Boundedness of Calderón-Zygmund operators in Sobolev spaces of a Lipschitz domain

PHD thesis in progress, directed by Xavier Tolsa

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June 13th, 2014

Introduction.

The Beurling transform.

The Beurling transform of a function $f \in L^p(\mathbb{C})$ is:

$$\mathcal{B}f(z) = c_0 \lim_{\varepsilon \to 0} \int_{|w-z|>\varepsilon} \frac{f(w)}{(z-w)^2} dm(w).$$

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Also $\mathcal{B} : W^{n,p}(\mathbb{C}) \to W^{n,p}(\mathbb{C})$ is bounded for $1 < p < \infty$ and $n > 0$.

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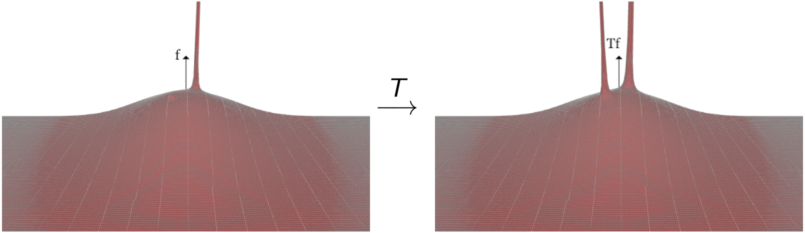
In general, if $x \notin \mathrm{supp}(f) \subset \mathbb{R}^d$ then a convolution CZO of order n is

$$Tf(x) = \int K(x-y) f(y)$$

with

$$|\nabla^j K(x)| \le \frac{1}{|x|^{d+j}} \qquad \text{for } j \le n.$$

The problem we face.



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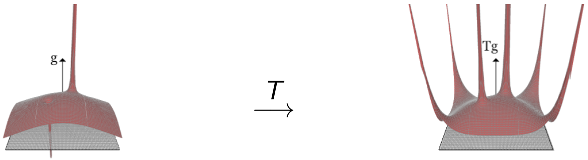

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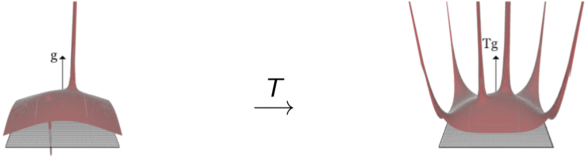

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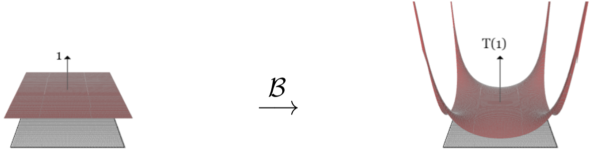

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For Ω a rectangle, $\mathcal{B}\chi_\Omega$ is in every $L^p(\Omega)$ but not in $W^{1,p}(\Omega)$ for $p \geq 2$.

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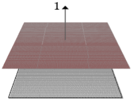
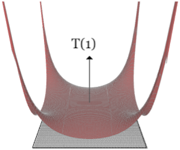


$\xrightarrow{\mathcal{B}}$



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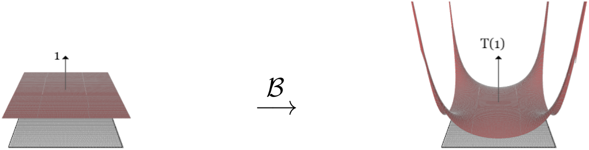

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When is $T : W^{n,p}(\Omega) \to W^{n,p}(\Omega)$ bounded?
We seek for answers in terms of test functions and in terms of the geometry of the boundary.

Test function conditions.

Results.

Theorem (Cruz, Mateu, Orobitg, 2013)

Given a $C^{1+\epsilon}$ domain $\Omega \subset \mathbb{R}^d$, T even and $p > d$.
If $T(\chi_\Omega) \in W^{1,p}(\Omega)$, then T is bounded in $W^{1,p}(\Omega)$.

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Given a Lipschitz domain $\Omega \subset \mathbb{R}^d$ and $p > d$. If $T(P) \in W^{n,p}(\Omega)$ for polynomials $P \in \mathcal{P}^{n-1}(\Omega)$, then T is bounded in $W^{n,p}(\Omega)$.

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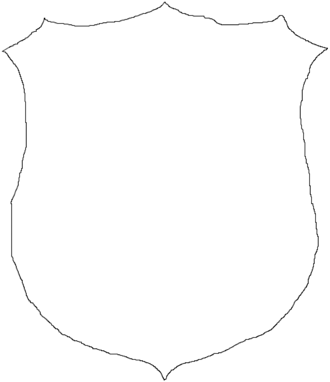

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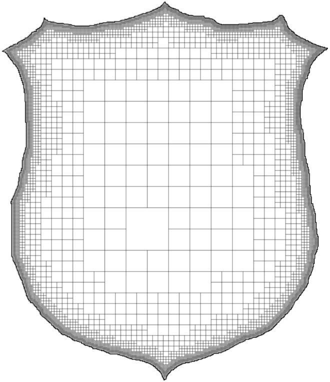

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We perform a Whitney covering $\mathcal{W}$ such that

- $\operatorname{dist}(Q, \partial\Omega) \approx \ell(Q)$.
- $\{5Q\}_{Q\in\mathcal{W}}$ has finite superposition.

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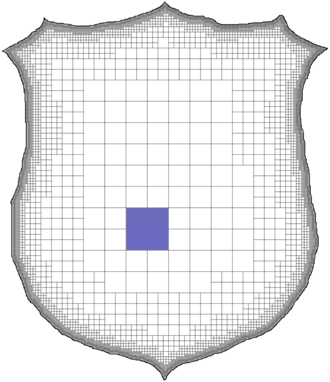

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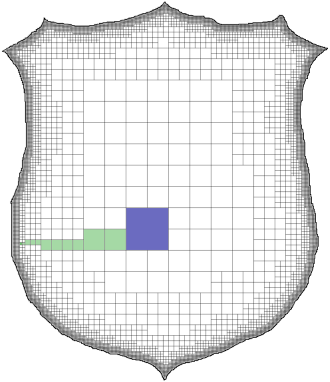

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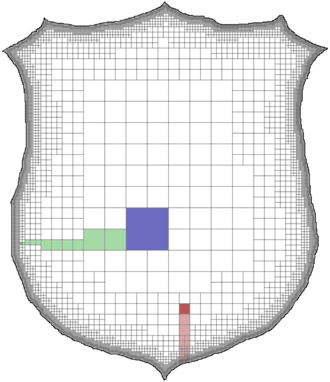

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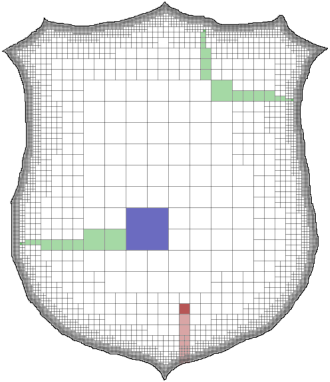

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We can think on Harnack chains.

The key point: approximating by polynomials.

A new approach for the case $n = 1$:

Key Lemma

The following are equivalent:

- $\|\nabla Tf\|_{L^p(\Omega)}^p \leq C\|f\|_{W^{1,p}(\Omega)}^p$.
- $\sum_{Q\in\mathcal{W}} |f_{3Q}|^p \|\nabla T(\chi_\Omega)\|_{L^p(Q)}^p \leq C\|f\|_{W^{1,p}(\Omega)}^p$.

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Enough to prove

$$\sum_Q \|\nabla T(f - f_{3Q}\chi_\Omega)\|^p_{L^p(Q)} \lesssim \|\nabla f\|^p_{L^p(\Omega)}.$$

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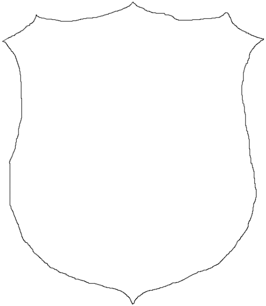
Local part is a good function, in $W^{1,p}(\mathbb{R}^d)$.
For the non-local part, we use a Harnack chain of cubes.
Ingredients: bounds for the kernel, Poincaré inequality and Hölder.

Proof of the T(P) theorem ($p > d$).



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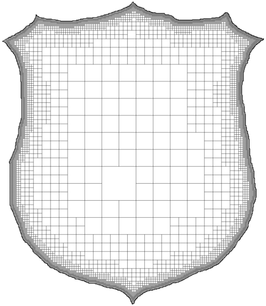

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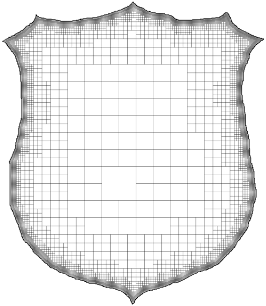

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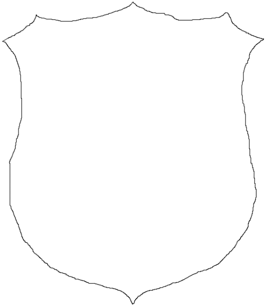

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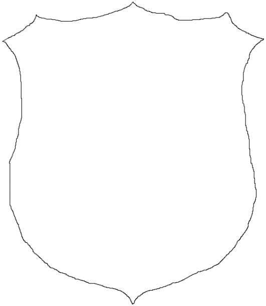

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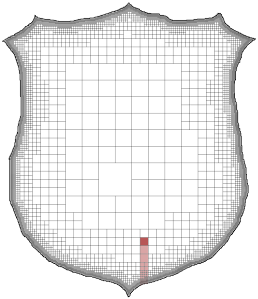
Since $p > d$, by the Sobolev Embedding Theorem

$$\|f\|_{L^\infty} \leq C\|f\|_{W^{1,p}(\Omega)}.$$

□

The Carleson measures.



According to carleson measures for Besov space of analytic functions $B_p(\rho)$,

Definition

We say that μ is p-Carleson for $\Omega \subset \mathbb{R}^d$ iff for every Whitney cube P,

$$\sum_{Q \subset \mathbf{Sh}(P)} \mu(\mathbf{Sh}(Q))^{p'} \ell(Q)^{\frac{p-d}{p-1}} \leq C\mu(\mathbf{Sh}(P)).$$

Proof of Carleson $\Rightarrow$ boundedness ($p \leq d$).

Assume that $n = 1$ and

$$\mu(x) = |\nabla T\chi_\Omega(x)|^p dx$$

is p-Carleson for Ω. We want

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But, by Poincaré inequalities and some p-Carleson measure properties,

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Hypothesis: T bounded in $W^{1,p}(\Omega)$. Then the averaging function

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by the Key Lemma, is also bounded $\mathcal{A} : W^{1,p}(\Omega) \to L^p(\mu)$ for

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By duality, $\mathcal{A}^* : L^p(\mu) \to (W^{1,p}(\Omega))^*$ is also bounded.

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By duality, $\mathcal{A}^* : L^p(\mu) \to (W^{1,p}(\Omega))^*$ is also bounded.
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$\mathcal{A}^*(g)$ solves a Neumann problem $\Delta h = \widetilde{g}$.

A geometric condition.

Results.

Theorem (P., 2013)

For $\Omega \subset \mathbb{C}$ smooth enough, if the vector normal to the boundary of Ω is in the Besov space $B^{n-\frac{1}{p}}_{p,p}(\partial\Omega)$ then $\mathcal{B}(\chi_\Omega) \in W^{n,p}(\Omega)$, with

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V. Cruz and X. Tolsa proved the case $n = 1$.
Tolsa proved a converse for $n = 1$ and Ω smooth enough.

Skip proof

Ingredients for the proof.

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- Beurling of characteristic functions of circles, half-planes, polynomials, ...

See details

Conclusions.

- For $p > d$ we have a $T(P)$ theorem for any Calderón-Zygmund operator of convolution type in any ambient space as long as we have uniform bounds in the derivatives of its kernel.

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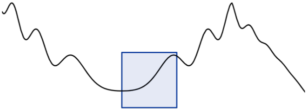
- Next steps:
  - Proving analogous results for any $s \in \mathbb{R}_+$.
  - Looking for a more general set of operators where the Besov condition on the boundary implies Sobolev boundedness.
  - Sharpness of all those results.

The end.

Moltes gràcies!!
Děkuji!!

Defining some generalized betas of David-Semmes.



A measure of the flatness of a set Γ:

Ending

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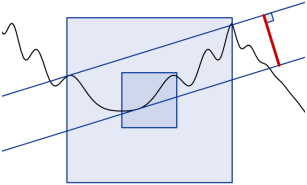

A measure of the flatness of a set Γ:

Definition (P. Jones)

$\beta_\Gamma(Q) = \inf_V \frac{w(V)}{\ell(Q)}$

Ending

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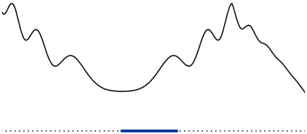

The graph of a function $y = A(x)$:
Consider $I \subset \mathbb{R}$, and define

Ending

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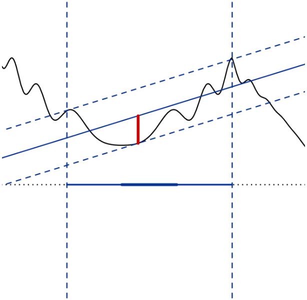

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Definition

$$\beta_\infty(I, A) = \inf_{P \in \mathcal{P}^1} \left\| \frac{A-P}{\ell(I)} \right\|_\infty$$

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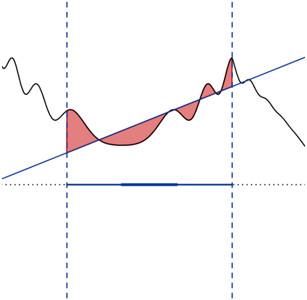

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Definition

$$\beta_p(I, A) = \inf_{P \in \mathcal{P}^1} \frac{1}{\ell(I)^{\frac{1}{p}}} \left\| \frac{A-P}{\ell(I)} \right\|_p$$

Ending

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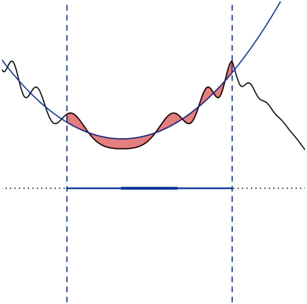

Ending

The graph of a function $y = A(x)$:
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Definition

$$\beta_{(n)}(I, A) = \inf_{P \in \mathcal{P}^n} \frac{1}{\ell(I)} \left\| \frac{A - P}{\ell(I)} \right\|_1$$

If there is no risk of confusion,
we will write just $\beta_{(n)}(I)$.

Geometric condition in terms of betas: The Besov space.

Definition

For $0 < s < \infty$, $1 \leq p < \infty$, $f \in B^s_{p,p}(\mathbb{R})$ if

$$\|f\|_{B^s_{p,p}} = \|f\|_{L^p} + \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \left| \frac{\Delta_h^{[s]+1} f(x)}{h^s} \right|^p \frac{dm(h)}{|h|} dm(x) \right)^{1/p} < \infty.$$

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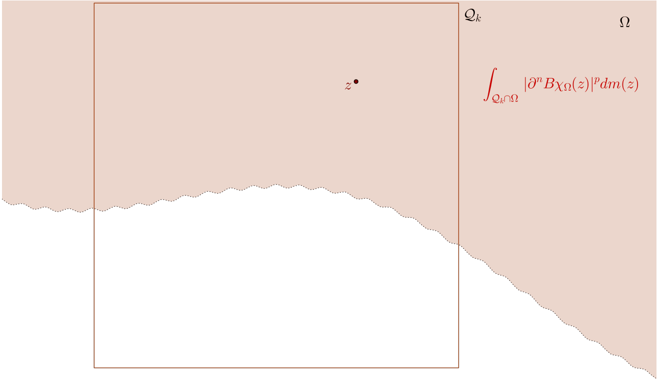
Theorem (Dorronsoro)

Let $f : \mathbb{R} \to \mathbb{R}$ be a function in the Besov space $B^s_{p,p}$. Then, for any $n \geq [s]$,

$$\|f\|^p_{B^s_{p,p}} \approx \|f\|_{L^p} + \sum_{I \in \mathcal{D}} \left(\frac{\beta_{(n)}(I)}{\ell(I)^{s-1}} \right)^p \ell(I).$$

Ending

Main idea: projecting cubes to the boundary.



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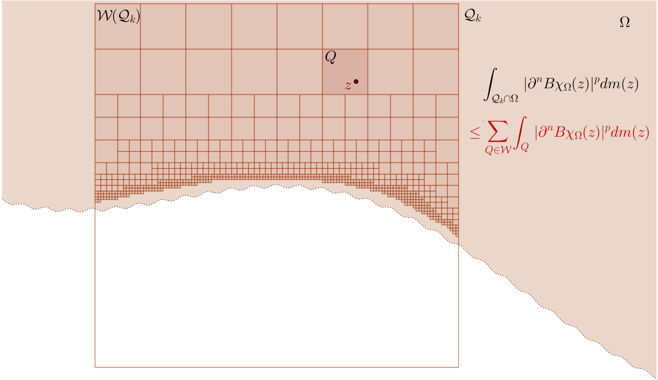

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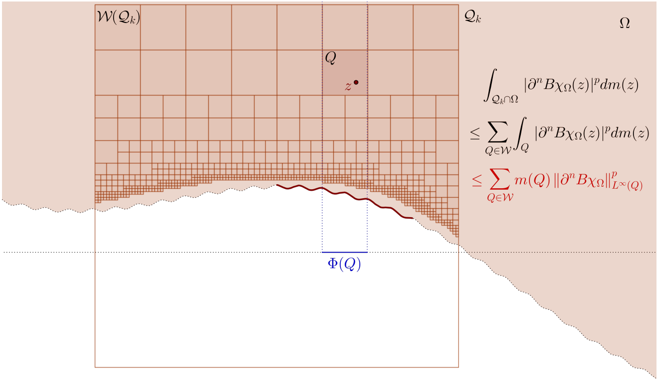

$$\int_{\mathcal{Q}_k\cap\Omega} |\partial^n B\chi_\Omega(z)|^p dm(z)$$
$$\leq \sum_{Q\in\mathcal{W}} \int_Q |\partial^n B\chi_\Omega(z)|^p dm(z)$$
$$\leq \sum_{Q\in\mathcal{W}} m(Q)\, \|\partial^n B\chi_\Omega\|^p_{L^\infty(Q)}$$

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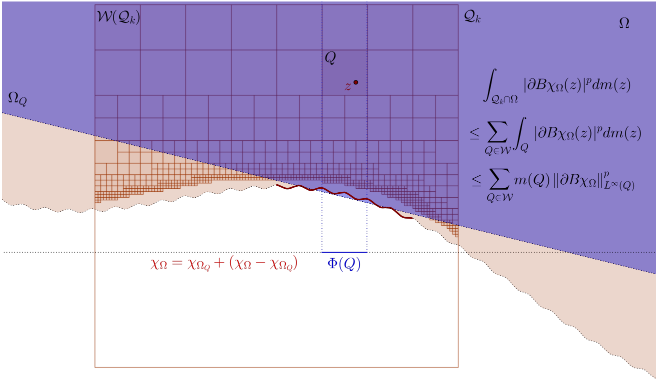

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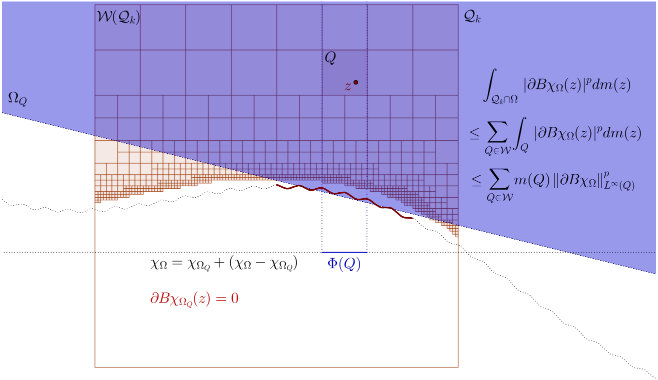

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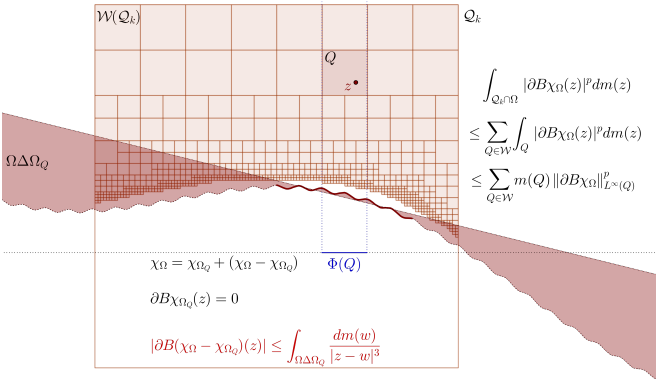

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