A T(1) theorem for Sobolev spaces on domains

PHD thesis in progress, directed by Xavier Tolsa

Martí Prats

Universitat Autònoma de Barcelona

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Introduction

The Beurling transform

The Beurling transform of a function $f \in L^p(\mathbb{C})$ is:

$$Bf(z) = c_0 \lim_{\varepsilon \to 0} \int_{|w-z|>\varepsilon} \frac{f(w)}{(z-w)^2} dm(z).$$

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Recall that $B : L^p(\mathbb{C}) \to L^p(\mathbb{C})$ is bounded for $1 < p < \infty$.
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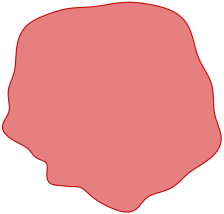
In particular, if $z \notin \mathrm{supp}(f)$ then Bf is analytic in an ε-neighborhood of z and

$$\partial^n Bf(z) = c_n \int_{|w-z|>\varepsilon} \frac{f(w)}{(z-w)^{n+2}} dm(z).$$

back to T(P)

The problem we face

Let Ω be a Lipschitz domain.



When is $B : W^{s,p}(\Omega) \to W^{s,p}(\Omega)$ bounded?
We want an answer in terms of the geometry of the boundary.

Known facts, part 1

In a recent paper, Cruz, Mateu and Orobitg proved that for $0 < s \leq 1$, $2 < p < \infty$ with $sp > 2$, and $\partial\Omega$ smooth enough,

Theorem

$$B : W^{s,p}(\Omega) \to W^{s,p}(\Omega) \quad \textit{is bounded}$$

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One can deduce regularity of a quasiregular mapping in terms of the regularity of its Beltrami coefficient.

Introducing the Besov spaces $B^s_{p,p}$

The geometric answer will be given in terms of Besov spaces $B^s_{p,p}$.
$B^s_{p,p}$ form a family closely related to $W^{s,p}$. They coincide for $p = 2$.
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Definition

For $0 < s < \infty$, $1 \leq p < \infty$, $f \in \dot{B}^s_{p,p}(\mathbb{R})$ if

$$\|f\|_{\dot{B}^s_{p,p}} = \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \left| \frac{\Delta_h^{[s]+1} f(x)}{h^s} \right|^p \frac{dm(h)}{|h|} dm(x) \right)^{1/p} < \infty.$$

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Furthermore, $f \in B^s_{p,p}(\mathbb{R})$ if

$$\|f\|_{B^s_{p,p}} = \|f\|_{L^p} + \|f\|_{\dot{B}^s_{p,p}} < \infty.$$

We call them homogeneous and non-homogeneous Besov spaces respectively.

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If the normal vector N belongs to $B^{1-1/p}_{p,p}(\partial\Omega)$, then $B(\chi_\Omega) \in W^{1,p}(\Omega)$ with

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Tolsa proved a converse for Ω flat enough.

Main results

T(P) Theorem

Let $2 < p < \infty$ and $1 \leq n < \infty$. Let Ω be a Lipschitz domain. Then the Beurling transform is bounded in $W^{n,p}(\Omega)$ if and only if for any polynomial of degree less than n restricted to the domain, $P = P\chi_\Omega$, $B(P) \in W^{n,p}(\Omega)$.

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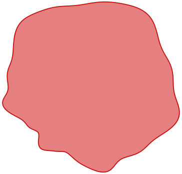
Theorem (Geometric condition on the boundary)

Let Ω be smooth enough. Then we can write

$$\|\partial^n B\chi_\Omega\|^p_{L^p(\Omega)} \lesssim \|N\|^p_{B^{n-1/p}_{p,p}(\partial\Omega)} + \mathcal{H}^1(\partial\Omega)^{2-np}.$$

Proof of the T(P) Theorem

Local charts



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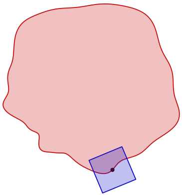

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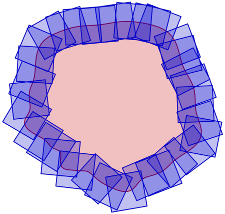

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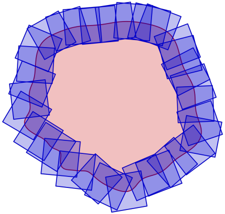

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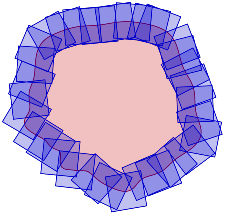

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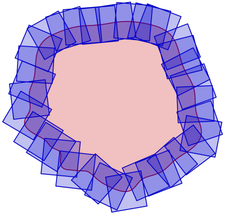

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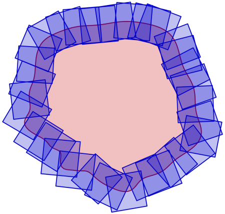

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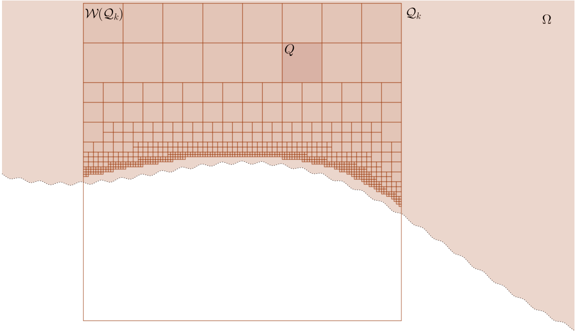
- The restriction to the inner region is always bounded: $f\psi_0 \in W^{n,p}(\mathbb{C})$.

Local charts: Whitney decomposition



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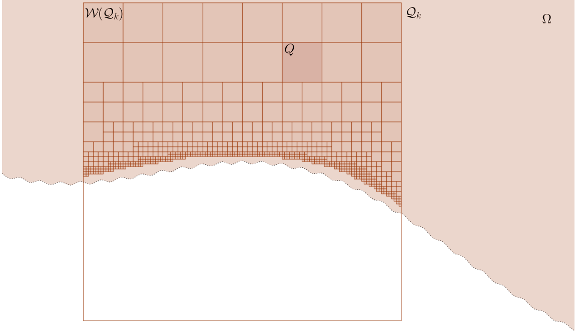

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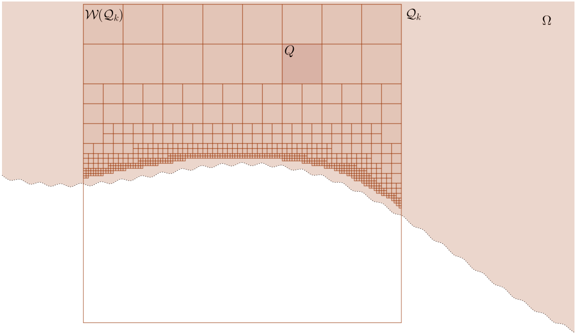

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- $\mathrm{dist}(Q, \partial\Omega \cap \mathcal{Q}) \approx \ell(Q)$ for every $Q \in \mathcal{W}$.
- The family $\{5Q\}_{Q\in\mathcal{W}}$ has finite superposition.
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A necessity arises: approximating polynomials

We will use the Poincaré inequality, that is, given $f \in W^{1,p}(Q)$, $1 \leq p \leq \infty$,

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Definition

Given $f \in W^{n,p}(\Omega)$ and a cube Q, we call $\mathfrak{P}^n_Q f$ to the polynomial of degree smaller than n restricted to Ω such that for any multiindex β with $|\beta| < n$,

$$\fint_{3Q} D^\beta \mathfrak{P}^n_Q f = \fint_{3Q} D^\beta f.$$

Properties of approximating polynomials

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$\left\| \mathfrak{P}^n_{Q_1} f - \mathfrak{P}^n_{Q_2} f \right\|_{L^\infty(3Q_1 \cap 3Q_2)} \lesssim \ell(Q_1)^{n-\frac{2}{p}} \|\nabla^n f\|_{L^p(3Q_1 \cup 3Q_2)}$.

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$|m_{Q,\gamma}| \lesssim \sum_{j=|\gamma|}^{n-1} \left\| \nabla^j f \right\|_{L^\infty(3Q)} \ell(Q)^{j-|\gamma|}$.

The proof: $BP \in W^{n,p}(\Omega) \Rightarrow \|Bf\|^p_{W^{n,p}(\Omega)} \lesssim \|f\|^p_{W^{n,p}(\Omega)}$

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so

$$D^\alpha B(\mathfrak{P}^n_Q f)(z) = \sum_{|\gamma|<n} m_{Q,\gamma} \sum_{(0,0)\leq\lambda\leq\gamma} \binom{\gamma}{\lambda} (x_0 - x_Q)^{\gamma-\lambda} D^\alpha (BP_\lambda)(z)$$

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where, by P5,

$$|m_{Q,\gamma}| \lesssim \sum_{j=|\gamma|}^{n-1} \left\|\nabla^j f\right\|_{L^\infty(3Q)} \ell(Q)^{j-|\gamma|}.$$

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Adding with respect to $Q \in \mathcal{W}$, by the Sobolev Embedding Theorem ($\left\|\nabla^j f\right\|_{L^\infty(\mathcal{Q}\cap\Omega)} \leq C\left\|\nabla^j f\right\|_{W^{1,p}(\mathcal{Q}\cap\Omega)}$ when $p > 2$), we get

$$\begin{aligned}\sum_{Q\in\mathcal{W}} \left\|D^\alpha B(\mathfrak{P}^n_Q f)\right\|^p_{L^p(Q)} &\lesssim \sum_{j<n} \left\|\nabla^j f\right\|^p_{W^{1,p}(\mathcal{Q}\cap\Omega)} \sum_{0\leq\lambda\leq\gamma} \|B P_\lambda\|^p_{W^{n,p}(\Omega)}\\ &\lesssim \|f\|^p_{W^{n,p}(\mathcal{Q}\cap\Omega)}.\end{aligned}$$

Key Lemma: sticking to the essential

Lemma

Let Ω be a Lipschitz domain, $\mathcal{Q}$ a window, $\psi \in \mathcal{C}^\infty(\frac{99}{100}\mathcal{Q})$ with $\left\|\nabla^j \psi\right\|_{L^\infty} \lesssim \frac{1}{R^j}$ for $j \geq 0$. Then, for any $|\alpha| = n$ and $f = \psi \cdot \widetilde{f}$ with $\widetilde{f} \in W^{n,p}(\Omega)$, TFAE:

- $\|D^\alpha Bf\|^p_{L^p(\mathcal{Q})} \lesssim \|f\|^p_{W^{n,p}(\mathcal{Q}\cap\Omega)}$.
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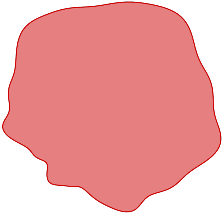
Idea of the proof: separate local and non-local parts of the error term,

$$\begin{aligned} &D^\alpha Bf(z) - D^\alpha B(\mathfrak{P}^n_Q f)(z) \\ &\qquad = D^\alpha B(\chi_{2Q}(f - \mathfrak{P}^n_Q f))(z) + D^\alpha B((1-\chi_{2Q})(f - \mathfrak{P}^n_Q f))(z). \end{aligned}$$

▸ Sketch of the proof

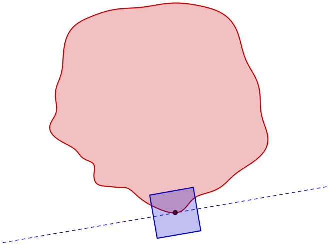
A geometric condition for the Beurling transform



Theorem (Geometric condition on the boundary)

Let Ω be smooth enough. Then we can write

$$\|\partial^n B\chi_\Omega\|^p_{L^p(\Omega)} \lesssim \|N\|^p_{B^{n-1/p}_{p,p}(\partial\Omega)} + \mathcal{H}^1(\partial\Omega)^{2-np}.$$



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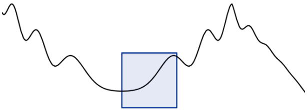
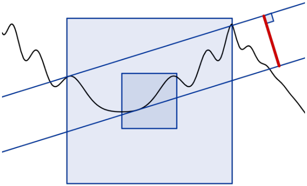
Defining some generalized betas of David-Semmes



A measure of the flatness of a set Γ:

Defining some generalized betas of David-Semmes

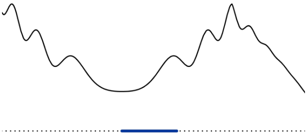


A measure of the flatness of a set Γ:

Definition (P. Jones)

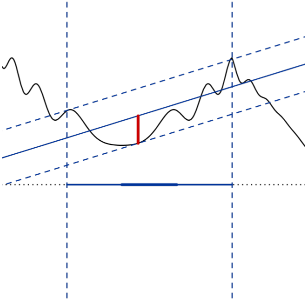
$\beta_\Gamma(Q) = \inf_V \frac{w(V)}{\ell(Q)}$

Defining some generalized betas of David-Semmes



The graph of a function $y = A(x)$:
Consider $I \subset \mathbb{R}$, and define

Defining some generalized betas of David-Semmes



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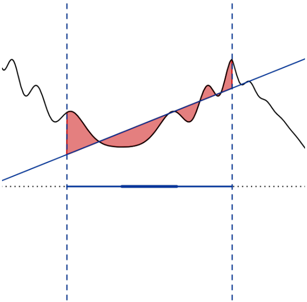
Definition

$$\beta_\infty(I, A) = \inf_{P \in \mathcal{P}^1} \left\| \frac{A - P}{\ell(I)} \right\|_\infty$$

Defining some generalized betas of David-Semmes



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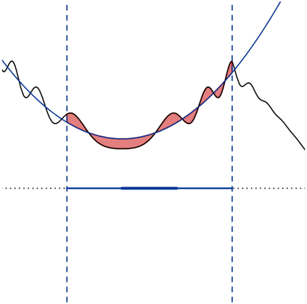
Definition

$$\beta_p(I, A) = \inf_{P \in \mathcal{P}^1} \frac{1}{\ell(I)^{\frac{1}{p}}} \left\| \frac{A-P}{\ell(I)} \right\|_p$$

Defining some generalized betas of David-Semmes



The graph of a function $y = A(x)$:
Consider $I \subset \mathbb{R}$, and define

Definition

$$\beta_{(n)}(I, A) = \inf_{P \in \mathcal{P}^n} \frac{1}{\ell(I)} \left\| \frac{A-P}{\ell(I)} \right\|_1$$

If there is no risk of confusion,
we will write just $\beta_{(n)}(I)$.

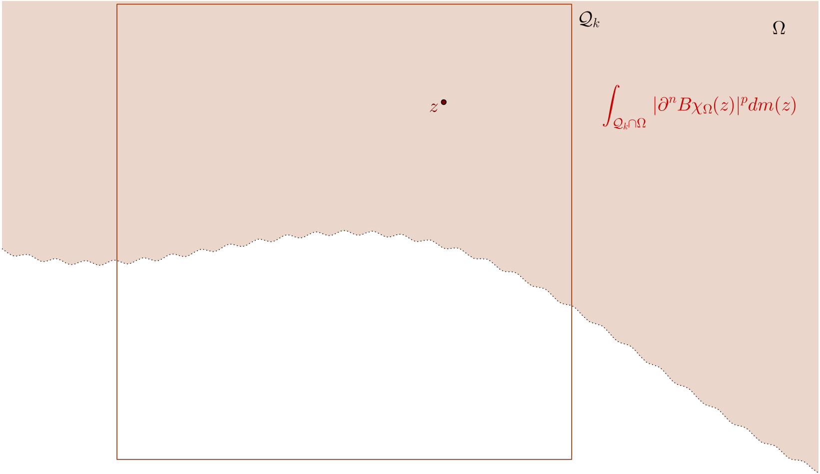
Relation between $\beta_{(n)}$ and $B^n_{p,p}$

Theorem (Dorronsoro)

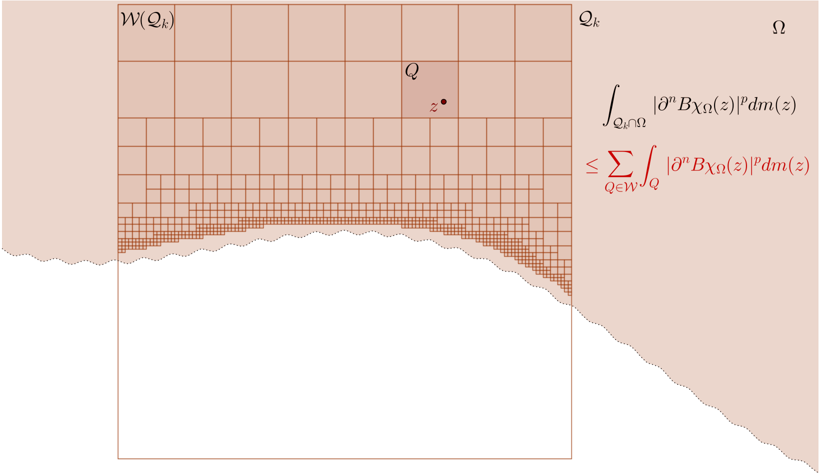
Let $f : \mathbb{R} \to \mathbb{R}$ be a function in the homogeneous Besov space $\dot{B}^s_{p,p}$. Then, for any $n \geq [s]$,

$$\|f\|^p_{\dot{B}^s_{p,p}} \approx \sum_{I \in \mathcal{D}} \left(\frac{\beta_{(n)}(I)}{\ell(I)^{s-1}} \right)^p \ell(I).$$

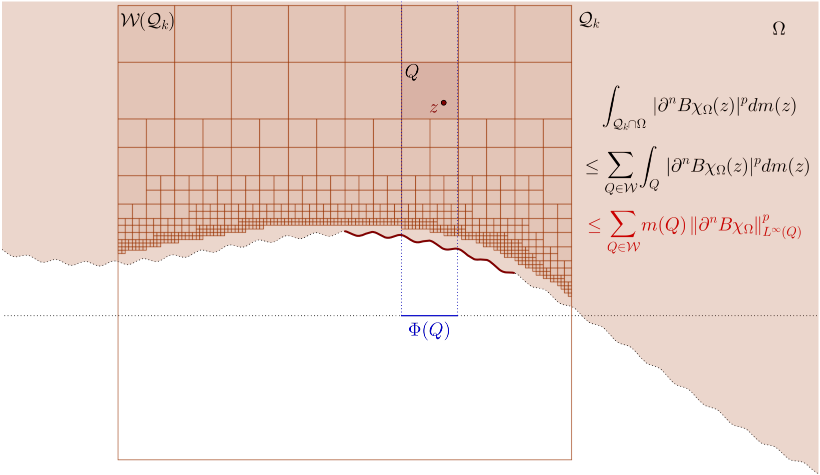
Local charts: Whitney decomposition



Local charts: Whitney decomposition

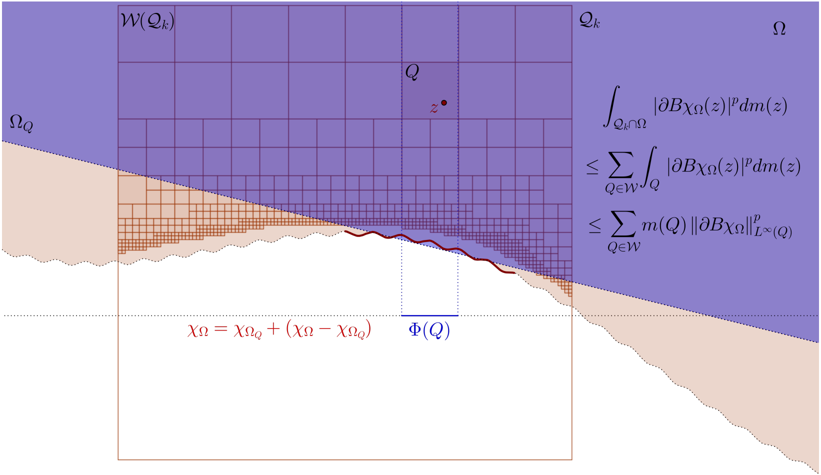


Local charts: Whitney decomposition



First order derivative | Second order derivative | Higher order derivatives | Skip higher order derivatives

Local charts: Bounds for the first derivative

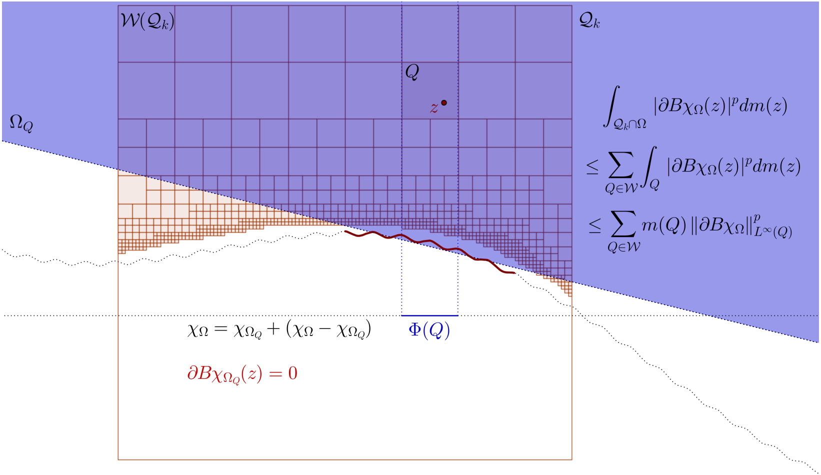


$$\int_{\mathcal{Q}_k\cap\Omega} |\partial B\chi_\Omega(z)|^p dm(z)$$

$$\leq \sum_{Q\in\mathcal{W}} \int_Q |\partial B\chi_\Omega(z)|^p dm(z)$$

$$\leq \sum_{Q\in\mathcal{W}} m(Q) \, \|\partial B\chi_\Omega\|^p_{L^\infty(Q)}$$

Local charts: Bounds for the first derivative



$$\int_{\mathcal{Q}_k\cap\Omega} |\partial B\chi_\Omega(z)|^p dm(z) \leq \sum_{Q\in\mathcal{W}} \int_Q |\partial B\chi_\Omega(z)|^p dm(z) \leq \sum_{Q\in\mathcal{W}} m(Q)\, \|\partial B\chi_\Omega\|^p_{L^\infty(Q)}$$

First order derivative | Second order derivative | Higher order derivatives | Skip higher order derivatives

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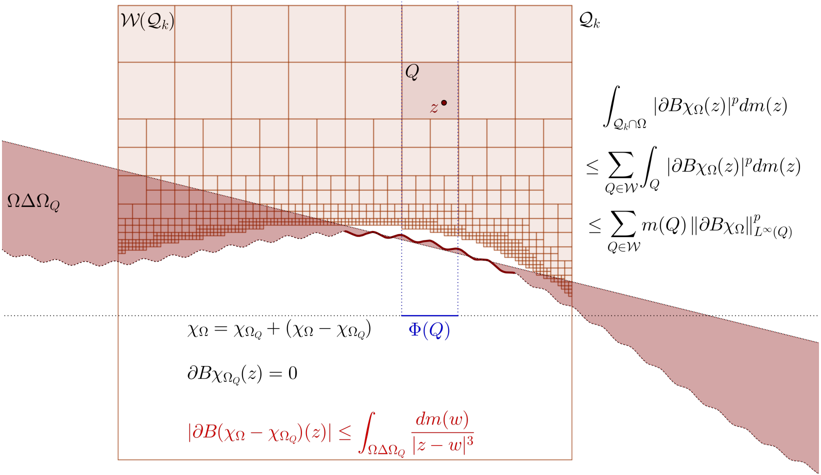

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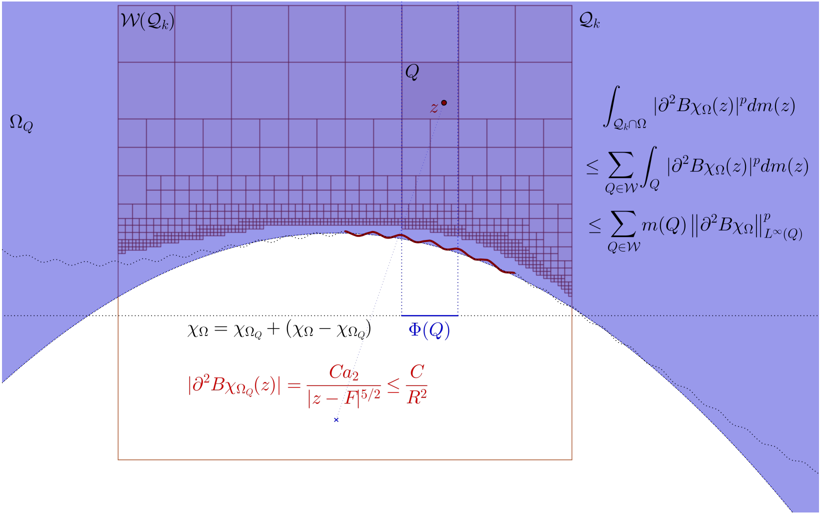
$$\chi_\Omega = \chi_{\Omega_Q} + (\chi_\Omega - \chi_{\Omega_Q})$$

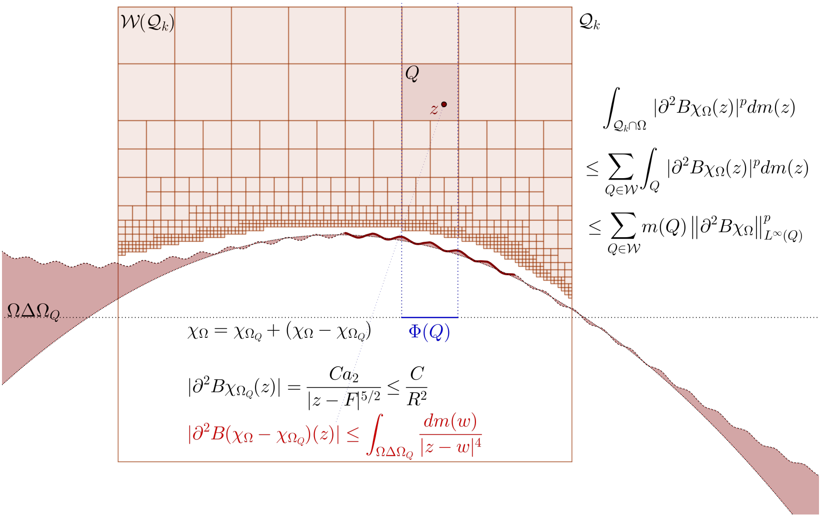
$$\partial B\chi_{\Omega_Q}(z) = 0$$

$$|\partial B(\chi_\Omega - \chi_{\Omega_Q})(z)| \leq \int_{\Omega\Delta\Omega_Q} \frac{dm(w)}{|z-w|^3}$$

Local charts: Second order derivative



Local charts: Second order derivative



$$\int_{\mathcal{Q}_k\cap\Omega} |\partial^2 B\chi_\Omega(z)|^p dm(z)$$

$$\leq \sum_{Q\in\mathcal{W}} \int_Q |\partial^2 B\chi_\Omega(z)|^p dm(z)$$

$$\leq \sum_{Q\in\mathcal{W}} m(Q) \left\|\partial^2 B\chi_\Omega\right\|^p_{L^\infty(Q)}$$

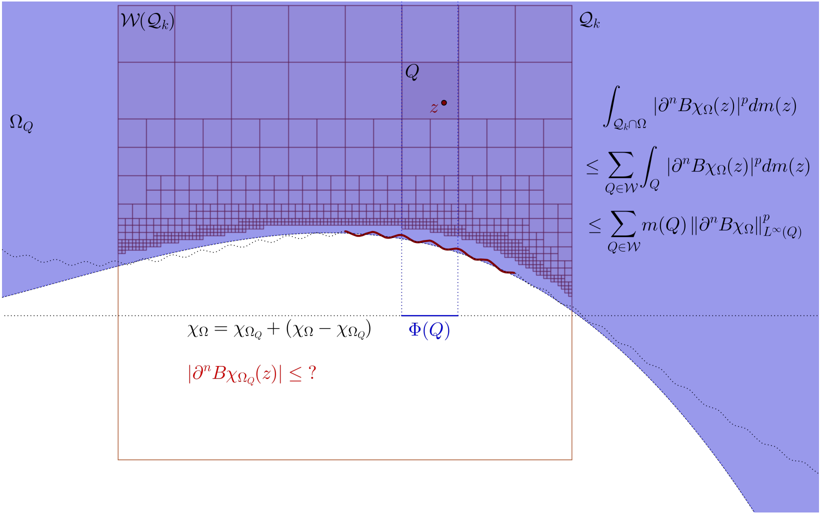
$$\chi_\Omega = \chi_{\Omega_Q} + (\chi_\Omega - \chi_{\Omega_Q})$$

$$|\partial^2 B\chi_{\Omega_Q}(z)| = \frac{Ca_2}{|z-F|^{5/2}} \leq \frac{C}{R^2}$$

$$|\partial^2 B(\chi_\Omega - \chi_{\Omega_Q})(z)| \leq \int_{\Omega\Delta\Omega_Q} \frac{dm(w)}{|z-w|^4}$$

First order derivative | Second order derivative | Higher order derivatives | Skip higher order derivatives

Local charts: Higher order derivatives



$$\int_{\mathcal{Q}_k\cap\Omega} |\partial^n B\chi_\Omega(z)|^p dm(z)$$

$$\leq \sum_{Q\in\mathcal{W}} \int_Q |\partial^n B\chi_\Omega(z)|^p dm(z)$$

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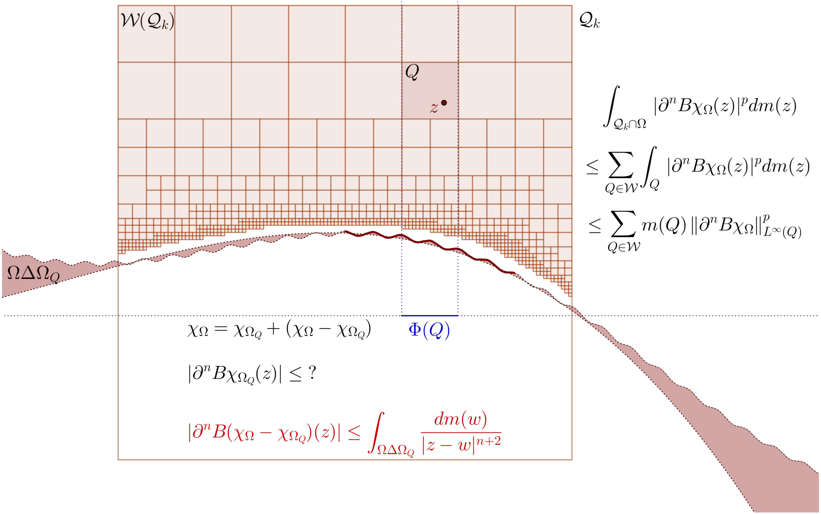
$$\chi_\Omega = \chi_{\Omega_Q} + (\chi_\Omega - \chi_{\Omega_Q})$$

$$|\partial^n B\chi_{\Omega_Q}(z)| \leq ?$$

Local charts: Higher order derivatives



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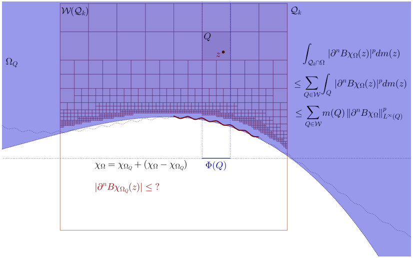
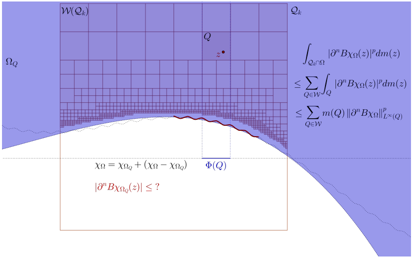
Bounding the polynomial region



We can choose the window length R small enough so that

Bounding the polynomial region



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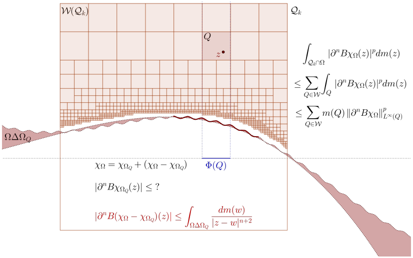
Proposition

If we denote by Ω_Q the region with boundary a minimizing polynomial for $\beta_{(n)}(\Phi(Q))$, we get

$$\left|\partial^n B\chi_{\Omega_Q}\right| \leq \frac{C}{R^n}.$$

Bounding the interstitial region

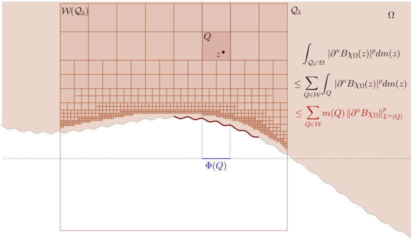


Proposition

Choosing a minimizing polynomial for $\beta_{(n)}(\Phi(Q))$, we get

$$\int_{\Omega\Delta\Omega_Q} \frac{dm(w)}{|z-w|^{n+2}} \lesssim \sum_{\substack{I\in\mathcal{D} \\ \Phi(Q)\subset I\subset\Phi(\mathcal{Q}_k)}} \frac{\beta_{(n)}(I)}{\ell(I)^n} + \frac{1}{R^n}.$$

Hölder inequalities do the rest



Theorem

Let Ω be a Lipschitz domain of order n. Then, with the previous notation,

$$\|\partial^n B\chi_\Omega\|^p_{L^p(\Omega)} \lesssim \sum_{k=1}^{N} \sum_{I\in\mathcal{D}^k} \left(\frac{\beta_{(n)}(I)}{\ell(I)^{n-1/p}}\right)^p \ell(I) + \mathcal{H}^1(\partial\Omega)^{2-np}.$$

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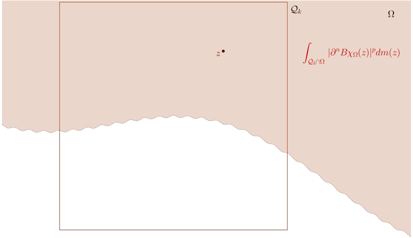

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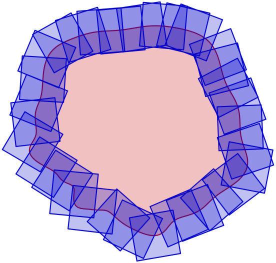

Using a decomposition in windows,

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Back

Conclusions

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  for any Calderon-Zygmund operator of convolution type
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Conclusions

- For $p > 2$ we have a $T(P)$ theorem for any Calderon-Zygmund operator of convolution type in any ambient space as long as we have uniform bounds in the derivatives of its kernel.
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- Next steps:
  - Proving analogous results for any $s \in \mathbb{R}_+$.
  - Looking for a more general set of operators where the Besov condition on the boundary implies Sobolev boundedness.
  - Giving a necessary condition for the boundedness of the Beurling transform when $p \leq 2$.
  - Sharpness of all those results.

Farewell

Thank you!

$\emptyset$

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We will see that

$$\left\|D^\alpha Bf - \sum_{Q\in\mathcal{W}} \chi_Q D^\alpha B(\mathfrak{P}^n_Q f)\right\|^p_{L^p(\mathcal{Q})} \lesssim \|f\|^p_{W^{n,p}(\mathcal{Q}\cap\Omega)}.$$

Breaking the integral into local and non-local parts

Take $\chi_{\frac{3}{2}Q} \leq \varphi_Q \leq \chi_{2Q}$ a smooth bump function.

Breaking the integral into local and non-local parts

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Breaking the integral into local and non-local parts

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Thus, we need to prove that the local part is bounded

$$\textcircled{1} = \sum_{Q \in \mathcal{W}} \left\| D^\alpha B(\varphi_Q(f - \mathfrak{p}_Q^n f)) \right\|_{L^p(Q)}^p \lesssim \|f\|_{W^{n,p}(\mathcal{Q} \cap \Omega)}^p$$

and the non-local part is bounded

$$\textcircled{2} = \sum_{Q \in \mathcal{W}} \left\| D^\alpha B((\chi_\Omega - \varphi_Q)(f - \mathfrak{p}_Q^n f)) \right\|_{L^p(Q)}^p \lesssim \|f\|_{W^{n,p}(\mathcal{Q} \cap \Omega)}^p$$

The local part ① $= \sum_{Q\in\mathcal{W}} \left\| D^\alpha B(\varphi_Q(f - \mathfrak{P}_Q^n f)) \right\|_{L^p(Q)}^p$

As $\varphi_Q(f - \mathfrak{P}_Q^n f) \in W^{n,p}(\mathbb{C})$, the Beurling transform commutes with the derivative

$$\left\| D^\alpha B(\varphi_Q(f - \mathfrak{P}_Q^n f)) \right\|_{L^p(Q)}^p = \left\| BD^\alpha(\varphi_Q(f - \mathfrak{P}_Q^n f)) \right\|_{L^p(Q)}^p$$

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Using P1 and P4, we get

$$\left\| D^\alpha B(\varphi_Q(f - \mathfrak{P}^n_Q f)) \right\|^p_{L^p(Q)} \lesssim \|\nabla^n f\|^p_{L^p(3Q)}.$$

Breaking the non-local part

② $= \sum_{Q\in\mathcal{W}} \left\| D^\alpha B((\chi_\Omega - \varphi_Q)(f - \mathfrak{P}^n_Q f)) \right\|^p_{L^p(Q)}$

In the non-local part we can take the derivative in the kernel of the transform (let's call it $B^{(-\alpha)}$)

Breaking the non-local part

$$\textcircled{2} = \sum_{Q\in\mathcal{W}} \left\| D^\alpha B((\chi_\Omega - \varphi_Q)(f - \mathfrak{P}^n_Q f))\right\|^p_{L^p(Q)}$$

In the non-local part we can take the derivative in the kernel of the transform (let's call it $B^{(-\alpha)}$) and use a partition of unity $\chi_\Omega - \varphi_Q = \sum_{S\in\mathcal{W}} \psi_{QS}$ related to the covering $\{2S\}_{S\in\mathcal{W}}$ with the usual bounds in their derivatives.

Breaking the non-local part
$\textcircled{2} = \sum_{Q\in\mathcal{W}} \left\| D^\alpha B((\chi_\Omega - \varphi_Q)(f - \mathfrak{p}_Q^n f))\right\|_{L^p(Q)}^p$

In the non-local part we can take the derivative in the kernel of the transform (let's call it $B^{(-\alpha)}$) and use a partition of unity $\chi_\Omega - \varphi_Q = \sum_{S\in\mathcal{W}} \psi_{QS}$ related to the covering $\{2S\}_{S\in\mathcal{W}}$ with the usual bounds in their derivatives.
Then,

$$\begin{aligned}\textcircled{2} &\leq \sum_{Q\in\mathcal{W}} \left(\sum_{S\in\mathcal{W}} \left\| B^{(-\alpha)}(\psi_{QS}(f - \mathfrak{p}_S^n f))\right\|_{L^p(Q)} \right)^p \\ &+ \sum_{Q\in\mathcal{W}} \left(\sum_{S\in\mathcal{W}} \left\| B^{(-\alpha)}(\psi_{QS}(\mathfrak{p}_S^n f - \mathfrak{p}_Q^n f))\right\|_{L^p(Q)} \right)^p \\ &= \textcircled{3} + \textcircled{4}.\end{aligned}$$

The non-local part re-localized ③

The re-localized sum is the easier to bound. Using the Hölder inequality and the Poincaré inequality, we get

$$|B^{(-\alpha)}(\psi_{QS}(f - \mathfrak{P}^n_S f))(z)| \lesssim \frac{\ell(S)^{n+\frac{2}{p'}}}{D(Q,S)^{n+2}} \|\nabla^n f\|_{L^p(3S)}.$$

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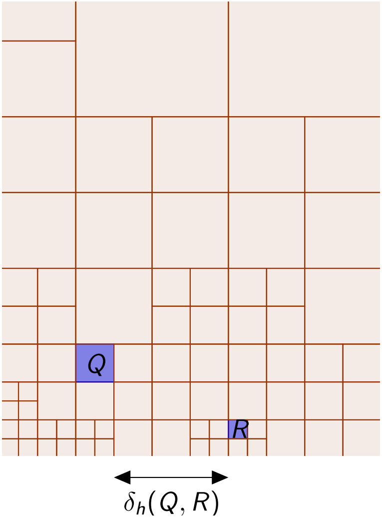
Using this uniform bound on Q, the properties of the covering and some Hölder inequalities, we bound

$$③ \lesssim \|\nabla^n f\|_{L^p(\mathcal{Q})}^p.$$

Proof

Internal structure of the cubes family



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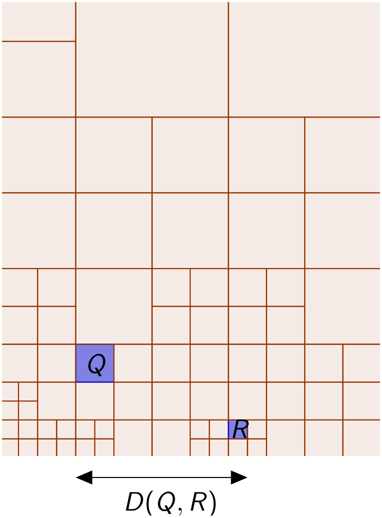

- We can define the long distance $D(Q,R) = \delta_h(Q,R) + \ell(Q) + \ell(R)$.

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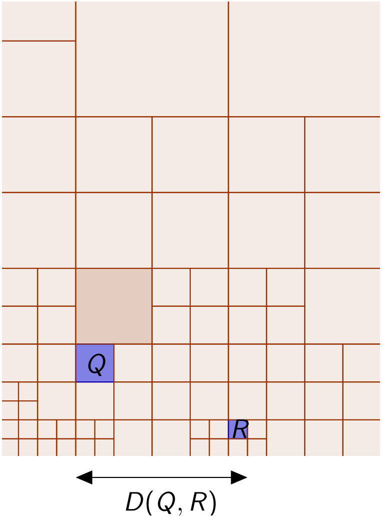

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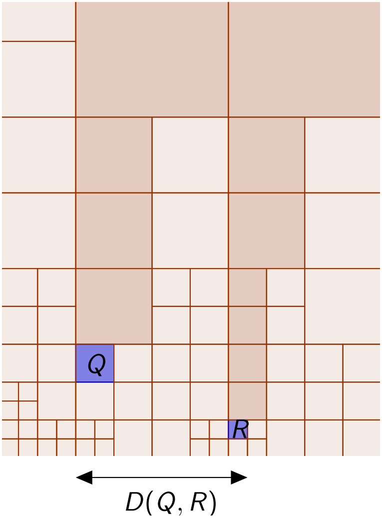

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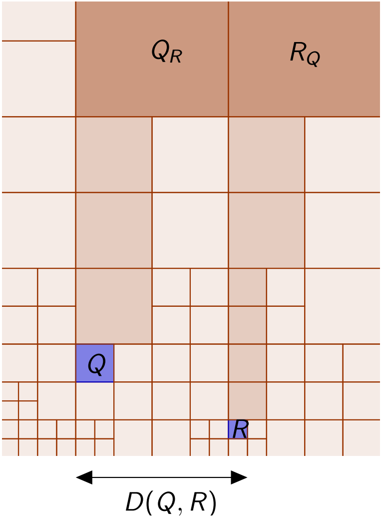

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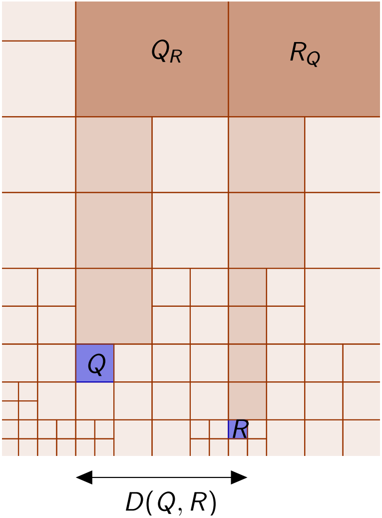

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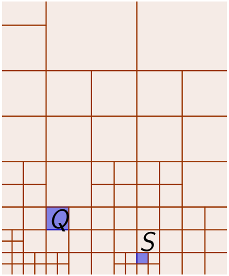
Sketch of the boundedness of the non-local part ④



- The difference between polynomials of distant cubes can be huge.

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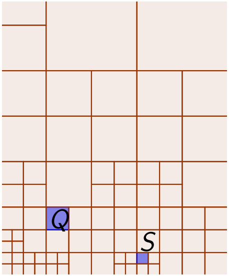

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$$\left|B^{(-\alpha)}\left[(\mathfrak{P}^n_S f - \mathfrak{P}^n_Q f)\psi_{QS}\right](z)\right| \lesssim \int_{2S} \frac{|\mathfrak{P}^n_S f(w) - \mathfrak{P}^n_Q f(w)|}{D(Q,S)^{n+2}} dm(w)$$

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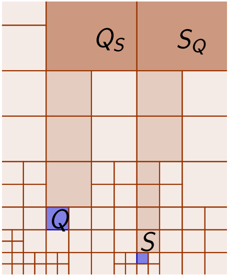

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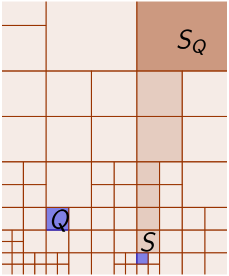

- The difference between polynomials of distant cubes can be huge.
- We take a tour changing between neighbor cubes.
- We apply the property P3 to one branch to illustrate.

$$\left|B^{(-\alpha)}\left[(\mathfrak{p}^n_S f - \mathfrak{p}^n_{S_Q} f)\psi_{QS}\right](z)\right| \lesssim \int_{2S} \frac{|\mathfrak{p}^n_S f(w) - \mathfrak{p}^n_{S_Q} f(w)|}{D(Q,S)^{n+2}} dm(w)$$

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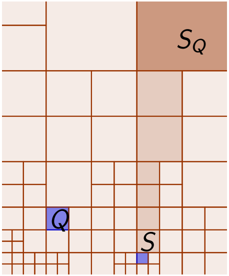

- The difference between polynomials of distant cubes can be huge.
- We take a tour changing between neighbor cubes.
- We apply the property P3 to one branch to illustrate.
- Hölder inequalities and the propeties of the Whitney covering will do the rest.

▸ Details

$$④^* \lesssim \sum_{P\in\mathcal{W}} \|\nabla^n f\|^p_{L^p(5P)} \ell(P)^{p-\frac{5}{2}} \sum_{Q\in\mathcal{W}} \frac{\ell(Q)^{\frac{3}{2}}}{D(Q,P)^{2p-\frac{1}{2}}} \sum_{S<P} \ell(S)^{p+\frac{1}{2}}$$

$$\lesssim \|\nabla^n f\|^p_{L^p(\mathcal{Q}\cap\Omega)}.$$

▸ Back

□

Bounding ③

The uniform bound in every square leads to

$$
③ \lesssim \sum_{Q\in\mathcal{W}} \left(\sum_{S\in\mathcal{W}} \frac{\ell(Q)^{\frac{2}{p}}\ell(S)^{n+\frac{2}{p'}}}{D(Q,S)^{n+2}} \|\nabla^n f\|_{L^p(3S)} \right)^p
$$

and, applying the Hölder inequality,

$$
③ \lesssim \sum_{Q\in\mathcal{W}} \sum_{S\in\mathcal{W}} \frac{\ell(Q)^2\ell(S)^{np}}{D(Q,S)^{\frac{3}{2}+np}} \|\nabla^n f\|^p_{L^p(3S)} \left(\sum_{S\in\mathcal{W}} \frac{\ell(S)^2}{D(Q,S)^{2+\frac{p'}{2p}}} \right)^{\frac{p}{p'}}
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Lemma

Let $b > a > 1$. Then,

$$\sum_{R\in\mathcal{W}} \frac{\ell(R)^a}{D(Q,R)^b} \leq C_{a,b} \ell(Q)^{a-b}.$$

Bounding ③

$$③ \leq C_p \sum_{S \in \mathcal{W}} \ell(S)^{np} \|\nabla^n f\|^p_{L^p(3S)} \sum_{Q \in \mathcal{W}} \frac{\ell(Q)^2}{D(Q,S)^{\frac{3}{2}+np}} \ell(Q)^{-\frac{1}{2}}$$

As $\frac{3}{2} + np > \frac{3}{2} > 1$, we can use the previous lemma again to get

$$③ \leq C_{n,p} \sum_{S \in \mathcal{W}} \ell(S)^{np} \|\nabla^n f\|^p_{L^p(3S)} \ell(S)^{-np}$$
$$\lesssim \|\nabla^n f\|^p_{L^p(\mathcal{Q} \cap \Omega)}$$

▸ Back

Bounding (4)

We have

$$\textcircled{4*_{QS}} = |B^{(-\alpha)}\left[(\mathfrak{p}^n_S f - \mathfrak{p}^n_{S_Q} f)\psi_{QS}\right](z)|$$

$$\lesssim \sum_{S<P\leq S_Q} \ell(S)^2 \frac{D(P,S)^{n-1}}{D(Q,S)^{n+2}} \ell(P)^{1-\frac{2}{p}} \|\nabla^n f\|_{L^p(5P)}.$$

On the other hand, as $S < P \leq S_Q$, we have

$$D(P,S) \approx \ell(P) \leq \ell(S_Q) \approx D(Q,S)$$

and

$$D(Q,S) \approx \ell(S_Q) = \ell(P_Q) \approx D(Q,P).$$

Thus,

$$\textcircled{4*_Q} \leq C \left(\sum_{S\in\mathcal{W}} \sum_{S<P\leq S_Q} \frac{\ell(S)^2 \ell(P)^{1-\frac{2}{p}} \|\nabla^n f\|_{L^p(5P)}}{D(Q,P)^3} \right)^p .$$

Bounding ④

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Using Hölder inequality, we get

$$\textcircled{4*_Q} \lesssim \left(\sum_S \left(\sum_P \frac{\ell(S)^{2p}\ell(P)^{p-\frac{5}{2}} \|\nabla^n f\|^p_{L^p(5P)}}{D(Q,P)^{3p}} \right)^{\frac{1}{p}} \left(\sum_P \ell(P)^{\frac{p'}{2p}} \right)^{\frac{1}{p'}} \right)^p.$$

Bounding ④

$$\text{④}_{QS} = |B^{(-\alpha)}\left[(\mathfrak{P}^n_S f - \mathfrak{P}^n_{S_Q} f)\psi_{QS}\right](z)|.$$

The sum $\sum_{S<P\leq S_Q} \ell(P)^{\frac{p'}{2p}}$ is geometric. Thus

$$\left(\sum_{S<P\leq S_Q} \ell(P)^{\frac{p'}{2p}}\right)^{\frac{1}{p'}} \lesssim \ell(S_Q)^{\frac{1}{2p}} \approx \frac{\ell(S)^{\frac{1}{2p}-1}}{D(Q,P)^{-\frac{1}{2p}-1}} \frac{\ell(S)^{1-\frac{1}{2p}}}{D(Q,S)}$$

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Then,

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$$\lesssim \left(\sum_S \left(\sum_{S<P\leq S_Q} \frac{\ell(S)^{p+\frac{1}{2}}\ell(P)^{p-\frac{5}{2}}\|\nabla^n f\|^p_{L^p(5P)}}{D(Q,P)^{2p-\frac{1}{2}}}\right)^{\frac{1}{p}} \frac{\ell(S)^{1-\frac{1}{2p}}}{D(Q,S)}\right)^p$$

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Then, we apply Hölder inequality

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$$\lesssim \sum_S \sum_{S<P\le S_Q} \frac{\ell(S)^{p+\frac{1}{2}}\ell(P)^{p-\frac{5}{2}}\|\nabla^n f\|^p_{L^p(5P)}}{D(Q,P)^{2p-\frac{1}{2}}} \left(\sum_S \frac{\ell(S)^{p'-\frac{p'}{2p}}}{D(Q,S)^{p'}}\right)^{\frac{p}{p'}}$$

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Then, we apply Hölder inequality and the properties of the covering,

$$\textcircled{4*_Q} \lesssim \sum_S \sum_{S<P\leq S_Q} \frac{\ell(S)^{p+\frac{1}{2}}\ell(P)^{p-\frac{5}{2}}\|\nabla^n f\|^p_{L^p(5P)}}{D(Q,P)^{2p-\frac{1}{2}}} \left(\sum_S \frac{\ell(S)^{p'-\frac{p'}{2p}}}{D(Q,S)^{p'}}\right)^{\frac{p}{p'}}$$

$$\lesssim \sum_{S\in\mathcal{W}} \sum_{S<P\leq S_Q} \frac{\ell(S)^{p+\frac{1}{2}}\ell(P)^{p-\frac{5}{2}}}{D(Q,P)^{2p-\frac{1}{2}}\ell(Q)^{\frac{1}{2}}} \|\nabla^n f\|^p_{L^p(5P)}.$$

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Bounding ④

$$\textcircled{4*} = \sum_{Q\in\mathcal{W}} \left(\sum_{S\in\mathcal{W}} \left\| B^{(-\alpha)}(\psi_{QS}(\mathfrak{P}_S^n f - \mathfrak{P}_{S_Q}^n f)) \right\|_{L^p(Q)} \right)^p$$

Then,

$$\textcircled{4*_Q} \lesssim \sum_{S\in\mathcal{W}} \sum_{S<P\leq S_Q} \frac{\ell(S)^{p+\frac{1}{2}} \ell(P)^{p-\frac{5}{2}}}{D(Q,P)^{2p-\frac{1}{2}} \ell(Q)^{\frac{1}{2}}} \|\nabla^n f\|_{L^p(5P)}^p.$$

Summing with respect to Q, we have

$$\textcircled{4*} \lesssim \sum_{Q\in\mathcal{W}} \ell(Q)^2 \sum_{S\in\mathcal{W}} \sum_{S<P\leq S_Q} \frac{\ell(S)^{p+\frac{1}{2}} \ell(P)^{p-\frac{5}{2}}}{D(Q,P)^{2p-\frac{1}{2}} \ell(Q)^{\frac{1}{2}}} \|\nabla^n f\|_{L^p(5P)}^p$$

$$\leq \sum_{P\in\mathcal{W}} \|\nabla^n f\|_{L^p(5P)}^p \ell(P)^{p-\frac{5}{2}} \sum_{Q\in\mathcal{W}} \frac{\ell(Q)^{\frac{3}{2}}}{D(Q,P)^{2p-\frac{1}{2}}} \sum_{S<P} \ell(S)^{p+\frac{1}{2}}.$$

Bounding $\textcircled{4}$

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We have found out

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Now, being $p+\frac{1}{2} > 1$ and $2p-\frac{1}{2} > \frac{3}{2} > 1$ imply

$$\sum \ell(S)^{p+\frac{1}{2}} \lesssim \ell(P)^{p+\frac{1}{2}} \quad \text{and} \quad \sum \frac{\ell(Q)^{\frac{3}{2}}}{D(Q,P)^{2p-\frac{1}{2}}} \lesssim \ell(P)^{-2(p-1)}$$

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so

$$④^* \lesssim \sum_{P\in\mathcal{W}} \ell(P)^{p-\frac{5}{2}-2p+2+p+\frac{1}{2}} \|\nabla^n f\|^p_{L^p(5P)} \lesssim \|\nabla^n f\|^p_{L^p(\mathcal{Q}\cap\Omega)}.$$