leq \log \frac{R}{|z|} + |\alpha| + \log R, \quad |z| < R.$$

Set $\beta = |\alpha| + \log R$, so that we obtain, for $\varepsilon < \frac{R}{2}$,

$$(14) \qquad (|W_\alpha| \star \varphi_\varepsilon)(z) \leq \left(\log \frac{R}{|z|} \star \varphi_\varepsilon \right)(z) + \beta \leq \log \frac{R}{|z|} + \beta, \quad |z| < \frac{R}{2},$$

invoking the fact that the function $-\log|z|$ is superharmonic and φ is radial (one writes the convolution in polar coordinates and then applies superharmonicity on each circle).

Note that

$$(15) \qquad (W_\alpha \star \varphi_\varepsilon)(z) \xrightarrow{\varepsilon \to 0} W_\alpha(z), \quad z \in \mathbb{R}^2,$$

because W_α is continuous as a function with values into $[0, +\infty]$.

We claim that ν has finite I_α energy (3), which can be translated into the condition

$$(16) \qquad \iint \log \frac{1}{|z - w|} \, d|\nu|(z) \, d|\nu|(w) < \infty.$$

Recall that $\nu = \mu_1 - \mu_2$ with μ_1 and μ_2 probability measures with finite energy and compact support. Thus $|\nu| \leq \mu_1 + \mu_2$ and (16) is a consequence of

$$(17) \qquad 2 \iint \log \frac{1}{|z - w|} \, d\mu_1(z) \, d\mu_2(w) \leq \iint \log \frac{1}{|z - w|} \, d\mu_1(z) \, d\mu_1(w)$$
$$+ \iint \log \frac{1}{|z - w|} \, d\mu_2(z) \, d\mu_2(w).$$

The inequality above is clearly equivalent to

$$(18) \qquad \iint \log \frac{1}{|z - w|} \, d(\mu_1 - \mu_2)(z) \, d(\mu_1 - \mu_2)(w) \geq 0,$$

which is proven in [ST, Lemma 1.8, p.29] and [L, Theorem 1.16, p.80]. A more self-contained proof of (16) is presented in Remark 3.

Combining (14), (15), (16) and the Dominated Convergence Theorem, we get

$$(19) \qquad \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (W_\alpha \star \varphi_\varepsilon)(z - w) \, d\nu(z) \, d\nu(w) \ \to \ \iint_{\mathbb{R}^2 \times \mathbb{R}^2} W_\alpha(z - w) \, d\nu(z) \, d\nu(w),$$

as $\varepsilon \to 0$, even if the right-hand side is infinite.

We now go back to the left-hand side of (12) and observe that

$$\int_{\mathbb{R}^2} (W_\alpha \star \nu_\varepsilon)(z)\nu_\varepsilon(z)\, dA(z) = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (W_\alpha \star \varphi_\varepsilon \star \varphi_\varepsilon)(z - w)\, d\nu(z)\, d\nu(w).$$

Note that $(\varphi_\varepsilon \star \varphi_\varepsilon)(z) = \varepsilon^{-2}(\varphi \star \varphi)(z/\varepsilon)$ and that $\varphi \star \varphi$ inherits the properties of φ: it is radial, belongs to $C_c^\infty(\mathbb{R}^2)$, and $\int_{\mathbb{R}^2}(\varphi \star \varphi)(z)\, dz = 1$. Therefore, (19) holds with φ_ε replaced by $\varphi_\varepsilon \star \varphi_\varepsilon$. $\qquad\square$

Remark 3 (Self-contained proof of (16))**.** *We provide now an alternative proof of (17) (and therefore of (16) and (18)), based on Fatou's lemma and on the superharmonicity of $-\log|z|$. To simplify the writing set $L(z) = \log(1/|z|)$. The mutual logarithmic energy of μ_1 and μ_2 can be written as*

$$\int (L \star \mu_1)\, d\mu_2 = (L \star \mu_1 \star \widetilde{\mu_2})\,(0),$$

where, given a positive Radon measure μ, $\widetilde{\mu}$ stands for the measure whose action on a test function φ is

$$\int \varphi(-z)\, d\mu(z).$$

Let φ_ε be an approximation of the identity as in the proof of the Lemma. Then $\varphi_\varepsilon \star \varphi_\varepsilon = (\varphi \star \varphi)_\varepsilon$. By Fatou's lemma

$$(L \star \mu_1 \star \widetilde{\mu_2})\,(0) = \left(\left(\lim_{\varepsilon \to 0} L \star \varphi_\varepsilon \star \varphi_\varepsilon\right) \star \mu_1 \star \widetilde{\mu_2}\right)(0)$$

$$\leq \liminf_{\varepsilon \to 0} (L \star \varphi_\varepsilon \star \varphi_\varepsilon \star \mu_1 \star \widetilde{\mu_2})\,(0)$$

$$= \liminf_{\varepsilon \to 0} \left(L \star (\varphi_\varepsilon \star \mu_1) \star (\widetilde{\varphi_\varepsilon \star \mu_2})\right)(0)$$

$$= \liminf_{\varepsilon \to 0} \iint L(z - w)\, \mu_{1\varepsilon}(w)\, dA(w)\mu_{2\varepsilon}(z)\, dA(z),$$

where $\mu_{i\varepsilon} = \varphi_\varepsilon \star \mu_i$, for $i = 1, 2$. In view of (12) for $\alpha = 0$ we have

$$\int_{\mathbb{R}^2} (L \star \nu_\varepsilon)\,(z)\, \nu_\varepsilon(z)\, dA(z) \geq 0,$$

and so

$$2\iint L(z - w)\, \mu_{1\varepsilon}(w)\, dA(w)\mu_{2\varepsilon}(z)\, dA(z) \leq \iint L(z - w)\mu_{1\varepsilon}(z)\mu_{1\varepsilon}(w)\, dA(z)\, dA(w)$$

$$+ \iint L(z - w)\mu_{2\varepsilon}(z)\mu_{2\varepsilon}(w)\, dA(z)\, dA(w).$$

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It remains to estimate the energies in the right-hand side of the previous inequality. Let μ a positive compactly supported Radon measure. We then have

$$\iint L(z-w)\mu_\varepsilon(z)\mu_\varepsilon(w)\,dA(z)\,dA(w) = \left(L \star \mu \star \widetilde{\mu} \star (\varphi \star \varphi)_\varepsilon\right)(0)$$

$$\leq \left(L \star \mu \star \widetilde{\mu}\right)(0),$$

appealing to the superharmonicity of $L \star \mu \star \widetilde{\mu}$ and the fact that $\varphi \star \varphi$ is non-negative, radial and with integral equal to 1. Therefore (17) holds.

5 The candidate ellipse

In this section we show that the potential of the normalised characteristic function of the domain enclosed by the ellipse with horizontal semi-axis $\sqrt{1-\alpha}$ and vertical semi-axis $\sqrt{1+\alpha}$ satisfies the first Euler–Lagrange condition. We need to work with a general ellipse with semi-axis a and b and the enclosed set

$$E = E(a,b) = \left\{(x,y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\right\}.$$

Let P be the potential of the normalised characteristic function of E defined as in (5). The first Euler–Lagrange equation states that P is constant on E, or equivalently, that its gradient is 0 on $\mathring{E}$. Expressing x as $(z+\overline{z})/2$ and recalling that $\nabla = 2\partial/\partial\overline{z}$ one obtains

$$(20) \qquad \nabla P(z) = \left(-\frac{1}{\overline{z}} + \frac{\alpha}{2}\left(\frac{1}{z} - \frac{z}{\overline{z}^2}\right)\right) \star \frac{1}{|E|}\chi_E + z, \quad z \in \mathbb{C}.$$

Note that the formula above, which holds in the sense of distributions, implies that ∇P is a continuous function, the first term being the convolution of a locally integrable kernel with a bounded compactly supported function. Hence P is of class C^1 in the whole plane. To check **EL1** one has to compute explicitly on E the potentials

$$\frac{1}{\overline{z}} \star \chi_E \qquad \text{and} \qquad \frac{z}{\overline{z}^2} \star \chi_E.$$

Once we have these explicit formulas we will set the equation $\nabla P = 0$ on E and solve it for a and b.

We start by computing the Cauchy potential of the characteristic function of E, following [HMV]. Recall that $1/\pi z$ is the fundamental solution of the operator $\overline{\partial} = \partial/\partial\overline{z}$. Hence

$$\left(\frac{1}{\pi z} \star \chi_E\right)(z) = \overline{z} + f(z), \quad z \in \mathring{E},$$

with f holomorphic on $\mathring{E}$. The function f has to be chosen so that on the boundary of E the function $\overline{z} + f(z)$ extends holomorphically to $\mathbb{C} \setminus E$. Writing the equation of the ellipse in the variables z and $\overline{z}$ and solving for $\overline{z}$ one obtains

$$\overline{z} = \lambda z + 2ab\, h(z), \quad z \in \partial E, \quad \lambda = \frac{a-b}{a+b},$$

where h is

$$h(z) = \frac{1}{z + \sqrt{z^2 + c^2}}, \quad \text{with} \quad c^2 = b^2 - a^2.$$

The domain of the holomorphic function h is the complement in the plane of the segment joining the foci of the ellipse. More explicitly, the domain of h is $\mathbb{C} \setminus [-\sqrt{a^2 - b^2}, \sqrt{a^2 - b^2}]$ if $a \geq b$ and $\mathbb{C} \setminus [-i\sqrt{b^2 - a^2}, i\sqrt{b^2 - a^2}]$ if $a \leq b$. Choosing $f(z) = -\lambda z$ one gets

$$(21) \qquad \left(\frac{1}{\pi z} \star \chi_E \right)(z) = \begin{cases} \overline{z} - \lambda z, & z \in E, \\ 2ab\, h(z), & z \in E^c. \end{cases}$$

The preceding identity follows from Liouville's theorem and the remark that both sides of (21) are continuous functions on the plane, vanishing at ∞, whose $\overline{\partial}$-derivative is the characteristic function of E. Taking conjugates

$$(22) \qquad \left(\frac{1}{\pi \overline{z}} \star \chi_E \right)(z) = \begin{cases} z - \lambda \overline{z}, & z \in E, \\ 2ab\, h(\overline{z}), & z \in E^c. \end{cases}$$

Now we reduce the computation of

$$(23) \qquad \frac{1}{\pi} \frac{z}{\overline{z}^2} \star \chi_E$$

to (22), by remarking that

$$\frac{1}{\pi} \frac{z}{\overline{z}^2} \star \chi_E = (-\overline{\partial}) \left(\frac{1}{\pi} \frac{z}{\overline{z}} \star \chi_E \right),$$

and

$$\partial \left(\frac{1}{\pi} \frac{z}{\overline{z}} \star \chi_E \right) = \frac{1}{\pi} \frac{1}{\overline{z}} \star \chi_E,$$

where we denote by $\partial = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$ the derivative with respect to the variable z. Hence to compute (23) one has to find a bounded primitive in z of (22) and then take $-\overline{\partial}$. A primitive in z of (22) in $\mathring{E} \cup (\mathbb{C} \setminus E)$ is

$$(24) \qquad \frac{1}{2}(z - \lambda \overline{z})^2 \chi_E(z) + (2abh(\overline{z})H(z) + \varphi(\overline{z})) \chi_{E^c}(z),$$

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where $H(z) = z - \lambda\overline{z} - 2abh(\overline{z})$ and $\varphi(z)$ is holomorphic on E^c. Since on ∂E

$$\frac{1}{2}(z - \lambda\overline{z})^2 = \frac{1}{2}(2abh(\overline{z}))^2$$

we choose $\varphi(z) = \frac{1}{2}(2abh(z))^2$ so that the function in (24) is continuous and bounded on $\mathbb{C}$. The function in (24) and $\frac{1}{\pi}\frac{z}{\overline{z}} \star \chi_E$ are bounded primitives in z of $\frac{1}{\pi}\frac{1}{\overline{z}} \star \chi_E$, and so the difference is a bounded function on $\mathbb{C}$ annihilated by the operator ∂. This means that the conjugate function is a bounded entire function. By Liouville's Theorem there is a constant C such that

$$(25) \qquad \frac{1}{\pi}\frac{z}{\overline{z}} \star \chi_E = \frac{1}{2}(z - \lambda\overline{z})^2\chi_E(z) + \left(2abh(\overline{z})H(z) + \frac{1}{2}(2abh(\overline{z}))^2\right)\chi_{E^c}(z) + C.$$

It can be readily checked, examining the expansion at ∞, that $C = \lambda ab$, but this precise value is not important here. Taking $-\overline{\partial}$ in (25) we get

$$(26) \qquad \left(\frac{1}{\pi}\frac{z}{\overline{z}^2} \star \chi_E\right)(z) = \lambda(z - \lambda\overline{z}), \quad z \in E.$$

Indeed, one can obtain an explicit (although complicated) expression for the potential above also off E, but precisely we want to show that it is not necessary to use it.

Plugging (21), (22) and (26) in the formula (20) for the gradient of P we get

$$\nabla P(z) = \left(\frac{1}{ab}(-1 - \alpha\lambda) + 1\right)z + \frac{1}{ab}\left(\lambda + \frac{\alpha}{2} + \frac{\alpha}{2}\lambda^2\right)\overline{z}, \quad z \in E.$$

Then ∇P vanishes on E if and only if a and b are solutions of the system

$$\begin{cases} ab = 1 + \alpha\lambda \\ \alpha\lambda^2 + 2\lambda + \alpha = 0. \end{cases}$$

Solving the system yields $a = \sqrt{1 - \alpha}$ and $b = \sqrt{1 + \alpha}$, which provides an ellipse such that the potential of the normalised characteristic function of the enclosed domain satisfies the first Euler–Lagrange condition. This is the candidate ellipse. Our task in the next section is to show that the corresponding potential P satisfies the second Euler–Lagrange condition.

6 The second Euler–Lagrange condition

Let P stand for the potential of the normalised characteristic function of the domain E enclosed by the candidate ellipse found in the previous section. We know that $P(z) = C_0$ for $z \in E$, and we have to prove that $P(z) \geq C_0$ for $z \notin E$. The proof proceeds in two steps. Our first task is to show the following result.

Lemma. *We have*

$$(27) \qquad \lim_{E\not\ni w\to z} \Delta P(w) \geq \frac{2}{ab}(1 - |\alpha|) > 0, \quad z \in \partial E.$$

Proof. Taking 2∂ in (20) one obtains

$$(28) \qquad \Delta P = -2\pi\frac{\chi_E}{|E|} - \alpha \ \mathrm{p.\,v.}\left(2\operatorname{Re}\left(\frac{1}{z^2}\right)\right) \star \frac{\chi_E}{|E|} + 2.$$

The jump of a function f defined on $\mathring{E} \cup E^c$ at a point $z \in \partial E$ is

$$\mathrm{jump}(f)(z) := \lim_{\mathring{E}\ni w\to z} f(w) - \lim_{E\not\ni w\to z} f(w).$$

Since $\Delta P = 0$ on $\mathring{E}$ because of the first Euler–Lagrange equation, we have

$$\lim_{E\not\ni w\to z} \Delta P(w) = -\mathrm{jump}(\Delta P)(z), \quad z \in \partial E.$$

To compute the jump of the Laplacian of P we first need to express the principal value integral in (28) in a more suitable form. We have

$$\mathrm{p.\,v.}\left(\frac{-2}{z^2}\right) \star \chi_E = \frac{1}{z} \star 2\partial\chi_E = \frac{1}{z} \star (-\overline{n}) \, d\sigma = \frac{1}{iz} \star \overline{\tau^2} \, dz_{\partial E},$$

where $d\sigma$ is the length measure on ∂E, n the exterior unit normal vector and τ the unit tangent vector.

By (28) and the Plemelj's jump formula (29) below we get

$$\lim_{E\not\ni w\to z} \Delta P(w) = \frac{2\pi}{|E|} + \frac{\alpha}{|E|} \operatorname{Re}\left(\mathrm{jump}\left(\frac{1}{iz} \star \overline{\tau^2} \, dz_{\partial E}\right)(z)\right)$$

$$= \frac{2}{ab}\left(1 - \alpha \operatorname{Re}\left(\tau(z)^2\right)\right)$$

$$\geq \frac{2}{ab}(1 - |\alpha|), \quad z \in \partial E.$$

$\square$

We turn now to the second step of the proof of the second Euler–Lagrange condition. Recall that P satisfies the first Euler–Lagrange condition, so that $P(z) = C_0$, $z \in E$. We want to show that $P(z) \geq C_0$, $z \notin E$, and for this we would like to apply the minimum principle. Note that the term $|z|^2/2$ makes P larger than C_0 at ∞. But P is not harmonic off E, nor superharmonic, and thus the minimum principle cannot be applied directly to

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P. Identity (28) shows that P is biharmonic off E, that is, the Laplacian is harmonic, and for these functions there are some well-known manipulations that allow an application of the minimum principle [Duf]. We found an inspiration in that paper to devise the argument below.

Take $a \notin E$ and let $a_0 \in \partial E$ stand for the projection of a into E. Let $\vec{n}$ be the exterior unit normal vector to ∂E at the point a_0, so that a belongs to the ray emanating from a_0 in the direction $\vec{n}$. If we set

$$h(z) = \langle \nabla P(z), \vec{n} \rangle - \frac{1}{2} \Delta P(z) \langle z - a, \vec{n} \rangle, \quad z \notin E,$$

then we get

$$\Delta h(z) = \langle \nabla \Delta P(z), \vec{n} \rangle - \langle \nabla \Delta P(z), \nabla (\langle z - a, \vec{n} \rangle) \rangle = 0, \quad z \notin E.$$

Since P is of class C^1 on $\mathbb{C}$ the first Euler–Lagrange condition implies that $\nabla P(z) = 0$ on E. Hence

$$\lim_{E \not\ni w \to z} h(w) = -\frac{1}{2} \lim_{E \not\ni w \to z} \Delta P(w) \langle z - a, \vec{n} \rangle \geq 0, \quad z \in \partial E.$$

This follows on the one hand by the Lemma, which yields $\lim_{E \not\ni w \to z} \Delta P(w) \geq 0$, $z \in \partial E$, and on the other hand by the inequality $\langle z - a, \vec{n} \rangle \leq 0$, $z \in \partial E$, which is due to the convexity of E.

For z large the behaviour of h is controlled by the function one gets by replacing P with the dominant term $|z|^2/2$ in the definition of h. This function is

$$\langle z, \vec{n} \rangle - \langle z - a, \vec{n} \rangle = \langle a, \vec{n} \rangle > 0,$$

which yields $h(z) > 0$, for z large enough. Therefore, by the minimum principle, $h(z) \geq 0$, $z \notin E$, and so $0 \leq h(a) = \langle \nabla P(a), \vec{n} \rangle$. One concludes that P is increasing along the ray starting at a_0 in the direction of $\vec{n}$ and so $P(a) \geq C_0$.

7 Appendix: The Plemelj jump formula

Let Γ be a smooth Jordan curve enclosing a domain D. Let f be a smooth function on the curve. Define the Cauchy integral of f as

$$C(f)(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta, \quad z \notin \Gamma.$$

The Plemelj formula (also called Plemelj–Sohotski) is the following:

$$(29) \qquad \lim_{D \ni z \to a} C(f)(z) - \lim_{\overline{D} \not\ni z \to a} C(f)(z) = f(a), \quad a \in \Gamma.$$

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Note that the limits in the left-hand side of (29) exist, because Γ and f are smooth.

There is a much more general version of the formula in which Γ is a rectifiable Jordan curve, f is integrable with respect to the arc length measure, the limits in the left-hand side are non-tangential and the identity holds a.e. with respect to the arc length measure on Γ.

We now prove (29) under the assumption that Γ is rectifiable and f is Lipschitz on Γ. Appealing to a well-known extension theorem we can further assume that f is defined and Lipschitz on the whole plane. Since the winding number of Γ with respect to a point in D is 1 and with respect to a point off $\overline{D}$ is 0, we have

$$C(f)(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta) - f(z)}{\zeta - z} \, d\zeta + f(z), \quad z \in D,$$

and

$$C(f)(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta) - f(z)}{\zeta - z} \, d\zeta, \quad z \notin \overline{D}.$$

Taking limits as z tends to a from D and from $\mathbb{C} \setminus \overline{D}$, we obtain

$$(30) \qquad \lim_{D \ni z \to a} C(f)(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta) - f(a)}{\zeta - a} \, d\zeta + f(a), \quad a \in \Gamma,$$

and

$$(31) \qquad \lim_{\overline{D} \not\ni z \to a} C(f)(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta) - f(a)}{\zeta - a} \, d\zeta, \quad a \in \Gamma,$$

by the Dominated Convergence Theorem. Subtracting (31) from (30) we get (29).

The interested reader may consult [Ver] for a relation with the boundary Cauchy singular integral, defined in terms of principal values, in a basic context as the one considered here. For more general results in higher dimensions one can see [HMT] and [T].

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