Regularity of planar quasiconformal mappings

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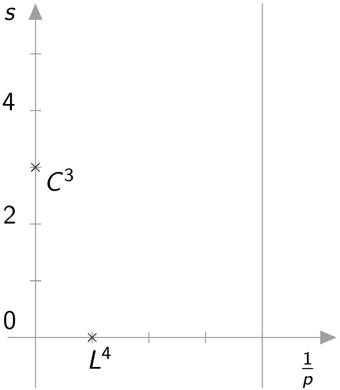
Introduction

Measuring smoothness and integrability in $\mathbb{R}^d$

Lebesgue spaces $\longrightarrow$ integrability.

- $\|f\|_{L^p} = \left(\int |f|^p\right)^{1/p}$,
  $\|f\|_{L^\infty} = \operatorname{ess\,sup}|f|$

Measuring smoothness and integrability in $\mathbb{R}^d$



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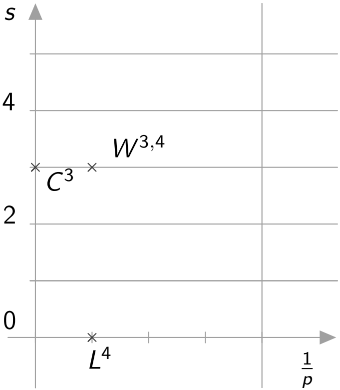

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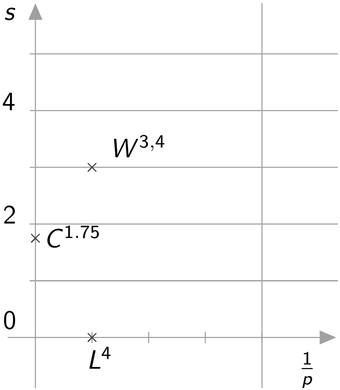

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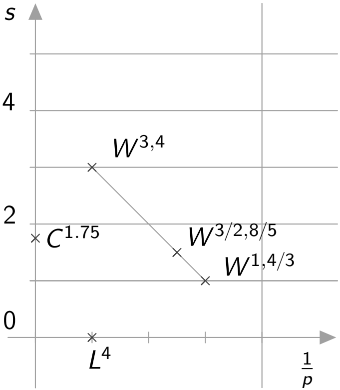

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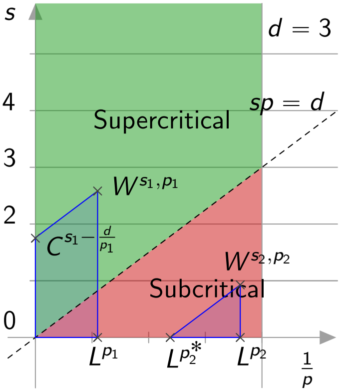

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By means of Sobolev embeddings, we have either continuity or extra integrability.

Quasiconformal mappings

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Conformal mappings
Preserves angles
"Circles to circles"
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Quasiconformal mappings
Angle distortion bounded.
"Circles to ellipses".
$|\bar{\partial} f| \leqslant k|\partial f|$

The Beurling transform

The Beurling transform of a function $f \in L^p(\mathbb{C})$ is:

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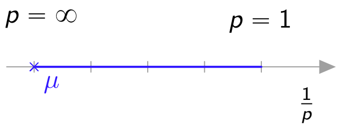
Recall that $\mathcal{B} : L^p(\mathbb{C}) \to L^p(\mathbb{C})$ is bounded for $1 < p < \infty$.
Also $\mathcal{B} : W^{s,p}(\mathbb{C}) \to W^{s,p}(\mathbb{C})$ is bounded for $1 < p < \infty$ and $s > 0$.

QC mappings of the whole plane

The Beltrami equation

Let $\mu \in L^\infty_c(\mathbb{C})$ with $\kappa := \|\mu\|_\infty < 1$.



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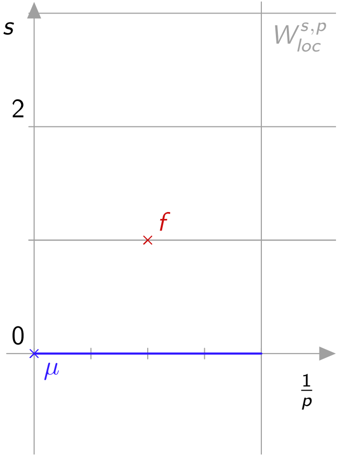

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$$\bar{\partial} f(z) = \mu(z) \partial f(z)$$

has a unique solution $f \in W^{1,2}_{loc}$ such that $f(z) = z + \mathcal{O}(1/z) \quad$ as $z \to \infty$.

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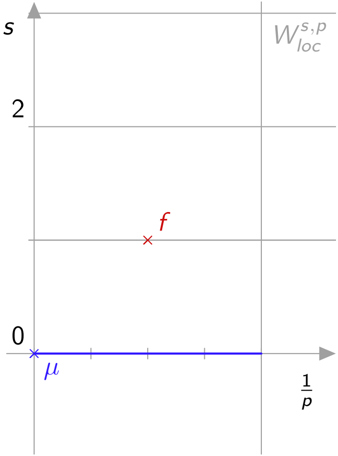

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Consider
$h := \mu + \mu\mathcal{B}(\mu) + \mu\mathcal{B}(\mu\mathcal{B}(\mu)) + \cdots$

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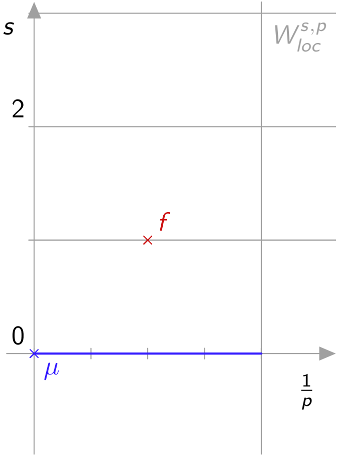

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$= (I - \mu\mathcal{B})^{-1}(\mu)$,

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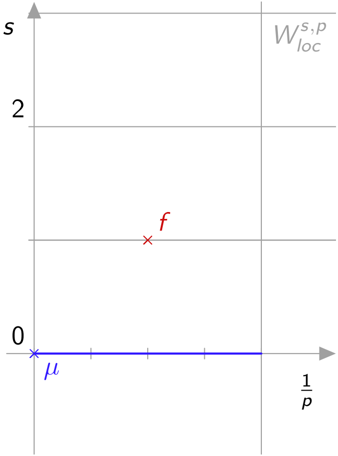

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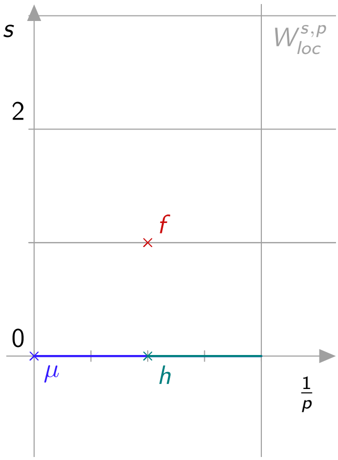

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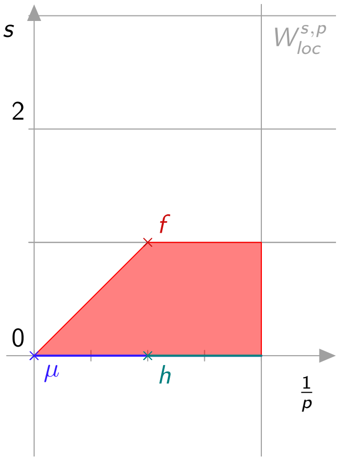

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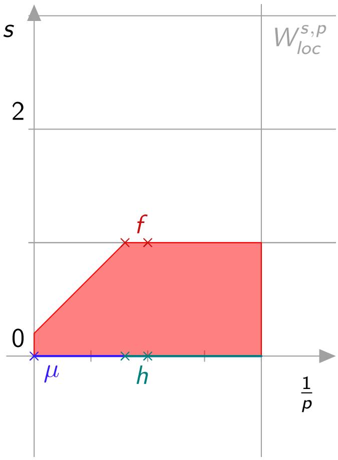

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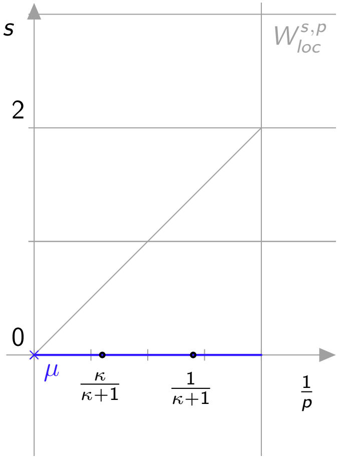
This remains true if $\|\mathcal{B}\|_{(p,p)} < 1/\kappa$.

Results without boundaries



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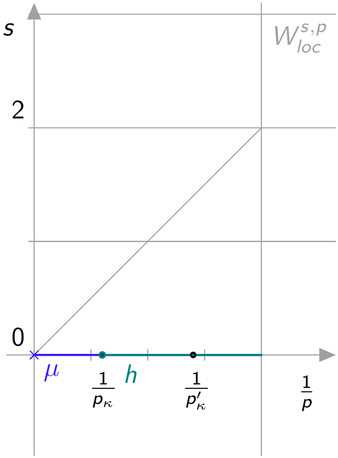

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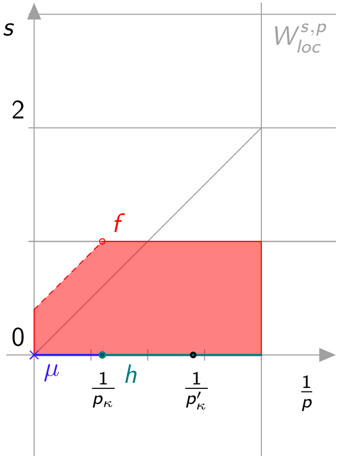

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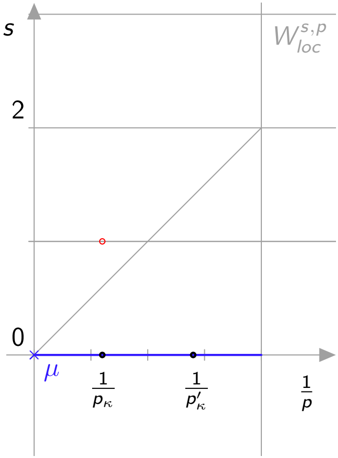

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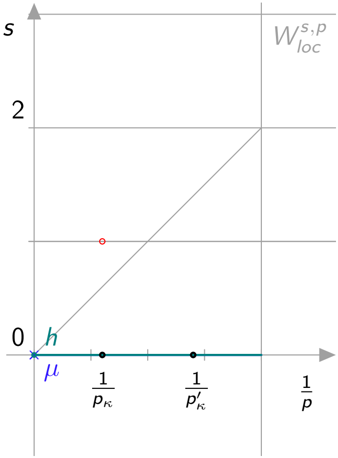

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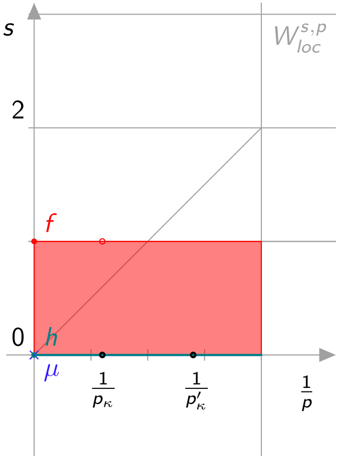

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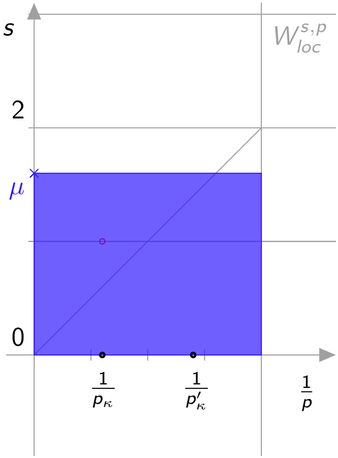

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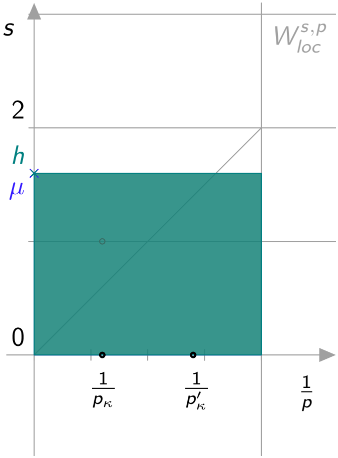

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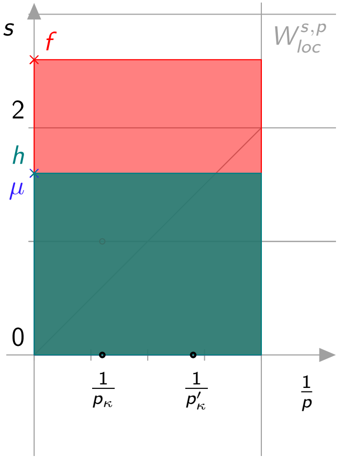

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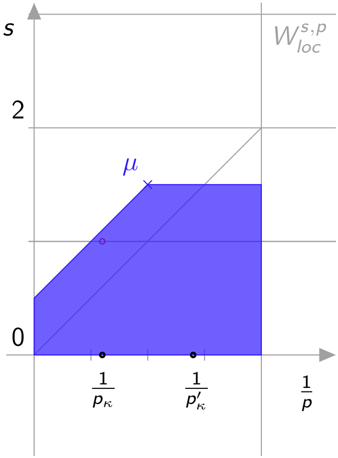

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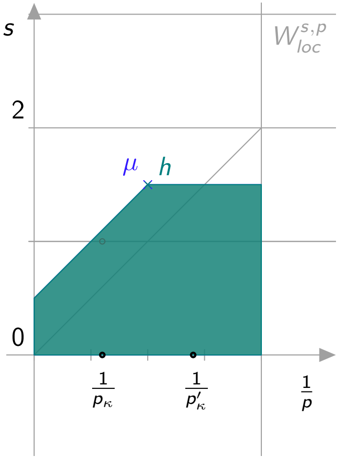

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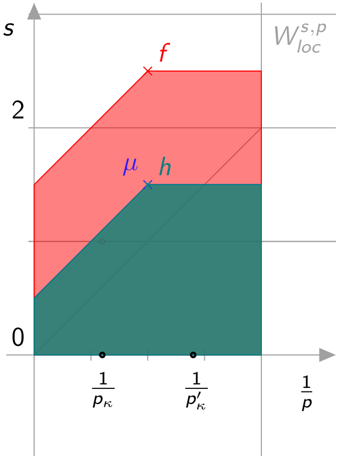

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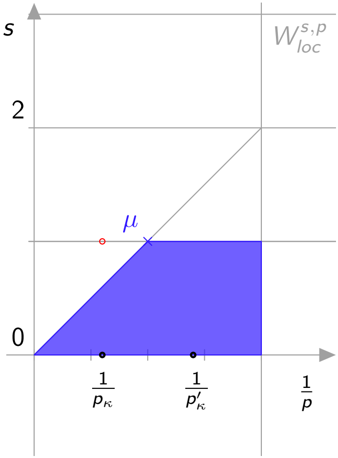

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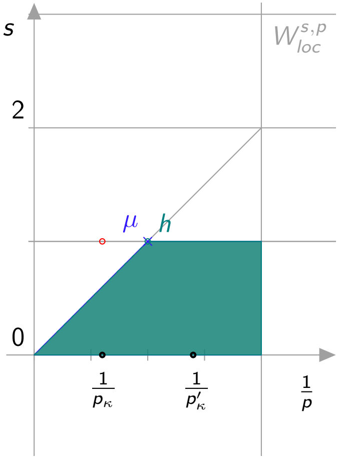

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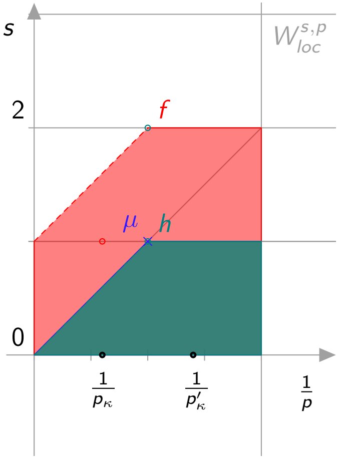

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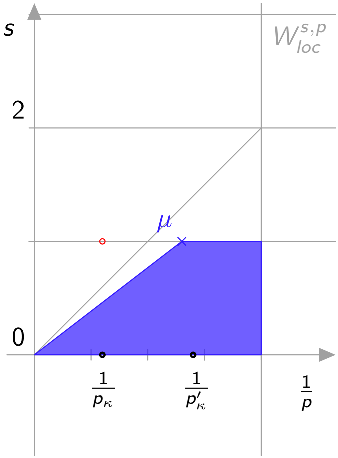
Results without boundaries



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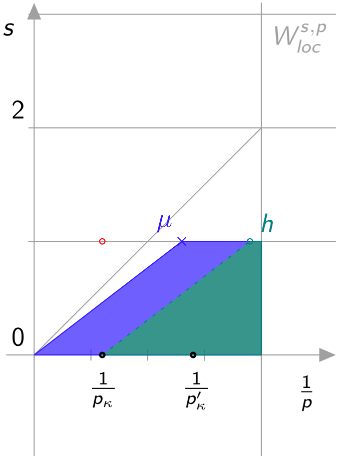

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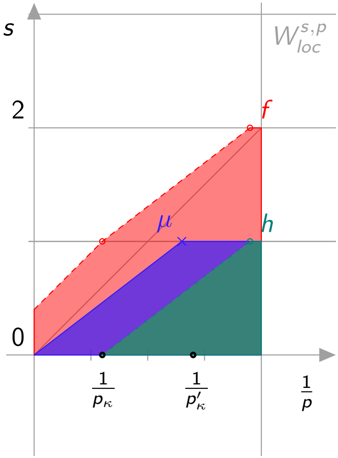

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Recent progress

Theorem (P.)

Let $0 < s < 2$, $1 < p < \infty$, *let* $\mu \in W^{s,p} \cap L^\infty$, *with* $\mu \leqslant \kappa\chi_{\mathbb{D}}$ *and let* f *be the principal solution to the Beltrami equation* $\bar\partial f = \mu\partial f$.
If $s = \frac{2}{p}$, *then*

$$\bar\partial f \in W^{s,q} \qquad \text{for every } \frac{1}{q} > \frac{1}{p}.$$

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See [Clop, Faraco, Ruiz] for previous weaker results and Baisón's thesis for a stronger result in the critical setting with $s > 1/2$.

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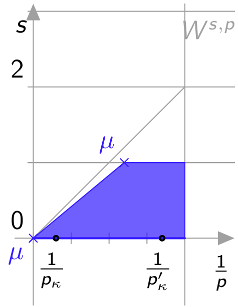
It remains unclear if the condition $\frac{1}{p} < \frac{1}{p'_\kappa} - \frac{1}{p_\kappa}$ can be replaced by $\frac{1}{p} < \frac{1}{p'_\kappa}$, which is more natural and is achieved for $s = 1$.

Idea



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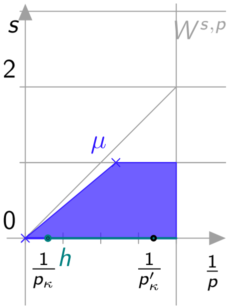

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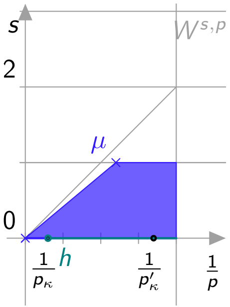

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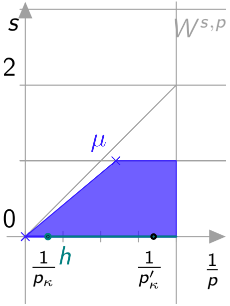

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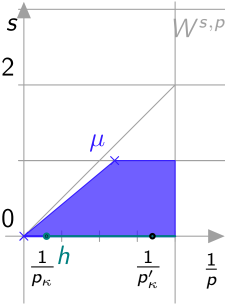

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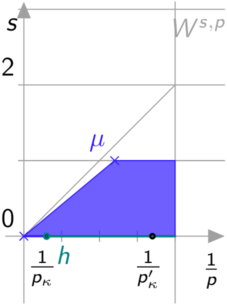

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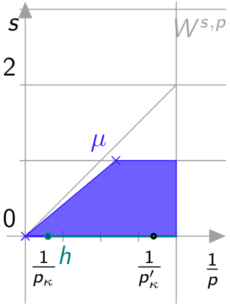

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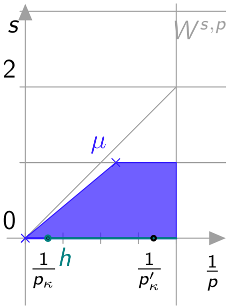

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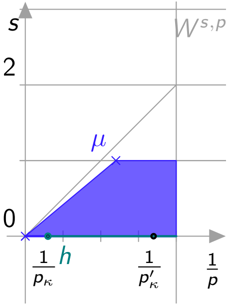

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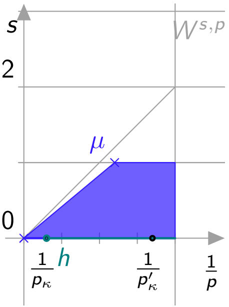

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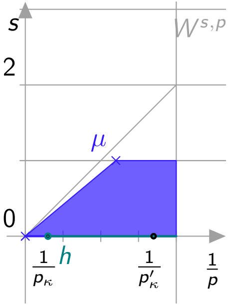

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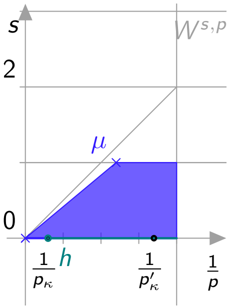

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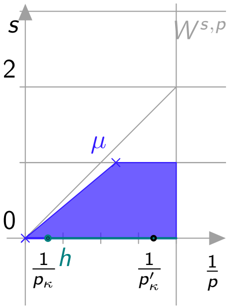

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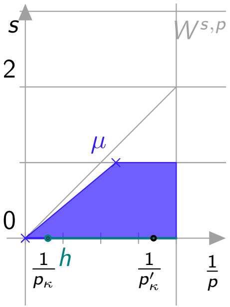

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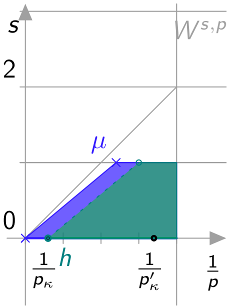

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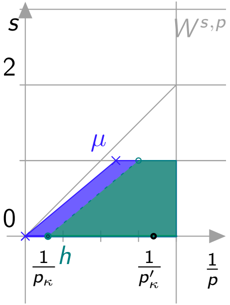
Idea



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Spare room

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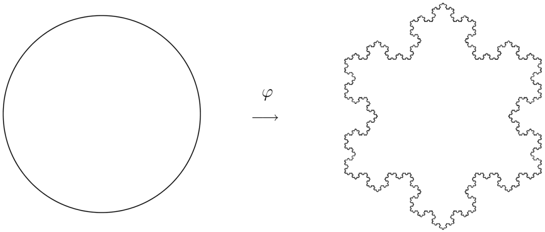
This technique cannot be used for fractional derivatives. Can we bypass it?
In the critical setting with fractional derivatives, Baisón et al. could do it combining the use of the logarithm with certain potentials to give some better results, namely $\log(\partial f) \in W^{s,p}$, but they were forced to work only with $s > 1/2$. Is this restriction natural? Can this procedure be adapted to the subcritical setting?

QC mappings on domains

What about quasiconformal mappings on domains?



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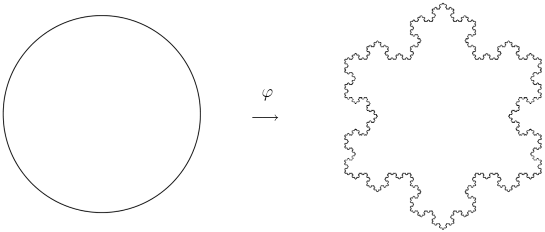

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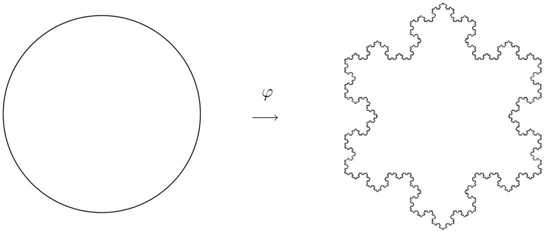

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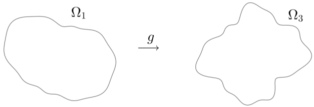
The moral is that in order to study the regularity of μ-quasiconformal mappings between domains we must take into account both the regularity of the boundary and the regularity of μ.

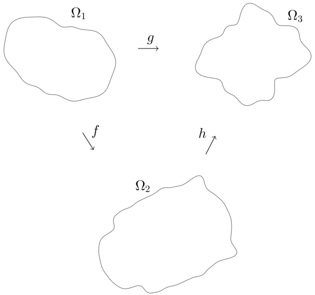
Idea

Let $g : \Omega_1 \to \Omega_3$ to be μ-QC, with $\mu \in W^{s,p}(\Omega_1)$ and $\partial\Omega_1$, $\partial\Omega_3$ regular enough. Can we say that $\partial g \in W^{s,p}(\Omega_1)$??



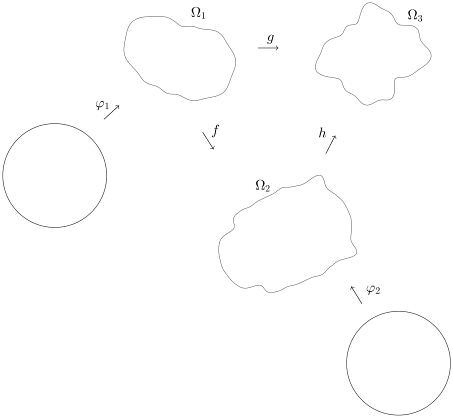
Idea

By Stoilow factorization, $g = h \circ f$ where $f : \mathbb{C} \to \mathbb{C}$ is the μ-principal mapping and $h : \Omega_2 \to \Omega_3$ is conformal.

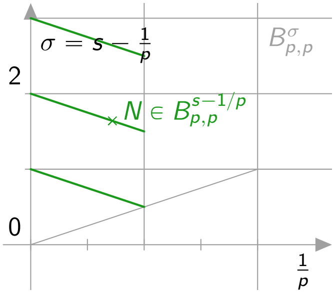


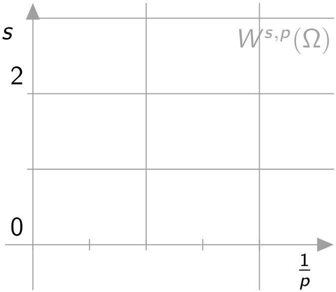
Idea

We can find Riemann mappings (conformal) if the domains are simply connected.



The principal mapping



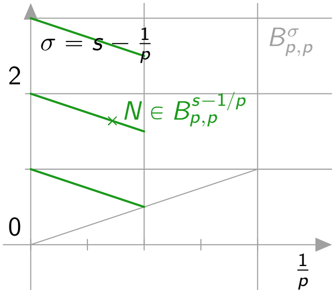


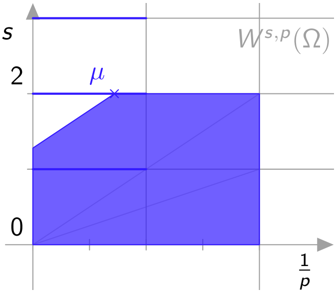
We study supercritical case.

Theorem (P)

Let $\Omega \subset \mathbb{C}$ be a bdd domain, with normal vector $N \in B^{s-1/p}_{p,p}(\partial\Omega)$, $s \in \mathbb{N}$ and $p > 2$.

The principal mapping





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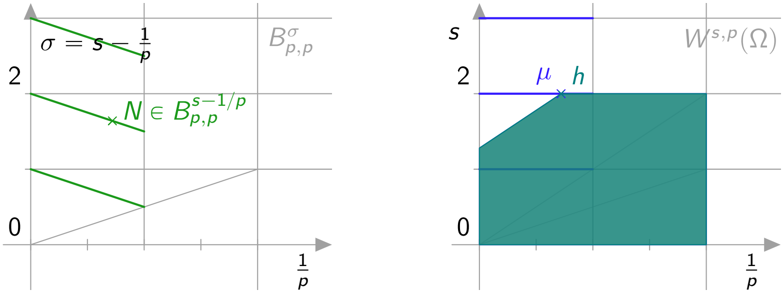

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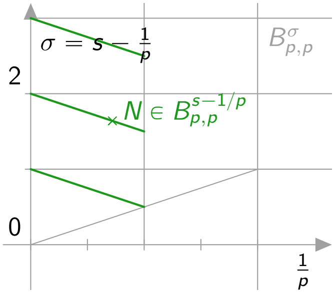
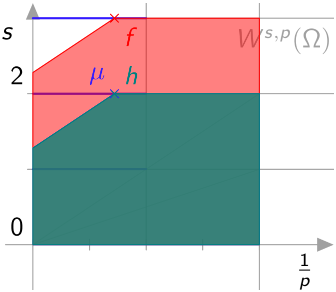
The principal mapping





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General case

Conjecture (Theorem in progress with K. Astala)

Let $s \in \mathbb{N}$ and $p > 2$. If Ω is a simply connected $B^{s+1-\frac{1}{p}}_{p,p}$-domain, then any Riemann mapping $\varphi : \mathbb{D} \to \Omega$ satisfies that $\varphi \in W^{s+1,p}(\mathbb{D})$ and it is bi-Lipschitz.

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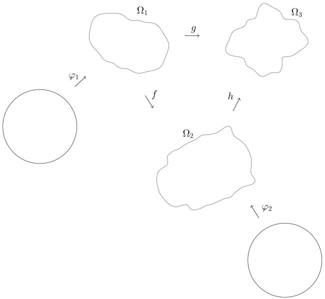
Conjectured corollary

Let $s \in \mathbb{N}$ and $p > 2$, let Ω_1 and Ω_3 be simply connected $B^{s+1-\frac{1}{p}}_{p,p}$-domains and let $g : \Omega_1 \to \Omega_3$ be a μ-quasiconformal mapping with $\mu \in W^{s,p}(\Omega_1)$. Then $g \in W^{s+1,p}(\Omega_1)$.

Idea of the proof of the corollary



$f \in W^{s+1,p}(\Omega)$ and it is bi-Lipschitz by the Theorem.

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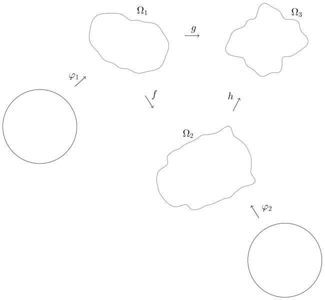

$f \in W^{s+1,p}(\Omega)$ and it is bi-Lipschitz by the Theorem. $\varphi_1 \in W^{s+1,p}(\mathbb{D})$ and it is bi-Lipschitz by the conjecture.

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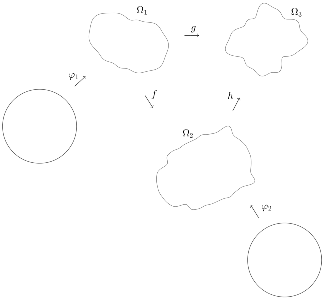

$f \in W^{s+1,p}(\Omega)$ and it is bi-Lipschitz by the Theorem. $\varphi_1 \in W^{s+1,p}(\mathbb{D})$ and it is bi-Lipschitz by the conjecture. By the trace condition, $f \circ \varphi_1$ is a $B_{p,p}^{s+1-\frac{1}{p}}$ parameterization of $\partial\Omega_2$.

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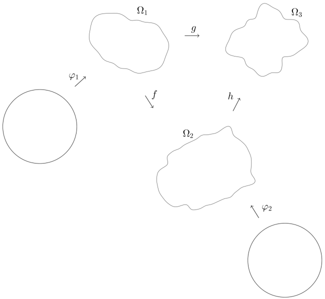

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By the trace condition, $f \circ \varphi_1$ is a $B^{s+1-\frac{1}{p}}_{p,p}$ parameterization of $\partial\Omega_2$.
By the conjecture, $h \circ \varphi_2$ and φ_2 are in $W^{s+1,p}(\mathbb{D})$.

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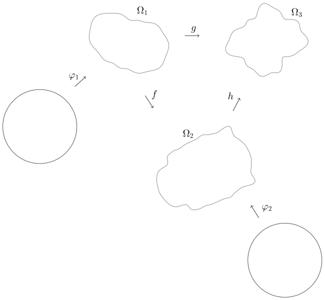

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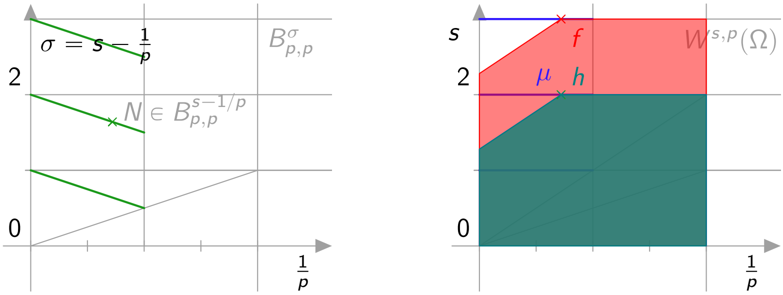
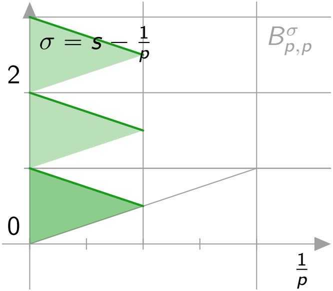
By the conjecture, $h \circ \varphi_2$ and φ_2 are in $W^{s+1,p}(\mathbb{D})$.
Then, $g = (h \circ \varphi_2) \circ (\varphi_2^{-1}) \circ f$.

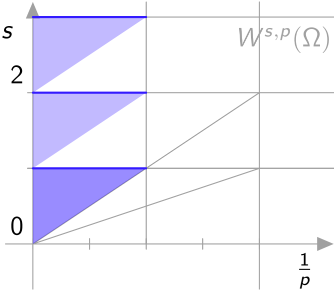
Conclusions



- In the complex plane, if $N \in B^{s-1/p}_{p,p}(\partial\Omega)$ and $p > 2$, then $\mu \in W^{s,p}(\Omega) \implies f, g \in W^{s+1,p}(\Omega)$.

Conclusions





- In the complex plane, if $N \in B^{s-1/p}_{p,p}(\partial\Omega)$ and $p > 2$, then $\mu \in W^{s,p}(\Omega) \implies f, g \in W^{s+1,p}(\Omega)$.
- Expected further results:
    - The results hold apparently for $0 < s < 1$, $sp > 2$ (work in progress with Eero Saksman) and for Hölder spaces.
    - Subcritical situation: is there any condition on $\partial\Omega$ which can lead to analogous results?

The end

Moltes gràcies!!
Muchas gracias!!