A NEW ELEMENTARY PROOF OF L^2 ESTIMATES FOR THE CAUCHY INTEGRAL ON LIPSCHITZ GRAPHS

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A new notion of integral curvature, arising in the study of analytic capacity, has been introduced by M. S. Melnikov in [3]. The purpose of this lecture is to show how this can be used to provide an elementary proof of the Coifman-McIntosh-Meyer Theorem on L^2 boundedness of the Cauchy Integral on Lipschitz graphs [2]. See [1] for the first two elementary proofs.

Let Γ be the graph of a function $A : \mathbb{R} \to \mathbb{R}$ satisfying $|A(x) - A(y)| \le M|x - y|$, x, $y \in \mathbb{R}$. For each $\varepsilon > 0$ set

$$C_\varepsilon f(x) = \int_{|y-x|>\varepsilon} \frac{f(y)}{\gamma(y) - \gamma(x)}\, dy, \quad f \in L^2(\mathbb{R}),$$

where $\gamma(x) = x + iA(x)$. The L^2 estimate mentioned in the title is

$$\int |C_\varepsilon f(x)|^2\, dx \le C \int |f(x)|^2\, dx, \tag{1}$$

where C is a positive constant independent of ε.

We need two lemmas.

Lemma 1 [3]. *Given three different points z_1, z_2 and z_3 in the plane, we have*

$$\sum_\sigma \frac{1}{z_{\sigma(2)} - z_{\sigma(1)}} \frac{1}{\overline{z_{\sigma(3)}} - \overline{z_{\sigma(1)}}} = \left(\frac{4S(z_1, z_2, z_3)}{|z_1 - z_2|\,|z_1 - z_3|\,|z_2 - z_3|} \right)^2 = \frac{1}{R^2}$$

where the sum is over the permutations of $\{1, 2, 3\}$, $S(z_1, z_2, z_3)$ is the area of the triangle with vertices at the z_j and R is the radius of the cercle passing through z_1, z_2 and z_3.

The proof of Lemma 1 is an extremely simple computation that will be omitted.

Let $c(z_1, z_2, z_3)$ stand for R^{-1} where R is as in Lemma 1. Using the elementary formula for the area of a triangle whose vertices are given in Cartesian coordinates we get

$$c(\gamma(x), \gamma(y), \gamma(z)) \le 4 \left| \frac{\frac{A(y)-A(x)}{y-x} - \frac{A(z)-A(x)}{z-x}}{z - y} \right|.$$

Lemma 2. *Let a be a (locally) absolutely continuous function on $\mathbb{R}$ such that $a' \in L^2(\mathbb{R})$. Then*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\frac{a(y)-a(x)}{y-x}-\frac{a(z)-a(x)}{z-x}}{z-y}\right)^2 dx\,dy\,dz = c\|a'\|_2^2,$$

where c is some numerical constant.

The proof of Lemma 2 consists (basically) in applying Plancherel in the x variable after having set $y = x+h$ and $z = x+k$.

Applying Lemma 2 to $a(x) = \chi_I(x)(A(x) - P_I(x))$, where I is the interval $[\alpha, \beta]$ and

$$P_I(x) = A(\alpha) + \frac{A(\beta)-A(\alpha)}{\beta-\alpha}(x-\alpha),$$

one gets the following localized version of Lemma 2.

Corollary. *For some constant $C = C(M)$ and any interval $I \subset \mathbb{R}$,*

$$\int_I\int_I\int_I c^2(\gamma(x),\gamma(y),\gamma(z))\,dx\,dy\,dz \le C|I|. \tag{2}$$

Proof of (1). We present a sketch of the argument for the particularly simple case $f = \chi_I$, I some interval in $\mathbb{R}$.

Clearly

$$\begin{aligned}\int_I |C_\varepsilon\chi_I|^2 &= \int_I C_\varepsilon\chi_I(x_1)\overline{C_\varepsilon\chi_I(x_1)}\,dx_1\\ &= \iiint_T \frac{1}{\gamma(x_2)-\gamma(x_1)}\,\overline{\frac{1}{\gamma(x_3)-\gamma(x_1)}}\,dx_1\,dx_2\,dx_3\end{aligned} \tag{3}$$

where $T = \{(x_1,x_2,x_3) \in I\times I\times I : |x_2-x_1| > \varepsilon,\ |x_3-x_1| > \varepsilon\}$. The last integral turns out to be

$$\iiint_{D_\varepsilon} \frac{1}{\gamma(x_2)-\gamma(x_1)}\,\overline{\frac{1}{\gamma(x_3)-\gamma(x_1)}}\,dx_1\,dx_2\,dx_3 + O(|I|),$$

where $D_\varepsilon = \{(x_1,x_2,x_3) \in I\times I\times I : |x_2-x_1| > \varepsilon,\ |x_3-x_1| > \varepsilon,\ |x_2-x_3| > \varepsilon\}$. Permutating the variables in the last integral in (3) in all possible ways we get

$$6\int_I |C_\varepsilon\chi_I|^2 = \iiint_{D_\varepsilon} c^2(\gamma(x_1),\gamma(x_2),\gamma(x_3))\,dx_1\,dx_2\,dx_3 + O(|I|),$$

and so, by (2),

$$\int_I |C_\varepsilon\chi_I|^2 \le C|I|. \tag{4}$$

Since (4) implies that $C_\varepsilon(1) \in BMO$ we could invoke the T1-Theorem of David and Journé to complete the proof. However, some additional simple manipulations show that

$$\int_I |C_\varepsilon b|^2 \leq C\|b\|_\infty^2 |I|, \quad b \in L^\infty(I).$$

Then C_ε sends boundedly $L^\infty(\mathbb{R})$ into $BMO(\mathbb{R})$ and $H^1_{at}(\mathbb{R})$ into $L^1(\mathbb{R})$. Interpolation now gives (1). □

References

[1] Coifman-Jones-Semmes, *Two elementary proofs of the L^2 boundedness of Cauchy integrals on Lipschitz curves*, J. Amer. Math. Soc. **2** (1989), 553–564.

[2] Coifman-McIntosh-Meyer, *L'integrale de Cauchy définit un operateur borné sur L^2 sur les courbes Lipschitziennes*, Ann. of Math. **116** (1982), 361–388.

[3] M. S. Melnikov, *A discrete approach to analytic capacity*, Preprint, 1994.