

# Choquet integrals, Hausdorff content and the Hardy-Littlewood maximal operator

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## Introduction

Using the BMO- $H^1$  duality (among other things), D. R. Adams proved in [1] the strong type inequality

$$(1) \quad \int Mf(x) dH^\alpha(x) \leq C \int |f(x)| dH^\alpha(x), \quad 0 < \alpha < n,$$

where  $C$  is some positive constant independent of  $f$ . Here  $M$  is the Hardy-Littlewood maximal operator in  $\mathbb{R}^n$ ,  $H^\alpha$  is  $\alpha$ -dimensional Hausdorff content and the integrals are taken in the Choquet sense. The Choquet integral of  $\varphi \geq 0$  with respect to a set function  $C$  is defined by

$$\int \varphi dC = \int_0^\infty C\{x \in \mathbb{R}^n : \varphi(x) > t\} dt.$$

Precise definitions of  $M$  and  $H^\alpha$  will be given below. For an application of (1) to the Sobolev space  $W^{1,1}(\mathbb{R}^n)$  see [1, p. 1]

The purpose of this note is to provide a selfcontained, direct proof of a result more general than (1).

**Theorem.** *Let  $0 < \alpha < n$ . Then, for some constant  $C$  depending only on  $\alpha$  and  $n$ , the following inequalities hold.*

- (i)  $\int (Mf)^p dH^\alpha \leq C \int |f|^p dH^\alpha, \quad \alpha/n < p.$
- (ii)  $H^\alpha\{x : Mf(x) > t\} \leq Ct^{-\alpha/n} \int |f|^{\alpha/n} dH^\alpha.$

The proof of the Theorem is described in the next section. An elementary argument gives (i) readily. For (ii), besides the classical line of reasoning to treat weak type inequalities, we need a covering lemma concerning Hausdorff content.

It is worthwhile mentioning that (i) follows also from (ii) and the standard argument to derive  $L^p$  inequalities,  $1 < p < \infty$ , for  $M$  from the weak  $L^1$  inequality (see [2, 2.5, p. 145]). We believe, however, that the independent simple proof of (i) we present is of some interest.

We proceed now to establish some notation and terminology and to recall some background facts.

Let  $f$  be a locally integrable function on  $\mathbb{R}^n$ . The Hardy-Littlewood maximal function of  $f$  is

$$(2) \quad Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f|,$$

where the supremum is taken over all cubes  $Q$  containing  $x$ , with sides parallel to the coordinate axes. We have denoted by  $|Q|$  the  $n$ -dimensional volume of  $Q$ . Very often it is much more convenient to work with the essentially equivalent *dyadic maximal function*  $M_d f(x)$ , which is defined by the right hand side of (2), but where now the supremum is taken only on the family of *dyadic cubes* containing  $x$ . Clearly  $M_d f \leq Mf$ . The reverse inequality fails, but some useful substitutes are available, as we will see below.

If  $E \subset \mathbb{R}^n$  and  $0 < \alpha \leq n$ , the  $\alpha$ -dimensional Hausdorff content of  $E$  is defined by

$$(3) \quad H^\alpha(E) = \inf \sum_{j=1}^{\infty} l(Q_j)^\alpha,$$

where the infimum is taken over all coverings of  $E$  by countable families of cubes  $Q_j$  with sides parallel to the coordinate axes. Here  $l(Q)$  denotes the side length of the cube  $Q$ . If we take the infimum in (3) only on coverings of  $E$  by dyadic squares, we get an equivalent quantity  $H_d^\alpha(E)$  called the *dyadic  $\alpha$ -dimensional Hausdorff content*.

A well-known argument [2, p. 136] gives

$$H^\alpha\{x : Mf(x) > t\} \leq 3^\alpha H^\alpha\{x : M_d f(x) > 4^{-n}t\},$$

which implies

$$(4) \quad \int Mf(x) dH^\alpha(x) \leq 3^\alpha 4^n \int M_d f(x) dH^\alpha(x).$$

Therefore, at least in the integral sense expressed by (4),  $Mf$  is dominated by  $M_d f$ .

A fundamental point in dealing with Choquet integrals with respect to Hausdorff content is that for non-negative functions  $f_j$  one has

$$(5) \quad \int \sum_{j=1}^{\infty} f_j dH^\alpha \leq C \sum_{j=1}^{\infty} \int f_j dH^\alpha$$

for some constant  $C$  depending only on  $\alpha$  and  $n$ . This follows from the nontrivial fact that the Choquet integral with respect to dyadic Hausdorff content is sublinear [1].

## 1 Proof of the Theorem

We need a Lemma.

**Lemma 1.** *Let  $\chi_Q$  be the characteristic function of the cube  $Q$ . Then*

$$\int M(\chi_Q)^p dH^\alpha \leq Cl(Q)^\alpha, \quad \frac{\alpha}{n} < p.$$

*Proof of Lemma 1.* Let  $x_Q$  be the center of  $Q$ . Then

$$M(\chi_Q)(x) \leq C \inf \left( 1, \left( \frac{l(Q)}{|x - x_Q|} \right)^n \right), \quad x \in \mathbb{R}^n.$$

Thus, since  $\alpha/np < 1$ ,

$$\int M(\chi_Q)^p dH^\alpha \leq Cl(Q)^\alpha + C \int_0^1 l(Q)^\alpha t^{-(\alpha/np)} dt = Cl(Q)^\alpha.$$

■

*Proof of (i).* In proving (i) we can assume that  $f \geq 0$ .

For each integer  $k$  let  $\{Q_j^{(k)}\}_j$  be a family of non-overlapping dyadic cubes  $Q_j^{(k)}$  such that

$$\{x : 2^k < f(x) \leq 2^{k+1}\} \subset \bigcup_j Q_j^{(k)}$$

and

$$\sum_j l(Q_j^{(k)})^\alpha \leq 2H_d^\alpha \{x : 2^k < f \leq 2^{k+1}\}.$$

Set  $g = \sum_k 2^{(k+1)p} \chi_{A_k}$ , where  $A_k = \bigcup_j Q_j^{(k)}$ . Thus  $f^p \leq g$ .

Assume first that  $1 \leq p$ . Then

$$(Mf)^p \leq M(f^p) \leq M(g) \leq \sum_k 2^{(k+1)p} \sum_j M(\chi_{Q_j^{(k)}}).$$

By (5) and Lemma 1 we get

$$\begin{aligned} \int (Mf)^p dH^\alpha &\leq C \sum_k 2^{(k+1)p} \sum_j \int M(\chi_{Q_j^{(k)}}) dH^\alpha \\ &\leq C \sum_k 2^{(k+1)p} \sum_j l(Q_j^{(k)})^\alpha \\ &\leq C \sum_k 2^{(k+1)p} H^\alpha \{x : 2^k < f(x) \leq 2^{k+1}\} \\ &\leq C \sum_k \frac{2^{2p}}{2^p - 1} \int_{2^{(k-1)p}}^{2^{kp}} H^\alpha \{x : f(x)^p > t\} dt \\ &\leq C \int f^p dH^\alpha, \end{aligned}$$

which concludes the proof in the case at hand.

Assume now that  $\frac{\alpha}{n} < p < 1$ .

Since  $f \leq \sum_k 2^{k+1} \chi_{A_k}$ ,

$$Mf \leq \sum_k 2^{k+1} \sum_j M(\chi_{Q_j^{(k)}}).$$

We have

$$(Mf)^p \leq \sum_k 2^{(k+1)p} \sum_j M(\chi_{Q_j^{(k)}})^p,$$

because  $p < 1$ , and hence

$$\int (Mf)^p dH^\alpha \leq C \sum_k 2^{(k+1)p} \sum_j l(Q_j^{(k)})^\alpha \leq C \int f^p dH^\alpha.$$

■

The main difficulty in the proof of part (ii) of the Theorem is that  $H^\alpha$  is not additive if  $\alpha < n$ . In particular is not true that there exists a constant

$C > 0$  such that if  $Q_1, \dots, Q_m$  are non-overlapping dyadic cubes and  $f \geq 0$ , then

$$(6) \quad \sum_{j=1}^m \int_{Q_j} f dH^\alpha \leq C \int_{\bigcup_j Q_j} f dH^\alpha.$$

This can be shown by subdividing the interval  $[0, 1]$  in  $2^m$  ( $m$  large enough) equal intervals, and taking  $f \equiv 1$ .

Nevertheless, if we assume that for some constant  $p$  we have

$$(7) \quad \sum_{Q_j \subset Q} l(Q_j)^\alpha \leq p l(Q)^\alpha, \text{ for each dyadic cube } Q,$$

then is not difficult to prove that (6) holds with some  $C = C(p)$  independent of  $f$ .

The next lemma, first appeared in [3], provides us with families of dyadic cubes satisfying (7) and thus (6). We include a short proof for the reader's convenience.

**Lemma 2 (Melnikov).** *Let  $\{Q_j\}$  be a family of non-overlapping dyadic cubes. Then there exists a subfamily  $\{Q_{j_\nu}\}$  such that*

$$(i) \quad \sum_{Q_{j_\nu} \subset Q} l(Q_{j_\nu})^\alpha \leq 2l(Q)^\alpha, \text{ for each dyadic square } Q,$$

and

$$(ii) \quad H^\alpha \left( \bigcup_{\nu} Q_{j_\nu} \right) \leq 2 \sum_{\nu} l(Q_{j_\nu})^\alpha.$$

*Proof.* Let  $\{Q_{j_\nu}\}$  be a maximal subfamily of  $\{Q_j\}$  satisfying (i). That is, we set  $j_1 = 1$  and if  $j_1, \dots, j_\nu$  have been chosen so that (i) holds, then we define  $j_{\nu+1}$  as the first index such that the family  $\{Q_1, \dots, Q_{j_{\nu+1}}\}$  satisfies (i). Therefore property (i) holds and we are left with the task of proving (ii). Take an index  $j$  such that  $j_m < j < j_{m+1}$  for some  $m$ . Then there exists a dyadic cube  $Q_j^* \supset Q_j$  such that

$$\sum_{Q_{j_\nu} \subset Q_j^*, \nu \leq m} l(Q_{j_\nu})^\alpha + l(Q_j)^\alpha > 2l(Q_j^*)^\alpha.$$

Then

$$l(Q_j^*)^\alpha \leq \sum_{Q_{j_\nu} \subset Q_j^*, \nu \leq m} l(Q_{j_\nu})^\alpha.$$

We can assume that  $\sum_{\nu} l(Q_{j\nu})^{\alpha} < \infty$ , because otherwise (ii) is obviously satisfied. Then the sequence  $l(Q_j^*)$  is bounded, and thus we can consider the family  $(\tilde{Q}_k)$  of maximal cubes of the family  $\{Q_j^*\}_j$ . Hence

$$\bigcup Q_j \subset \left( \bigcup_{\nu} Q_{j\nu} \right) \cup \left( \bigcup_k \tilde{Q}_k \right)$$

and, consequently,

$$H^{\alpha} \left( \bigcup Q_j \right) \leq 2 \sum_{\nu} l(Q_{j\nu})^{\alpha},$$

as desired. ■

We still need an auxiliary inequality.

**Lemma 3.** *For  $f \geq 0$  we have*

$$\int f(x) dx \leq \frac{n}{\alpha} \left( \int f(x)^{\alpha/n} dH^{\alpha}(x) \right)^{n/\alpha}.$$

*Proof.* Since for  $l_j \geq 0$

$$\left( \sum_j l_j^n \right)^{1/n} \leq \left( \sum_j l_j^{\alpha} \right)^{1/\alpha},$$

we get

$$H^n(E)^{1/n} \leq H^{\alpha}(E)^{1/\alpha}, \text{ for } E \subset \mathbb{R}^n.$$

Set

$$\lambda_{\beta}(t) = H^{\beta} \{x : f(x) > t\}.$$

Then

$$\lambda_n^{1/n}(t) \leq \lambda_{\alpha}^{1/\alpha}(t), \quad t > 0,$$

and so

$$\begin{aligned} \int_0^{\infty} f(x) dx &= \int_0^{\infty} \lambda_n(t) dt = \frac{n}{\alpha} \int_0^{\infty} \lambda_n(r^{n/\alpha}) r^{(n/\alpha)-1} dr \\ &\leq \frac{n}{\alpha} \int_0^{\infty} \lambda_{\alpha}(r^{n/\alpha})^{n/\alpha} r^{(n/\alpha)-1} dr \\ &\leq \frac{n}{\alpha} \left( \int_0^{\infty} f^{\alpha/n} dH^{\alpha} \right)^{(n/\alpha)-1} \int_0^{\infty} \lambda_{\alpha}(r^{n/\alpha}) dr \\ &\leq \frac{n}{\alpha} \left( \int_0^{\infty} f^{\alpha/n} dH^{\alpha} \right)^{n/\alpha}, \end{aligned}$$

where in the second inequality we used the fact that

$$\lambda_\alpha(r^{n/\alpha})r \leq \int_{\{x: f^{\alpha/n}(x) > r\}} f^{\alpha/n} dH^\alpha, \quad r > 0.$$

■

*Proof of (ii).* Given  $t > 0$  let  $\{Q_j\}$  be the family of maximal dyadic cubes  $Q_j$  such that  $\frac{1}{|Q_j|} \int_{Q_j} f > t$  (we assume again, without loss of generality, that  $f \geq 0$ ). Then

$$\{x : M_d f(x) > t\} = \bigcup_j Q_j.$$

By Lemma 3

$$l(Q_j)^\alpha \leq \left( \frac{1}{t} \int_{Q_j} f \right)^{\alpha/n} \leq C t^{-\alpha/n} \int_{Q_j} f^{\alpha/n} dH^\alpha.$$

Applying Lemma 2 to the  $\{Q_j\}$  we get some subfamily  $\{Q_{j_\nu}\}$  for which one can write

$$\begin{aligned} H^\alpha\{x : M_d f(x) > t\} &\leq 2 \sum_\nu l(Q_{j_\nu})^\alpha \\ &\leq C t^{-\alpha/n} \sum_\nu \int_{Q_{j_\nu}} f^{\alpha/n} dH^\alpha \\ &\leq C t^{-\alpha/n} \int f^{\alpha/n} dH^\alpha, \end{aligned}$$

where the last inequality is due to the packing condition (i) of Lemma 2. ■

## References

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