THE STRUCTURE OF COUNTABLY GENERATED PROJECTIVE MODULES OVER REGULAR RINGS

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Abstract. We prove that for every regular ring R there exists a monoid isomorphism between $V(\mathrm{End}_R(\aleph_0 R_R))$ and $V(\mathcal{M}(FM(R)))$. We use this result to give a precise description of the countably generated projective modules over simple regular rings and over regular rings satisfying s-comparability.

Introduction.

One of the most relevant topics in the theory of von Neumann regular rings is the study of the finitely generated projective modules. This study is usually done by using stable and non-stable K-theory. The arbitrary projective modules over a regular ring have also been object of interest, see for example [**Kad**, **Ku1**, **Ku2**, **Ku3**]. A fundamental result for this study is the fact that a projective module over a regular ring R satisfies the exchange property, and so it is a direct sum of cyclic projective modules, see [**Os**] and [**Sto**]. It follows that the ring $\mathrm{End}(P_R)$ is an exchange ring in the sense of Warfield [**War**]. If we concentrate attention on the countably generated projective R-modules, then we must consider the ring $\mathrm{End}(\aleph_0 R_R)$, since the category of finitely generated projective modules over it is equivalent to the category of countably generated projective R-modules. Recent results on the structure and K-theory of exchange rings (e.g. [**Aex**, **AGOP**, **Par**]) can then be applied.

On the other side/hand a detailed study of the structure of the multiplier rings of σ-unital (non-unital) regular rings has been done in [**AP**]. The multiplier ring of the (non-unital) regular ring $FM(R)$ of countably infinite matrices over R having only a finite number of nonzero entries is the ring of row- and column-finite matrices over R [**AP**, Proposition 1.1], which is related to the ring $\mathrm{End}(\aleph_0 R_R)$, being the latter the ring of column-finite matrices over R. Although important differences are detected between $\mathcal{M}(FM(R))$ and $\mathrm{End}(\aleph_0 R_R)$, we are able to prove that their respective monoids of isomorphism classes of finitely generated projective modules are isomorphic (Theorem 1.3). By using this, we apply the results in [**AP**] to obtain explicit information on the countably generated projective R-modules.

More accurate results can be obtained for particular classes of regular rings. In particular, we consider simple regular rings and regular rings satisfying s-comparability for

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some positive integer s. We see that the structure of the countably generated projective modules over a simple regular ring R depends heavily on the compact convex set of pseudo-rank functions on R. Moreover, we study how the known theory of comparison of finitely generated projective modules over a regular ring satisfying s-comparability extends to the countably generated ones, with respect to the relations $\lesssim$ and $\lesssim^{\oplus}$. We also extend results of Kutami [**Ku3**] concerning the behaviour of directly finite projective modules over regular rings satisfying s-comparability.

In outline the paper is as follows. In Section 1, we recall the necessary definitions, and we prove that for any regular ring R there exists a monoid isomorphism between $V(\mathrm{End}_R(\aleph_0 R_R))$ and $V(\mathcal{M}(FM(R)))$. Section 2 is devoted to the study of simple regular rings and regular rings with s-comparability for some positive integer s. In particular, we obtain our main result on the comparison theory for countably generated projective modules over regular rings satisfying s-comparability. In Section 3 we study the relations between the ideals of $\mathrm{End}(\aleph_0 R_R)$ and $\mathcal{M}(FM(R))$ for a regular ring R, and we obtain finer results in the case where R satisfies s-comparability. Finally, we deal in Section 4 with property (DF), which was introduced by Kutami in [**Ku3**]. We show that every regular ring with s-comparability satisfies property (DF), i.e., the class of directly finite projective R-modules is closed under finite direct sums. This extends a result of Kutami, who proved the same result under the additional hypothesis that R is unit-regular

1. Countably generated projective modules and intervals.

We start by fixing some notation and terminology. Throughout, R will denote a unital von Neumann regular ring (see [**vnrr**] for definitions and properties on this class of rings). For a ring T, let $\mathcal{C}_T$ denote the category of countably generated projective right T-modules, and let $FP(T)$ denote the category of finitely generated projective right T-modules. We denote by $V(T)$ the monoid of isomorphism classes of finitely generated projective right R-modules. In the sequel, for A, B arbitrary T-modules, we use $A \lesssim B$ to denote "A is isomorphic to a submodule of B", $A \lesssim^{\oplus} B$ to denote "A is isomorphic to a direct summand of B", $A \prec B$ to denote "A is isomorphic to a proper submodule of B", and $A \prec^{\oplus} B$ to denote "A is isomorphic to a proper direct summand of B". We will use the following important fact on regular rings [**vnrr**, Theorem 1.11]: If A is a projective right module over a regular ring R and $B \in FP(R)$, then $B \lesssim A$ if and only if $B \lesssim^{\oplus} A$.

For a cardinal number κ and a right T-module A, we will denote by κA the direct sum of κ copies of A.

Let $E = FCM(R)$ be the ring of column-finite matrices over R (of countably infinite size). It is well-known that $E = \mathrm{End}(\aleph_0 R_R)$. Let $B = FRCM(R)$ be the subring of E consisting of matrices in E with only finitely many nonzero entries in each row. There are ideals of B and E which play an important role. These are defined as follows:

$$F = FM(R) = \{x \in B \mid x(\aleph_0 R_R) \subseteq nR_R \quad \text{for some } n\},$$

$$G = FR(R) = \{x \in E \mid x(\aleph_0 R_R) \subseteq nR_R \quad \text{for some } n\}.$$

Note that F consists of the matrices in B with only finitely many nonzero entries, while G consists of the matrices in E with only finitely many nonzero rows. Of course, $FM(R) =$

$\varinjlim M_n(R)$, where the (non-unital) embeddings $M_n(R) \to M_{n+1}(R)$ are given by $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$.

We will use repeteadly the following well-known lemma:

Lemma 1.1. *Let E, B, F and G be as defined above. Then F is an ideal of B, G is an ideal of E, and $G = FE$ and $F = EF$.* $\square$

The ring B is the multiplier ring of the non-unital ring F, that is, B is the biggest unital ring containing F as an essential ideal [**AP**, Proposition 1.1]. Similarly, the ring E is the multiplier ring of G. However there are remarkable differences between the pairs (F, B) and (G, E). First of all, since all the matrix rings $M_n(R)$ over a regular ring R are regular rings [**vnrr**, Theorem 1.7], we see that F is a (nonunital) regular ring. This is not the case for G in general. For, let R be a regular ring which is not artinian. By [**vnrr**, Corollary 2.16], there exists a sequence (e_n) of nonzero orthogonal idempotents in R. Let X be the matrix in G such that all rows but the first are 0, and the first row of X is $[e_1, e_2, \dots]$. Then it is easy to see that X is not a von Neumann regular element. A second difference comes from the notion of σ-unital rings. A semiprime ring I is said to be σ-unital in case there is a sequence (x_n) which converges strictly to 1 in the multiplier ring $\mathcal{M}(I)$, see [AP] and [**Lo**, p.14]. The (nonunital) ring F is σ-unital, being $(\mathrm{diag}(1, \dots, 1, 0, 0, \dots))$ a σ-unit, but G is not σ-unital. The theory developed in [**AP**] works for multiplier rings of regular σ-unital rings, hence that theory encompases the ring B but not the ring E. Another important difference between B and E will be pointed out in Section 3. In spite of all these differences, the monoids $V(B)$ and $V(E)$ are isomorphic (Theorem 1.3).

If T is a ring and M_T a module, it is well-known that there is a categorical equivalence between the category $FP(\mathrm{End}(M_T))$ and the category $\mathrm{Add}(M_T)$ of T-modules which are direct summands of nM_T for some n. Since $\mathcal{C}_T = \mathrm{Add}(\aleph_0 T_T)$, we obtain the following:

Proposition 1.2. *Let T be a ring. Then there is a categorical equivalence between the category $\mathcal{C}_T$ of countably generated projective right T-modules and the category of finitely generated projective right modules over the ring End$(\aleph_0 T_T)$.*

Let $V(\mathcal{C}_T)$ be the set of isomorphism classes of objects from $\mathcal{C}_T$. Then $V(\mathcal{C}_T)$ has a natural structure of abelian monoid, induced by the direct sum of projective modules. By Proposition 1.2, we have $V(\mathcal{C}_T) \cong V(\mathrm{End}(\aleph_0 T_T))$. For $e = e^2 \in \mathrm{End}(\aleph_0 T_T)$, the class $[e\mathrm{End}(\aleph_0 T_T)]$ corresponds under this isomorphism to the class of the countably generated projective T-module $e(\aleph_0 T_T)$.

Let R be a regular ring. As before, set $E = FCM(R) = \mathrm{End}(\aleph_0 R_R)$ and $B = FRCM(R)$. Next, we will prove that $V(B)$ and $V(E)$ are isomorphic monoids. This contrasts with the fact that the categories $FP(E)$ and $FP(B)$ are never equivalent. For, if $FP(E)$ and $FP(B)$ were equivalent categories, then it is easy to check that E and B would be Morita-equivalent, which would contradict [**HRS**, Theorem 8].

Theorem 1.3. *Let R be a regular ring. Then the natural inclusion $i : B \to E$ induces a monoid isomorphism $V(i) : V(B) \to V(E)$.*

Proof. We shall use the idempotent picture of $V(-)$, see [**Ros**], so that, for a ring T, the monoid $V(T)$ is identified with the monoid of equivalence classes of idempotents in

$FM(T)$. Since $E_E \cong 2E_E$ and $B_B \cong 2B_B$, we see that every element in $V(E)$ (respectively, $V(B)$) is represented by an idempotent of E (respectively, B). Now the map $V(i)$ is defined by $V(i)([p]_B) = [p]_E$ for every idempotent $p \in B$. We first prove that $V(i)$ is surjective. Let p be an idempotent in E, and consider $P = p(\aleph_0 R_R)$. Then P is a countably generated projective module over the regular ring R, and so P has the exchange property. Consequently $P \cong \oplus_{n=1}^{\infty} e_n R$ for some idempotents $e_n \in R$. Now consider the following idempotent in B

$$q = \operatorname{diag}(e_1, e_2, e_3, \dots).$$

Since $p(\aleph_0 R_R) \cong q(\aleph_0 R_R)$, we have $p \sim_E q$, and so $V(i)([q]_B) = [p]_E$, proving the surjectivity of $V(i)$.

Now we will prove injectivity of $V(i)$. Let p, q be two idempotents in B such that $p \sim_E q$. We need to prove that $p \sim_B q$. Let F be the ideal of B consisting of the matrices with only a finite number of nonzero entries. Recall that B is the multiplier ring of F, and F is obviously a non-unital regular ring. Similarly, E is the multiplier ring of its ideal G, the subring of E consisting in the matrices with just a finite number of nonzero rows. By [**AP**, Lemma 2.1], there exist increasing sequences $(e_n)_{n\geq 1}$, $(f_n)_{n\geq 1}$ of idempotents in F, with $e_n \in pFp$ and $f_n \in qFq$, such that e_n converges to p and f_n converges to q in the strict topology. For $n \geq 1$, set $g_n = e_n - e_{n-1}$ and $h_n = f_n - f_{n-1}$ (here $e_0 = f_0 = 0$). Since p and q are equivalent in E, there exist $x \in pEq$ and $y \in qEp$ such that $p = xy$ and $q = yx$.

Put $g_1' = yg_1x$, and note that $g_1' \in G$. Since f_n converges strictly to q and $yg_1 \in EF = F$ there exists $n \geq 1$ such that $f_n(yg_1) = q(yg_1) = yg_1$. Consequently, $f_n g_1' = f_n(yg_1)x = (yg_1)x = g_1'$. Changing notation, we can assume that $n = 1$, so that $h_1 g_1' = f_1 g_1' = g_1'$. Write $g_1'' = g_1' h_1 \in EF = F$. Then $x_1 := g_1 x g_1'' \in F$ and $y_1 := g_1' y g_1 \in F$, and we have

$$x_1 y_1 = g_1 x g_1'' g_1' y g_1 = g_1 x g_1' y g_1 = g_1,$$

and $y_1 x_1 = g_1' g_1'' = g_1''$. Moreover $p - g_1 = x_1' y_1'$ and $q - g_1'' = y_1' x_1'$, where $x_1' = (1-g_1)x(1-g_1')$ and $y_1' = (1-g_1'')y(1-g_1)$. So $g_1, g_1'' \in F$, $g_1 \sim_B g_1''$ and $p - g_1 \sim_E q - g_1''$. Observe that $g_1'' \leq h_1$. Write $h_1' = x_1'(h_1 - g_1'')y_1' \leq p - g_1$. Since $e_2 - g_1, e_3 - g_1, e_4 - g_1, \dots$ converges in the strict topology induced by F to $p - g_1$, there exists $n \geq 2$ such that $(e_n - g_1)h_1' = h_1'$. Changing notation, we can assume that $n = 2$, so that $g_2 h_1' = (e_2 - g_1)h_1' = h_1'$. Set $h_1'' = h_1' g_2 \in EF = F$. By using the same argument as before, we have $h_1'', h_1 - g_1'' \in F$, $h_1'' \sim_B h_1 - g_1''$ and $p - (h_1'' + g_1) \sim_E q - h_1$. Observe that $h_1'' \leq g_2$. Continuing this process, we get sequences of idempotents in F

$$\begin{array}{lllll} g_1'' \leq h_1, & g_2'' \leq h_2, & \dots, & g_n'' \leq h_n, & \dots \\ h_1'' \leq g_2, & h_2'' \leq g_3, & \dots, & h_n'' \leq g_{n+1}, & \dots \end{array}$$

such that $g_1 \sim_B g_1''$, and $g_{n+1} - h_n'' \sim_B g_{n+1}''$ and $h_n - g_n'' \sim_B h_n''$ for all $n \geq 1$. Since $p = g_1 + \sum_{n=1}^{\infty}(g_{n+1} - h_n'') + \sum_{n=1}^{\infty} h_n''$ and $q = \sum_{n=1}^{\infty} g_n'' + \sum_{n=1}^{\infty}(h_n - g_n'')$, we obtain from [**AP**, Lemma 1.6] that $p \sim_B q$, as desired.

This proves that $V(i)$ is injective and so we conclude that it is a monoid isomorphism from $V(B)$ onto $V(E)$. $\square$

Recall that a ring R is said to be *unit-regular* if for each $x \in R$ there is a unit $u \in R$ such that $x = xux$. The unit-regular rings are exactly the regular rings with stable rank one [**vnrr**, Proposition 4.12]. Also, we infer from [**vnrr**, Theorem 4.5] that a regular ring R is unit-regular if and only if $V(R)$ is a cancellative monoid. By combining Theorem 1.3 with the results in [**AP**] we will obtain a description of $V(\mathcal{C}_R)$ in terms of intervals in $V(R)$ for any unit-regular ring R. To this end we recall the definition of an interval in a monoid M.

Definition. *Let M be an abelian monoid. An interval in M is a nonempty, hereditary, upward directed subset I of M. An interval I in a monoid M is said to be countably generated provided that I has a countable cofinal subset.*

Intervals have been extensively used in the theory of multiplier C^*-algebras, e.g. [**gKth**], [**GH**], [**Per**], and recently in the study of multiplier rings of regular rings [**AP**].

Given an abelian monoid M we denote by $\Lambda_\sigma(M)$ the abelian monoid of countably generated intervals in M, with the sum defined by

$$X + Y = \{z \in M \mid z \leq x + y \text{ for some } x \in X \text{ and some } y \in Y\},$$

where $X, Y \in \Lambda_\sigma(M)$.

Theorem 1.4. *Let R be a unit-regular ring and let $\mathcal{C}_R$ be the category of countably generated projective right R-modules. Then there is a monoid isomorphism $\Phi : V(\mathcal{C}_R) \to \Lambda_\sigma(V(R))$ such that $\Phi([P])$ is the interval in $V(R)$ generated by the increasing sequence $\{[e_1 \oplus e_2 \oplus \cdots \oplus e_n] \mid n = 1, 2, \dots\}$, for any $P \in \mathcal{C}_R$ and any decomposition $P \cong \oplus_{i=1}^\infty e_i R$ with $e_i = e_i^2 \in R$.*

Proof. Set $E = FCM(R) = \mathrm{End}(\aleph_0 R_R)$, and $B = FRCM(R)$, and recall that B is the multiplier ring of $FM(R)$ [**AP**, Proposition 1.1]. By Proposition 1.2 and Theorem 1.3 we have a monoid isomorphism $\tau : V(\mathcal{C}_R) \to V(B)$. This isomorphism sends $[P]$ to the class in $V(B)$ of the idempotent $e = \mathrm{diag}\,(e_1, e_2, \dots) \in B$, where $P \cong \oplus_{i=1}^\infty e_i R$ and e_i are idempotents in R. By [**AP**, Theorem 2.7], there is a monoid isomorphism $\mu : V(B) \cong \Lambda_\sigma(V(R))$ which sends $[p] \in V(B)$ to the interval in $V(R)$ generated by $\{[p_n]\}$, where $(p_n)_{n\geq 1}$ is an approximate unit for $pFM(R)p$ consisting of idempotents. Define $\Phi = \mu \circ \tau$. Then Φ is a monoid isomorphism from $V(\mathcal{C}_R)$ onto $V(B)$. Now, let $P \in \mathcal{C}_R$ and let $P \cong \oplus_{i=1}^\infty e_i R$ with e_i idempotents in R, and write $e = \mathrm{diag}\,(e_1, e_2, \dots)$. Since $\mathrm{diag}\,(e_1, e_2, \dots, e_n, 0, 0 \dots)$ is an approximate unit consisting of idempotents for $eFM(R)e$, it follows from the above description that $\Phi([P])$ is the interval of $\Lambda_\sigma(V(R))$ generated by the countable set $\{[e_1 \oplus \cdots \oplus e_n] \mid n \geq 1\}$. $\square$

Proposition 1.5. *Let R be a unit-regular ring, and let $\Phi : V(\mathcal{C}_R) \to \Lambda_\sigma(V(R))$ be the natural isomorphism. Let P and Q be countably generated projective R-modules. Then*

(a) *$P \lesssim Q$ if and only if $\Phi([P]) \subseteq \Phi([Q])$.*

(b) *$P \lesssim^{\oplus} Q$ if and only if there is $Z \in \Lambda_\sigma(V(R))$ such that $\Phi([P]) + Z = \Phi([Q])$.*

Proof. (a) Write $P \cong \oplus_{i=1}^\infty P_i$ and $Q \cong \oplus_{i=1}^\infty Q_i$, for $P_i, Q_i \in FP(R)$. By [**vnrr**, Proposition 4.8], we have $P \lesssim Q$ if and only if $P_1 \oplus \cdots \oplus P_n \lesssim Q$ for all $n \geq 1$. Since each P_i is finitely generated, this holds if and only if for each $n \geq 1$ there exists $m \geq 1$ such that

$P_1 \oplus \cdots \oplus P_n \lesssim Q_1 \oplus \cdots \oplus Q_m$. By the description of Φ in Theorem 1.4, the latter statement holds if and only if $\Phi([P]) \subseteq \Phi([Q])$.

(b) This is clear from Theorem 1.4. $\square$

Remark 1.6. *If M is any abelian monoid, the algebraic pre-order on M is defined by the rule $x \leq y$ iff there is $z \in M$ such that $x + z = y$. Note that Proposition 1.5(b) says that the algebraic pre-order on $\Lambda_\sigma(V(R))$ corresponds to the pre-order relation $\lesssim^\oplus$ on $\mathcal{C}_R$. Similarly, Proposition 1.5(a) says that the order induced by the inclusion of intervals corresponds to the relation $\lesssim$ on $\mathcal{C}_R$.*

Say that an abelian pre-ordered monoid $(M, \leq)$ is unperforated in case $nx \leq ny$ implies $x \leq y$ for all $n \geq 1$ and all $x, y \in M$. The following lemma is well-known. Although stated for Riesz groups in [**gKth**], the translation to cancellative Riesz monoids is immediate.

Lemma 1.7. [**gKth**, Lemma 2.3] *Let M be a cancellative Riesz monoid.*

(a) *If M is unperforated, then $\Lambda_\sigma(M)$ is unperforated for the algebraic pre-order, and also for the order given by set inclusion.*

(b) *If $nx = ny$ implies $x = y$ for all $n \geq 1$ and all $x, y \in M$, then $nX = nY$ implies $X = Y$ for all $n \geq 1$ and $X, Y \in \Lambda_\sigma(M)$.*

As an immediate consequence of Proposition 1.5 and Lemma 1.7 we obtain

Proposition 1.8. *Let R be a unit-regular ring.*

(a) *If $V(R)$ is unperforated, then $(V(\mathcal{C}_R), [\lesssim])$ and $(V(\mathcal{C}_R), [\lesssim^\oplus])$ are also unperforated.*

(b) *Assume that $nP \cong nQ$ implies $P \cong Q$ for all $n \geq 1$ and $P, Q \in FP(R)$. Then the same property holds for all $P, Q \in \mathcal{C}_R$.* $\square$

An immediate consequence of Proposition 1.8 is that, in case R is a regular ring satisfying hypothesis (b), both E and B satisfies a weak cancellation property, called separativity, which has been proved to be very useful in the study of some questions on exchange rings (see [**AGOP**]).

Recall that a ring T is *separative* provided the following cancellation property holds for finitely generated projective right (equivalently, left) T-modules A and B:

$$A \oplus A \cong A \oplus B \cong B \oplus B \quad \Longrightarrow \quad A \cong B.$$

See [**AGOP**] for the origin of this terminology and for a number of equivalent conditions.

Corollary 1.9. *Let R be a unit-regular ring. If $nP \cong nQ$ implies $P \cong Q$ for all $n \geq 1$ and $P, Q \in FP(R)$, then both E and B are separative rings.* $\square$

Proof. The result follows from Proposition 1.8(b), Proposition 1.2 and Theorem 1.3. $\square$

2. Simple rings and rings with s-comparability.

In this section we will give a precise description of the structure of the countably generated projective modules over some special types of regular rings. Our approach is based on the reduction to the simple case.

Let s be a positive integer. A regular ring is said to satisfy *s-comparability* in case for every $x, y \in R$, either $xR \lesssim s(yR)$ or $yR \lesssim s(xR)$. Directly finite regular rings with

s-comparability are not always unit-regular [**AOT**, Example 3.2], but so are in the simple case, by a result of O'Meara [**OM**, Corollary 2].

The following facts will be used repeteadly, see [**AOT**]. Let R be a nonzero regular ring satisfying s-comparability. Then the lattice $L(R)$ of two-sided ideals of R is totally ordered. In particular there exists a unique maximal ideal M. If, in addition, R is directly finite, then R/M is a simple directly finite regular ring satisfying s-comparability, and so it is unit-regular by O'Meara's result, see [**AOT**, Corollary 2.7].

Say that R is *strictly unperforated* in case $nP \prec nQ$ implies $P \prec Q$ for all $n \geq 1$ and $P, Q \in FP(R)$.

Let $\mathbb{P}(R)$ be the compact convex set of pseudo-rank functions defined on R [**vnrr**, Chapter 16]. As for pseudo-rank functions on non-unital regular rings, we will follow the conventions used in [**AP**].

Let K be a compact convex set. We denote by $\mathrm{LAff}(K)$ the monoid of all affine and lower semicontinuous functions on K with values on $\mathbb{R} \cup \{+\infty\}$. Let $\mathrm{LAff}_\sigma(K)$ denote the submonoid of $\mathrm{LAff}(K)$ whose elements are pointwise suprema of increasing sequences of affine real-valued continuous functions on K. The semigroup of strictly positive elements in $\mathrm{LAff}_\sigma(K)$ will be denoted by $\mathrm{LAff}_\sigma(K)^{++}$.

Theorem 2.1. *Let R be a simple, nonartinian, strictly unperforated, unit-regular ring. Then there exists a monoid isomorphism $\mu : V(\mathcal{C}_R) \to V(R) \sqcup \mathrm{LAff}_\sigma(\mathbb{P}(R))^{++}$. This isomorphism is the identity on $V(R)$ and it is given by the formula*

$$\mu([P])(N) = sup\{N(e_1) + \cdots + N(e_n) \mid n \geq 1\},$$

where $[P] \in V(\mathcal{C}_R) \setminus V(R)$, $N \in \mathbb{P}(R)$ and $P \cong \oplus_{i=1}^{\infty} e_i R$ with $e_i = e_i^2 \in R$.

Proof. By Theorem 1.4, there is a monoid isomorphism $\Phi : V(\mathcal{C}_R) \to \Lambda_\sigma(V(R))$. Write $M = V(FM(R)) = V(R)$, and note that M is a conical simple refinement monoid. Since R is simple and nonartinian, M has no atoms. Furthermore, M is cancellative and strictly unperforated because R is strictly unperforated and unit-regular. So, it follows from [**Per**, Theorem 3.9] that there is a monoid isomorphism $\varphi : \Lambda_\sigma(M) \to M \sqcup \mathrm{LAff}_\sigma(S_u)^{++}$, where S_u is the state space $St(M, u)$ for a given nonzero element $u \in M$. Here, the semigroup operation in $M \sqcup \mathrm{LAff}_\sigma(S_u)^{++}$ is the one obtained by extending the given ones in M and $\mathrm{LAff}_\sigma(S_u)^{++}$, and, for $x \in M$ and $f \in \mathrm{LAff}_\sigma(S_u)^{++}$, by putting $x + f = \phi_u(x) + f$, where $\phi_u : M \to \mathrm{Aff}(S_u)$ is the natural representation homomorphism.

Now fix $u = [R_R] \in M$, and note that u is the class of the idempotent $e := \mathrm{diag}\,(1, 0, \dots)$ in $FM(R)$.

Notice that, by [**vnrr**, Proposition 16.8], every $N \in \mathbb{P}(R)$ can be uniquely extended to an unnormalized pseudo-rank function on $FM(R)$, also denoted by N, such that $N(\mathrm{diag}(x_1, \dots, x_n, 0, 0, \dots)) = N(x_1) + \cdots + N(x_n)$. As in [**AP**], we denote by $\mathbb{P}(FM(R))_e$ the compact convex set of all the pseudo-rank functions N on $FM(R)$ such that $N(e) = 1$. By the previous observation, we can identify $\mathbb{P}(FM(R))_e$ with $\mathbb{P}(R)$. Hence, by [**AP**, Proposition 3.4], there is an affine homeomorphism $\alpha : \mathbb{P}(R) \to S_u$ such that $\alpha(N)([f]) = N(f)$ for every $N \in \mathbb{P}(FM(R))_e$ and every idempotent $f \in FM(R)$. This affine homeomorphism induces a monoid isomorphism $M \sqcup \mathrm{LAff}_\sigma(S_u)^{++} \to M \sqcup \mathrm{LAff}_\sigma(\mathbb{P}(R))^{++}$. Composing Φ with this isomorphism we get a monoid isomorphism $\mu : V(\mathcal{C}_R) \to M \sqcup$

$\mathrm{LAff}_\sigma(\mathbb{P}(R))^{++}$. From the description of Φ given in Theorem 1.4 and the description of the above maps we get the desired properties of the map μ. $\square$

Note that if R is a simple artinian ring then $V(\mathcal{C}_R) \cong V(R) \sqcup \{\infty\} = \mathbb{Z} \sqcup \{\infty\}$.

Corollary 2.2. *Let R be a simple regular ring satisfying s-comparability for some $s \geq 1$.*

(a) *If R is either artinian or directly infinite then $V(\mathcal{C}_R) \cong V(R) \sqcup \{\infty\}$, that is, there is a unique $P \in \mathcal{C}_R \setminus FP(R)$ up to isomorphism, namely $P = \aleph_0 R_R$.*

(b) *If R is nonartinian and directly finite, then $V(\mathcal{C}_R) \cong V(R) \sqcup \mathbb{R}^{++} \sqcup \{\infty\}$.*

Proof. (a) The simple artinian case is clear, so assume that R is a simple directly infinite regular ring with s-comparability. By [**Par**, Proposition 1.7(3)], R is purely infinite, that is, $P \prec Q$ for every two nonzero finitely generated projective R-modules. Therefore, we conclude from Proposition 1.2, Theorem 1.3 and [**AP**, Proposition 2.12] that $V(\mathcal{C}_R) \cong V(R) \sqcup \{\infty\}$.

(b) Assume now that R is a nonartinian, directly finite, simple regular ring satisfying s-comparability. By [**OM**, Corollary 2], R is unit-regular. By [**AGPT**, Corollary 4.5], R has a unique pseudo-rank function and is strictly unperforated. Since $\mathbb{P}(R)$ is a singleton, we have $\mathrm{LAff}_\sigma(\mathbb{P}(R))^{++} = \mathbb{R}^{++} \sqcup \{\infty\}$. So, Theorem 2.1 gives $V(\mathcal{C}_R) \cong V(R) \sqcup \mathbb{R}^{++} \sqcup \{\infty\}$. $\square$

Remark 2.3. *Note that in Corollary 2.2(b) the unique (up to isomorphism) directly infinite module that appears is that corresponding to $\aleph_0 R_R$.*

For a right R-module X define $\mathrm{tr}(X) = \sum_{f \in X^*} f(X)$, where $X^* = \mathrm{Hom}(X_R, R_R)$. The set $\mathrm{tr}(X)$ is always a two-sided ideal of R, called the *trace ideal* of X, and it is a principal two-sided ideal in case $X \in FP(R)$ and R is a regular ring. If R is a regular ring with s-comparability, then it is not true in general that R satisfies full comparability, see for example [**vnrr**, Example 18.19]. However, by [**AOT**, Proposition 2.3(b)], two finitely generated projectives P and Q are always comparable provided their trace ideals are different.

In order to proof our main result on comparison for countably generated projective modules, we need some preliminaries.

Definition. *Let R be a regular ring satisfying s-comparability. Let M be the unique maximal ideal of R. For $A, B \in FP(R)$, write $A \prec_M B$ in case $A/AM \prec B/BM$.*

Lemma 2.4. *Let R be a regular ring satisfying s-comparability for some $s \geq 1$. Let M be the unique maximal two-sided ideal of R.*

(1) If $A, B \in FP(R)$ and $A \prec_M B$, then there is $C \in FP(R)$ such that $C \neq CM$ and $B \cong A \oplus C$.

(2) Set $P = \oplus_{i=1}^{\infty} P_i$ and $Q = \oplus_{i=1}^{\infty} Q_i$, for $P_i, Q_i \in FP(R)$. Assume that $P_1 \prec_M Q_1 \prec_M P_1 \oplus P_2 \prec_M Q_1 \oplus Q_2 \prec_M \cdots$ and that $P_i \neq P_i M$ and $Q_i \neq Q_i M$ for all $i \geq 1$. Then $P \cong Q$.

Proof. (1) The proof is the same as that of [**AOT**, Proposition 2.3(c)].

(2) Notice first that the condition $P \neq PM$ for $P \in FP(R)$ means that P is a generator for the category $\mathrm{Mod} - R$.

Since $P_1 \prec_M Q_1$ there exists by (1) a decomposition $Q_1 = Q_1' \oplus Q_1''$ such that $P_1 \cong Q_1'$ and $Q_1'' \neq Q_1'' M$. Further, $Q_1 \prec_M P_1 \oplus P_2$ and so we have an isomorphism $Q_1' \oplus Q_1'' \oplus T_1 \cong$

$P_1 \oplus P_2 \cong Q_1' \oplus P_2$ for some T_1 such that $T_1 \neq T_1M$. Since both $Q_1'' \oplus T_1$ and P_2 are generators, and R is separative by [**Par**, Theorem 2.2], we conclude that $Q_1'' \oplus T_1 \cong P_2$. So we can write $P_2 = P_2' \oplus P_2''$ with $P_2' \cong Q_1''$ and $P_2'' \neq P_2''M$. Since $P_1 \oplus P_2 \prec_M Q_1 \oplus Q_2$, we can find by (1) a decomposition $Q_1 \oplus Q_2 \cong P_1 \oplus P_2 \oplus T_2$ with $T_2 \neq T_2M$. Now note that

$$(Q_1' \oplus Q_1'') \oplus Q_2 \cong Q_1 \oplus Q_2 \cong P_1 \oplus P_2' \oplus P_2'' \oplus T_2 \cong (Q_1' \oplus Q_1'') \oplus P_2'' \oplus T_2.$$

By using again that R is separative, we get $Q_2 \cong P_2'' \oplus T_2$. So we obtain a decomposition $Q_2 = Q_2' \oplus Q_2''$ such that $Q_2' \cong P_2''$ and $Q_2'' \neq Q_2''M$. Continuing in this way we obtain decompositions $P_n = P_n' \oplus P_n''$ and $Q_n = Q_n' \oplus Q_n''$ for all $n \geq 1$ such that $P_1' = 0$ and $Q_n' \cong P_n''$ and $Q_n'' \cong P_{n+1}'$ for all $n \geq 1$. Finally we get

$$P = \oplus_{n=1}^{\infty} P_n = \oplus_{n=1}^{\infty}(P_n' \oplus P_n'') \cong \oplus_{n=1}^{\infty}(Q_n' \oplus Q_n'') = Q,$$

as desired. $\square$

For an ideal I of a ring T, let $FP(I)$ (respectively, $\mathcal{C}_I$) denote the full subcategory of $FP(T)$ (respectively, $\mathcal{C}_T$) whose objects are the finitely generated (respectively, countably generated) projective modules A such that $A = AI$.

Lemma 2.5. *Let R be a regular ring and let e be an idempotent in R. Set $I = ReR$, and let M be an ideal of R such that $M \subseteq I$. Then there are equivalences of categories $\mathcal{C}_I \to \mathcal{C}_{eRe}$ and $\mathcal{C}_{I/M} \to \mathcal{C}_{eRe/eMe}$ such that the following diagram commutes*

$$\begin{array}{ccc} \mathcal{C}_I & \longrightarrow & \mathcal{C}_{eRe} \\ \big\downarrow & & \big\downarrow \\ \mathcal{C}_{I/M} & \longrightarrow & \mathcal{C}_{eRe/eMe}. \end{array}$$

Proof. Write $S = eRe/eMe$. It is well-known that there is a categorical equivalence between $FP(eRe)$ and $FP(I)$, see for example [**AGOP**, Lemma 1.5(c)]. Indeed, the equivalence is given by the functors $(-) \otimes_{eRe} eR$ from $FP(eRe)$ to $FP(I)$ and $(-) \otimes_R Re$ from $FP(I)$ to $FP(eRe)$. Since every projective R-module (respectively, projective eRe-module) is a direct sum of modules in $FP(R)$ (respectively, $FP(eRe)$), the above equivalence extends to an equivalence between the category of countably generated projective R-modules A such that $AI = A$ and the category $\mathcal{C}_{eRe}$. For $P \in \mathcal{C}_I$, we have $(P \otimes_R Re)/(P \otimes_R Re)(eMe) \cong P/PM \otimes_{R/M} (R/M)(e + M)$, so that the stated diagram is commutative. $\square$

Theorem 2.6. *Let R be a regular ring satisfying s-comparability for some $s \geq 1$, and let P and Q be countably generated projective right R-modules.*

(a) *If* $\mathrm{tr}(P) \subset \mathrm{tr}(Q)$ *then* $P \prec Q$.

(b) *If* $\mathrm{tr}(P) = \mathrm{tr}(Q)$ *and* $\mathrm{tr}(P)$ *is not a principal two-sided ideal, then* $P \cong Q$.

(c) *Assume that* $\mathrm{tr}(P) = \mathrm{tr}(Q)$ *is a principal two-sided ideal, and let* M *be the unique maximal ideal of* $\mathrm{tr}(P)$. *Then we have:*

(c1) *If either P/PM or Q/QM is not finitely generated, then P and Q are comparable with respect to $\lesssim$.*

(c2) *If both P/PM and Q/QM are not finitely generated and $P/PM \cong Q/QM$, then $P \cong Q$.*

(c3) *If P/PM and Q/QM are both finitely generated and $P/PM \prec Q/QM$, then $P \prec Q$.*

Proof. Write $P \cong \oplus_{i=1}^{\infty} P_i$ and $Q \cong \oplus_{i=1}^{\infty} Q_i$, where $P_i, Q_i \in FP(R)$.

(a) If $\mathrm{tr}(Q_i) \subseteq \mathrm{tr}(P)$ for all i, then $\mathrm{tr}(Q) = \sum \mathrm{tr}(Q_i) \subseteq \mathrm{tr}(P)$, a contradiction. So there is $i \geq 1$ such that $\mathrm{tr}(P) \subset \mathrm{tr}(Q_i)$. Now by using the technique in [**AOT**, Proposition 2.5(1)(2)] we get $\oplus_{n=1}^{\infty} P_n \prec Q_i$, so that $P \prec Q$.

(b) By using repeteadly the hypothesis that $\mathrm{tr}(P)$ is not principal, we can arrange the decompositions of P and Q in such a way that

$$(*) \quad \mathrm{tr}(P_1 \oplus \cdots \oplus P_n) \subset \mathrm{tr}(Q_1 \oplus \cdots \oplus Q_n) \subset \mathrm{tr}(P_1 \oplus \cdots \oplus P_{n+1})$$

for all $n \geq 1$. Now we will define inductively a sequence of homomorphisms $\varphi_n : P_1 \oplus \cdots \oplus P_n \to Q_1 \oplus \cdots \oplus Q_n$ and $\psi_n : Q_1 \oplus \cdots \oplus Q_n \to P_1 \oplus \cdots \oplus P_{n+1}$ such that $\psi_n \circ \varphi_n = \iota_n$ and $\varphi_{n+1} \circ \psi_n = \varsigma_n$, where $\iota_n : P_1 \oplus \cdots \oplus P_n \to P_1 \oplus \cdots \oplus P_{n+1}$ and $\varsigma_n : Q_1 \oplus \cdots \oplus Q_n \to Q_1 \oplus \cdots \oplus Q_{n+1}$ are the natural inclusion maps. Set $P_0 = Q_0 = 0$ and $\varphi_0 = \psi_0 = 0$. Let $n \geq 0$, and assume we have constructed φ_k and ψ_k for $0 \leq k \leq n$. Write $P_1 \oplus \cdots \oplus P_{n+1} = \psi_n(Q_0 \oplus Q_1 \oplus \cdots Q_n) \oplus P'_{n+1}$. Taking into account (*), we get $\mathrm{tr}(P'_{n+1}) \subseteq \mathrm{tr}(P_1 \oplus \cdots \oplus P_{n+1}) \subset \mathrm{tr}(Q_1 \oplus \cdots \oplus Q_{n+1}) = \mathrm{tr}(Q_{n+1})$. (The latter equality follows from the relation $\mathrm{tr}(Q_1 \oplus \cdots \oplus Q_n) \subset \mathrm{tr}(Q_1 \oplus \cdots \oplus Q_{n+1})$ and comparability of ideals.) Therefore there exists an injective homomorphism $\theta : P'_{n+1} \to Q_{n+1}$. Let $\varphi_{n+1} : P_1 \oplus \cdots \oplus P_{n+1} \to Q_1 \oplus \cdots \oplus Q_{n+1}$ be defined as $\varphi_{n+1}(\psi_n(x_1 + \cdots + x_n) + x'_{n+1}) = x_1 + \cdots + x_n + \theta(x'_{n+1})$, for $x_i \in Q_i$ and $x'_{n+1} \in P'_{n+1}$. Clearly $\varphi_{n+1} \circ \psi_n = \varsigma_n$. The map ψ_{n+1} is defined similarly.

Note that we have $\varphi_{n+1} \circ \iota_n = \varphi_{n+1} \circ \psi_n \circ \varphi_n = \varsigma_n \circ \varphi_n$ and similarly $\psi_{n+1} \circ \varsigma_n = \psi_{n+1} \circ \varphi_{n+1} \circ \psi_n = \iota_{n+1} \circ \psi_n$. We conclude that we can define homomorphisms $\varphi : P \to Q$ and $\psi : Q \to P$ such that $\psi \circ \varphi = \mathrm{id}_P$ and $\varphi \circ \psi = \mathrm{id}_Q$. So $P \cong Q$, as desired.

(c) Assume that $I := \mathrm{tr}(P) = \mathrm{tr}(Q)$ is a principal two-sided ideal of R, and let M be the unique maximal ideal of I. Let $e = e^2 \in I \setminus M$, and note that $I = ReR$. By using Lemma 2.5, we can reduce to the case where $I = R$ and M is the unique maximal ideal of R. Set $S = R/M$, and note that S is a simple regular ring satisfying s-comparability.

(c1) Assume that either P/PM or Q/QM is not finitely generated. Since S is a simple regular ring with s-comparability, we get from Corollary 2.2 that P/PM and Q/QM are comparable. Without loss of generality, we can assume $P/PM \lesssim Q/QM$.

There are now two cases to be considered. Assume first that P/PM is not finitely generated. Clearly $P_1 \oplus \cdots \oplus P_n \prec_M Q$ for all $n \geq 1$, and we can assume that $P_i \neq P_i M$ for all i. By Lemma 2.4(1), there is a decomposition $Q = A_1 \oplus B_1$ such that $A_1 \cong P_1$ and $B_1 \neq B_1 M$. Since $P_1 \oplus P_2 \prec_M Q$, there is by Lemma 2.4(1) an isomorphism $P_1 \oplus P_2 \oplus X_2 \cong Q$, where $X_2 \neq X_2 M$. So we obtain $A_1 \oplus (P_2 \oplus X_2) \cong A_1 \oplus B_1$. Since both $P_2 \oplus X_2$ and B_1 are generators, we can apply separative cancellation [**Par**, Theorem 2.2], to get $P_2 \oplus X_2 \cong B_1$. Therefore, we obtain a decomposition $B_1 = A_2 \oplus B_2$ such that $A_2 \cong P_2$ and $B_2 \cong X_2$. Note that in particular $B_2 \neq B_2 M$. Continuing in this way, we obtain submodules A_n and B_n of Q such that $Q = A_1 \oplus \cdots \oplus A_n \oplus B_n$ for all $n \geq 1$. So we get $P \cong \oplus_{i=1}^{\infty} P_i \cong \oplus_{i=1}^{\infty} A_i \leq Q$. This shows that $P \lesssim Q$, as desired.

Now, assume that P/PM is finitely generated and that Q/QM is infinitely generated. Then there is some $n \geq 1$ such that $P/PM \cong P_1/P_1M \oplus \cdots \oplus P_n/P_nM$, and, since Q/QM is not finitely generated, there is $m \geq 1$ such that $P_1 \oplus \cdots \oplus P_n \prec_M Q_1 \oplus \cdots \oplus Q_m$. By Lemma 2.4(1), we have $Q_1 \oplus \cdots \oplus Q_m \cong P_1 \oplus \cdots \oplus P_n \oplus C$ for some C with $C \neq CM$. Since $\mathrm{tr}(P_{n+1} \oplus \cdots) \subseteq M \subset \mathrm{tr}(C)$ we conclude from (a) that $\oplus_{i=n+1}^{\infty} P_i \prec C$, and therefore $P = (P_1 \oplus \cdots \oplus P_n) \oplus (\oplus_{i=n+1}^{\infty} P_i) \prec (P_1 \oplus \cdots \oplus P_n) \oplus C \cong Q_1 \oplus \cdots \oplus Q_m \lesssim Q$, showing $P \prec Q$.

(c2) Since P/PM and Q/QM are both infinitely generated and $P/PM \cong Q/QM$, for each $n \geq 1$ there exist positive integers $t(n)$ and $s(n)$ such that $P_1 \oplus \cdots \oplus P_n \prec_M Q_1 \oplus \cdots \oplus Q_{t(n)}$ and $Q_1 \oplus \cdots \oplus Q_n \prec_M P_1 \oplus \cdots \oplus P_{s(n)}$. Therefore, changing notation we can assume that $P_1 \prec_M Q_1 \prec_M P_1 \oplus P_2 \prec_M Q_1 \oplus Q_2 \prec_M \cdots$. and that $P_i \neq P_iM$ and $Q_i \neq Q_iM$ for all $i \geq 1$. By Lemma 2.4(2), we get $P \cong Q$.

(c3) The proof is similar to the last case in (c1). □

Corollary 2.7.

(a) *Let R be a regular ring satisfying s-comparability for some $s > 1$, and let P and Q be countably generated projective modules. Then either $P \lesssim 2Q$ or $Q \lesssim 2P$.*

(b) *Let R be a regular ring satisfying comparability, and let P and Q be countably generated projective modules. Then either $P \lesssim Q$ or $Q \lesssim P$.*

Proof. (a) If $\mathrm{tr}(P) \neq \mathrm{tr}(Q)$, then the result follows from Theorem 2.6(a) (by using comparability of ideals). So, assume that $\mathrm{tr}(P) = \mathrm{tr}(Q)$. If $\mathrm{tr}(P)$ is not a principal ideal, then the result follows from Theorem 2.6(b). So, we have reduced the problem to the case where $\mathrm{tr}(P) = \mathrm{tr}(Q)$ and $\mathrm{tr}(P)$ is a principal two-sided ideal of R. Let M be the unique maximal ideal of $\mathrm{tr}(P)$, let $e = e^2 \in \mathrm{tr}(P) \setminus M$, and set $S = eRe/(eMe)$, a simple regular ring satisfying s-comparability.

If either P/PM or Q/QM is not finitely generated, then P and Q are comparable by Theorem 2.6(c1), so clearly either $P \lesssim 2Q$ or $Q \lesssim 2P$.

If P/PM and Q/QM are both finitely generated, then P/PM and Q/QM are comparable in case S is either artinian or purely infinite, and P/PM and Q/QM are almost-comparable in case S is directly finite and non-artinian, see [**AGPT**, Corollary 4.5]. In either case we obtain that either $P/PM \prec (2Q)/(2Q)M$ or $Q/QM \prec (2P)/(2P)M$. By Theorem 2.6(c3), we get that either $P \prec 2Q$ or $Q \prec 2P$.

(b) We give a direct proof, which is a slight modification of the proof for the directly finite case, given in [**Ku2**, Theorem 2.1(a)]. Assume that P and Q are countably generated projective modules over a regular ring with comparability. First consider the case where P is finitely generated and Q is infinitely generated. We can assume $P \not\lesssim Q$. Proceeding by induction, assume we have for some $n \geq 1$ submodules $P_1, \ldots, P_n$ of P such that $P = P_1 \oplus \cdots \oplus P_n \oplus P'_n$ with $P_i \cong Q_i$ for $i = 1, \ldots, n$. If $P'_n \lesssim Q_{n+1}$, then $P \lesssim Q$, a contradiction, so that $Q_{n+1} \lesssim P'_n$. Write $P'_n = P_{n+1} \oplus P'_{n+1}$ with $P_{n+1} \cong Q_{n+1}$ and note that $P = P_1 \oplus \cdots \oplus P_{n+1} \oplus P'_{n+1}$. This completes the induction argument.

Assume now that both P and Q are infinitely generated, and assume that $Q \not\lesssim P$. Write $P = \oplus_{i=1}^{\infty} A_i$ with $A_i \in FP(R)$ for all i. We have seen before that A_1 and Q are comparable, but we cannot have $Q \lesssim A_1$, so $A_1 \lesssim Q$. Write $Q = B_1 \oplus C_1$ with $A_1 \cong B_1$.

We cannot have $C_1 \lesssim A_2$, so that we have $A_2 \lesssim C_1$. Continuing in this way, we obtain submodules $\{B_n\}$ of Q such that $Q = B_1 \oplus \cdots \oplus B_n \oplus C_n$ for some C_n, and $A_n \cong B_n$ for all $n \geq 1$. We conclude that $P \lesssim Q$, as desired. $\square$

3. The lattices of ideals.

Let R be a regular ring. Recall from Section 1 that we denote by E the ring $\mathrm{End}(\aleph_0 R_R) = FCM(R)$ and by B the ring $FRCM(R)$. For a ring T, we denote by $L(T)$ the lattice of (two-sided) ideals of T. In this Section we will obtain some general information on the lattices $L(B)$ and $L(E)$, and then we will carefully study the special situation in which R is a regular ring satisfying s-comparability.

The ring B is the multiplier ring of the σ-unital regular ring $F = FM(R)$, and so every ideal in B is generated by idempotents [**AP**, Theorem 2.5]. Moreover, the ideals of B correspond to certain subsets of $V(B)$, called order-ideals. To define them, let us consider an abelian monoid M, endowed with the algebraic pre-order (see Remark 1.6). An order-ideal of M is a submonoid S of M such that S is hereditary with respect to the algebraic pre-ordering, i.e., $y \leq x$ for $y \in M$ and $x \in S$ implies $y \in S$. We denote by $L(M)$ the lattice of order-ideals of a monoid M. By [**AP**, Theorem 2.7] we have a lattice isomorphism $L(B) \cong L(V(B))$. The situation with the ring E is somewhat different, since the ideals of E need not be generated by idempotents. For example, let R be a commutative non-artinian regular ring, so that R has a sequence (e_n) of nonzero orthogonal idempotents. Consider the matrix X having all rows but the first one 0, and with first row $[e_1, e_2, \ldots]$. Then the ideal generated by X in E cannot be generated by idempotents. However we can explote the fact that E is an exchange ring to obtain some useful information on the ideals of E.

Theorem 3.1. *Let R be a regular ring. Consider the maps $\alpha : L(B) \to L(E)$ and $\beta : L(E) \to L(B)$ defined by $\alpha(I) = EIE$ and $\beta(I') = I' \cap B$ for $I \in L(B)$ and $I' \in L(E)$. Then $\beta \circ \alpha = Id_{L(B)}$, so that α is injective and β is surjective. Moreover, for $I \in L(B)$, we have $\beta^{-1}(I) = [\alpha(I), \pi^{-1}_{\alpha(I)}(J(E/\alpha(I)))]$, being $\pi_{\alpha(I)} : E \to E/\alpha(I)$ the canonical projection.*

Proof. Let $i : B \to E$ be the canonical inclusion. By Theorem 1.3, the induced map $V(i) : V(B) \to V(E)$ is a monoid isomorphism. Hence, we get a lattice isomorphism $L(V(i)) : L(V(B)) \to L(V(E))$. On the one hand, by [**AP**, Theorem 2.7] we have a lattice isomorphism $L(B) \to L(V(B))$ which sends an ideal I of B to the order-ideal $V(I)$ of $V(B)$. On the other hand, since E is an exchange ring, we obtain from [**tePa**, Teorema 4.1.7(i)] a surjective lattice homomorphism $L(E) \to L(V(E))$ sending $I' \in L(E)$ to the order ideal $V(I')$ of $V(E)$. The composition $L(E) \to L(V(E)) \to L(V(B)) \to L(B)$ gives a surjective lattice homomorphism from $L(E)$ onto $L(B)$, which is easily seen to agree with β. Now, we infer from [**tePa**, Theorem 4.1.7(ii)] that, for $I \in L(B)$, we have $\beta^{-1}(I) = [\alpha(I), \pi^{-1}_{\alpha(I)}(J(E/\alpha(I)))]$. In particular we get $\beta \circ \alpha = \mathrm{Id}_{L(B)}$. $\square$

Following [**Par**], we define s-comparability for a general ring T in terms of its monoid $V(T)$, as follows. First, say that the monoid M satisfies s-comparability if for any $p, q \in M$ either p is a summand of sq or q is a summand of sp. If T is any ring, say that T satisfies

s-comparability provided $V(T)$ satisfies s-comparability. Of course, comparability stands for 1-comparability.

Let R be a regular ring satisfying s-comparability. In view of Corollary 2.7, it seems reasonable to ask whether the rings E and B satisfy s-comparability. However this is not true, as can be seen from easy examples. (Note that the relation appearing in Corollary 2.7 is $\lesssim$ while to get comparability in E we need the relation $\lesssim^{\oplus}$.) Similarly the lattices $L(B)$ and $L(E)$ need not be totally ordered. We will prove that things become better when we consider the rings B/F and E/G.

Theorem 3.2. *Let R be a regular ring satisfying s-comparability for some $s \geq 1$. Then B/F and E/G satisfy comparability.*

Proof. Since E is an exchange ring we have $V(E/G) \cong V(E)/V(G)$ by [**AGOP**, Proposition 1.4]. Since F is a regular ideal we can lift idempotents from B/F to B [**Men**, Lemma 3]. Thus the natural map $V(B)/V(F) \to V(B/F)$ is surjective. By using [**Aex**, Lemma 2.1], we see that the above map is injective as well (cf. [**AGOP**, proof of 1.4]). It follows from the above observations and Proposition 1.2 and Theorem 1.3 that $V(E/G) \cong V(B/F) \cong V(\mathcal{C}_R)/V(R)$. Therefore in order to prove that B/F and E/G satisfy comparability, it is enough to prove that, given two countably generated projective R-modules P and Q, there are finitely generated projective R-modules A and B such that either $P \oplus A \lesssim^{\oplus} Q \oplus B$ or $Q \oplus B \lesssim^{\oplus} P \oplus A$. Obviously, we can assume that both P and Q are not finitely generated.

Instead of working in the monoid $V(\mathcal{C}_R)/V(R)$ we will work with countably generated projective modules "modulo $FP(R)$".

It is easy to see that P (respectively, Q), falls modulo $FP(R)$ into exactly one the following classes:

(a) The class of those $A \in \mathcal{C}_R$ such that $\mathrm{tr}(A)$ is not a principal two-sided ideal.

(b) The class of those $A \in \mathcal{C}_R$ which admit a decomposition $A = \oplus_{i=1}^{\infty} A_i$, where $\mathrm{tr}(A_i) = \mathrm{tr}(A_j)$ for all i, j.

(c) The class of those $A \in \mathcal{C}_R$ admitting a decomposition $A = \oplus_{i=1}^{\infty} A_i$ with $\mathrm{tr}(A_n) \supset \mathrm{tr}(A_{n+1})$ for all $n \geq 1$.

Assume first that P and Q fall in class (a). We can assume $\mathrm{tr}(P) \subseteq \mathrm{tr}(Q)$. Then there are decompositions $P = \oplus_{i=1}^{\infty} P_i$ and $Q = \oplus_{i=1}^{\infty} Q_i$ such that $\mathrm{tr}(P_n) \subset \mathrm{tr}(P_{n+1})$ and $\mathrm{tr}(Q_n) \subset \mathrm{tr}(Q_{n+1})$ for all $n \geq 1$, and $\mathrm{tr}(P_n) \subset \mathrm{tr}(Q_n)$ for all n. By [**AOT**, Proposition 2.5(b)], we have $P_n \lesssim^{\oplus} Q_n$ for all n, and so $P \lesssim^{\oplus} Q$.

Assume that P falls in class (a) and Q falls in class (b). Then $\mathrm{tr}(P) \neq \mathrm{tr}(Q)$. By the same trick as in the above paragraph one obtains that either $P \lesssim^{\oplus} Q$ or $Q \lesssim^{\oplus} P$, depending on whether $\mathrm{tr}(P) \subset \mathrm{tr}(Q)$ or $\mathrm{tr}(Q) \subset \mathrm{tr}(P)$.

The case where P falls in class (b) and Q falls in class (c) is similar to the above case.

Assume now that P falls in class (a) and Q falls in class (c). Write $P = \oplus_{i=1}^{\infty} P_i$ and $Q = \oplus_{i=1}^{\infty} Q_i$, with $\mathrm{tr}(P_n) \subset \mathrm{tr}(P_{n+1})$ and $\mathrm{tr}(Q_n) \supset \mathrm{tr}(Q_{n+1})$ for all n. If $\mathrm{tr}(P_n) \supseteq \mathrm{tr}(Q_n)$ for some n, then $Q \lesssim^{\oplus} P$ modulo $FP(R)$. So we can assume that $\mathrm{tr}(P_n) \subset \mathrm{tr}(Q_n)$ for all n and so $P_n \lesssim^{\oplus} Q_n$ for all n, which gives $P \lesssim^{\oplus} Q$.

Consider now the case where P and Q fall both in class (b). Clearly we can assume that $\mathrm{tr}(P) = \mathrm{tr}(Q)$. Let M be the unique maximal ideal of $\mathrm{tr}(P)$. Since both P/PM and Q/QM are both infinitely generated, it follows from Corollary 2.2 that P/PM and

Q/QM are comparable with repect to $\lesssim^{\oplus}$. Assume that $P/PM \lesssim^{\oplus} Q/QM$. Then there is $T \in \mathcal{C}_R$ such that $(P \oplus T)/(P \oplus T)M \cong Q/QM$. By Theorem 2.6(c2), we get $P \oplus T \cong Q$ and so $P \lesssim^{\oplus} Q$.

Finally consider the case where both P and Q fall in class (c). Write $P = \oplus_{i=1}^{\infty} P_i$ and $Q = \oplus_{i=1}^{\infty} Q_i$, with $\mathrm{tr}(P_n) \supset \mathrm{tr}(P_{n+1})$ and $\mathrm{tr}(Q_n) \supset \mathrm{tr}(Q_{n+1})$ for all $n \geq 1$. We can assume that $\cap_{i=1}^{\infty} \mathrm{tr}(P_i) \subseteq \cap_{i=1}^{\infty} \mathrm{tr}(Q_i)$. If $\mathrm{tr}(Q_1) \subseteq \mathrm{tr}(P_i)$ for all i, then $\mathrm{tr}(Q_1) \subseteq \cap_{i=1}^{\infty} \mathrm{tr}(P_i) \subseteq \cap_{i=1}^{\infty} \mathrm{tr}(Q_i)$, a contradiction. So there is i_1 such that $\mathrm{tr}(P_{i_1}) \subset \mathrm{tr}(Q_1)$. Now by the same argument, there is $i_2 > i_1$ such that $\mathrm{tr}(P_{i_2}) \subset \mathrm{tr}(Q_2)$. In this way we obtain a strictly increasing sequence $i_1 < i_2 < \cdots$ such that $\mathrm{tr}(P_{i_n}) \subset \mathrm{tr}(Q_n)$ for all n. Now note that since $\mathrm{tr}(P_{i_n} \oplus \cdots \oplus P_{i_{n+1}-1}) \subset \mathrm{tr}(Q_n)$ we get $P_{i_n} \oplus \cdots \oplus P_{i_{n+1}} \lesssim^{\oplus} Q_n$ and so we get $P_{i_1} \oplus P_{i_1+1} \oplus \cdots \lesssim^{\oplus} Q_1 \oplus Q_2 \oplus \cdots$, showing that $P \lesssim^{\oplus} Q$ modulo $FP(R)$. $\square$

For a ring T, denote by $L_0(T)$ (respectively, $L_1(T)$) the subset of $L(T)$ consisting in the ideals of T which are generated by idempotents (respectively, the semiprimitive ideals of T).

Corollary 3.3. *Let R be a regular ring satisfying s-comparability for some $s \geq 1$. Then $L(B/F)$ is a totally ordered lattice, and $L_0(E/G)$ and $L_1(E/G)$ are totally ordered.*

Proof. By Theorem 3.2, B/F satisfies comparability, so by [**Par**, Lemma 1.5] the lattice $L(V(B/F))$ is totally ordered by inclusion. Now we have observed in the proof of Theorem 3.2 that $V(B/F) \cong V(B)/V(F)$. Therefore, by using [**AP**, Theorem 2.7], we obtain a lattice isomorphism $L(B/F) \to L(V(B/F))$ which sends I/F to $V(I)/V(F)$. We conclude that $L(B/F)$ is totally ordered by inclusion.

By Theorem 3.2, E/G satisfies comparability, so by [**Par**, Lemma 1.5] the lattice $L(V(E/G))$ is totally ordered. Since E is an exchange ring we have $V(E/G) \cong V(E)/V(G)$ [**AGOP**, Proposition 1.4]. So we obtain an order-preserving bijection from $L_0(E/G) \to L(V(E/G))$ given by the rule $I/G \mapsto V(I)/V(G)$. It follows that $L_0(E/G)$ is totally ordered by inclusion.

Since E/G is an exchange ring, there is an order-preserving bijection from $L_0(E/G)$ onto $L_1(E/G)$, see [**tePa**, Theorem 4.1.7]. Therefore $L_1(E/G)$ is also totally ordered by inclusion. $\square$

4. Property (DF).

Following Kutami [**Ku3**], we say that a ring S satisfies property (DF) provided $P \oplus Q$ is directly finite for every directly finite projective right S-modules P and Q. Kaplansky's classical result [**Kap**], stating that every projective module is a direct sum of countably generated ones, suggests that property (DF) could be equivalent to the statement that finite direct sums of countably generated directly finite projectives are again directly finite. This is indeed the case, as we prove below.

Lemma 4.1. *Let S be any ring and let P be a projective S-module. If P is directly infinite, then there is a countably generated direct summand of P which is also directly infinite.*

Proof. Let P be a directly infinite projective module. By Kaplansky's Theorem, there exist nonzero submodules X and P_1 of P such that $P = X \oplus P_1$, X is countably generated, and there is an injective homomorphism $\varphi : P \to P_1$ such that $\varphi(P)$ is a direct summand of P_1.

By Kaplansky's Theorem, we have $P_1 = \oplus_{i\in I} C_i$, where C_i are countably generated submodules of P_1. For any subset $L \subseteq I$, put $C_L = \oplus_{i\in L} C_i$. Since X is countably generated, there is a countable subset I_0 of I such that $\varphi(X) \subseteq C_{I_0}$. Since $X \oplus C_{I_0}$ is countably generated, there is a countable subset I_1 of I such that $I_0 \subseteq I_1$ and $\varphi(X\oplus C_{I_0}) \subseteq C_{I_1}$. Continuing in this way, we obtain a sequence (I_n) of countable subsets of I such that $I_n \subseteq I_{n+1}$ and $\varphi(X \oplus C_{I_n}) \subseteq C_{I_{n+1}}$ for all $n \geq 0$. Set $J = \cup_{n=0}^{\infty} I_n$, a countable subset of I. Set $P' = X \oplus C_J$, and note that P' is a countably generated direct summand of P. It remains to prove that P' is directly infinite. Clearly $\varphi(X \oplus C_J) \subseteq C_J$. Since φ is an injective homomorphism from P onto a direct summand of P_1 and $X \oplus C_J$ is a direct summand of P, we conclude that $\varphi(X \oplus C_J)$ is a direct summand of P_1. Therefore, we see from the modular law that $\varphi(X \oplus C_J)$ is a direct summand of C_J. This proves that $X \oplus C_J$ is directly infinite, as desired. □

Proposition 4.2. *Let S be any ring. Then S satisfies property (DF) if and only if, for every directly finite countably generated projective modules P and Q, the direct sum $P \oplus Q$ is also directly finite.*

Proof. Assume that the class of directly finite countably generated projectives is closed under finite direct sums. Let P and Q be projective right R-modules such that $P \oplus Q$ is directly infinite. By Lemma 4.1, there is a countably generated direct summand A of $P\oplus Q$ such that A is directly infinite. By Kaplansky's Theorem, there are countably generated direct summands P_1 and Q_1 of P and Q respectively, such that $P_1 \oplus Q_1 = A \oplus B$ for some B. Since A is directly infinite, $A \oplus B$ is directly infinite and so, either P_1 or Q_1 is directly infinite by hypothesis. Therefore, either P or Q is directly infinite, and S satisfies property (DF). □

Our next goal in this Section is to characterize the simple, strictly unperforated, regular rings which satisfy property (DF). We remark that there are no known examples of simple regular rings which do not satisfy strict unperforation. We need a technical lemma.

Lemma 4.3. *Let K be a Choquet simplex and let s and t be two distinct extreme points of K. Then there exist $f_1, f_2 \in \mathrm{LAff}_\sigma(K)^{++}$ such that $f_1(s) = f_2(t) = 1$ and $f_1 + f_2 = \infty$.*

Proof. Consider the discrete compact subset $\{s,t\}$ of $\partial_e(K)$, the extreme boundary of K. Define a continous function $g_0 : \{s,t\} \to \mathbb{R}$ by $g_0(s) = 0$ and $g_0(t) = 1$. By [**poag**, Theorem 11.14] there exists $g \in \mathrm{Aff}(K)$ such that $0 \leq g \leq 1$ and $g(s) = g_0(s) = 0$ and $g(t) = g_0(t) = 1$.

Write $F_1 = g^{-1}(\{0\})$. By [**poag**, Lemma 5.16], F_1 is a closed face of K. Note that $s \in F_1$ and $t \notin F_1$. Set $F_2 = \{t\}$. Then F_1 and F_2 are disjoint closed faces of K. Define $g_i \in \mathrm{Aff}(F_i)$ for $i = 1, 2$ by setting $g_1 = 1$ and $g_2 = 0$. By [**poag**, Theorem 11.22] there exists $h \in \mathrm{Aff}(K)$ such that $0 \leq h \leq 1$ and $h_{|F_i} = g_i$ for $i = 1, 2$. Note that $h(t) = 0$ and $h(s) = 1$.

Now define $\overline{g} = \sup_n ng$ and $\overline{h} = \sup_n nh$. Since $\overline{g}$ and $\overline{h}$ are pointwise suprema of sequences of continuous affine functions on K, we have $\overline{g}, \overline{h} \in \mathrm{LAff}_{sigma}(K)$. Note that $\overline{g}$ and $\overline{h}$ only take the values 0 and ∞, and $\overline{g}(x) = 0$ (respectively, $\overline{h}(x) = 0$) if and only if $g(x) = 0$ (respectively, $h(x) = 0$). Let us see that $\overline{g} + \overline{h} = \infty$. Take first $x \in F_1$. Then $h(x) = 1$ and so $\overline{h}(x) = \infty$. Take now $x \notin F_1 (= g^{-1}(\{0\})$. Then $g(x) > 0$ and so $\overline{g}(x) = \infty$. This shows that $\overline{g} + \overline{h} = \infty$.

Finally, set $f_1 = \overline{g}+1$ and $f_2 = \overline{h}+1$. Then $f_1, f_2 \in \mathrm{LAff}_\sigma(K)^{++}$ and $f_1(s) = f_2(t) = 1$ and $f_1 + f_2 = \infty$, as required. □

Theorem 4.4. *Let R be a simple, strictly unperforated, regular ring. Then R satisfies property (DF) if and only if R satisfies s-comparability for some $s \geq 1$.*

Proof. Assume first that R is directly infinite. Then R satisfies comparability. In fact, R satisfies the following property, which clearly implies comparability: Given two nonzero elements $x, y \in R$ then $xR \prec yR$. To see this, let x and y be two nonzero elements of R. By simplicity, there is $n \geq 1$ such that $R_R \lesssim n(yR_R)$, so that $n(yR_R)$ is directly infinite. Hence, there is a nonzero $z \in R$ such that $m(zR_R) \lesssim n(yR_R)$ for all $m \geq 1$, and again by simplicity of R, we obtain that $A \prec n(xR_R)$ for every $A \in FP(R)$. In particular, we have $n(xR_R) \prec n(yR_R)$ and so, since R is strictly unperforated, we get $xR \prec yR$. By Corollary 2.2(a), $V(\mathcal{C}_R) \cong V(R) \sqcup \{\infty\}$. So the only directly finite countably generated projective module is 0, hence R satisfies property (DF).

If R is artinian, then $V(\mathcal{C}_R) \cong V(R) \sqcup \{\infty\}$ by Corollary 2.2(a), so property (DF) is clear in this case, as is comparability.

Finally, assume that R is nonartinian and directly finite. By [**AGPT**, Theorem 4.3] and [**OM**, Theorem 1], R is unit-regular. By Theorem 2.1, there exists a monoid isomorphism $\mu : V(\mathcal{C}_R) \to V(R) \sqcup \mathrm{LAff}_\sigma(\mathbb{P}(R))^{++}$. By [**AGPT**, Corollary 4.5] and Proposition 4.2, it suffices to see that the directly finite elements of the monoid $M := V(R) \sqcup \mathrm{LAff}_\sigma(\mathbb{P}(R))^{++}$ form a submonoid if and only if $\mathbb{P}(R)$ is a singleton. If $\mathbb{P}(R)$ is a singleton, then $M = V(R) \sqcup \mathbb{R}^{++} \sqcup \{\infty\}$, so the set of directly finite elements is $V(R) \sqcup \mathbb{R}^{++}$, which is a submonoid of M. Assume now that $\mathbb{P}(R)$ is not a singleton, and note that the only directly infinite element of M is the constant function ∞ on $\mathbb{P}(R)$. By the Krein-Milman Theorem there are two different extreme points in $\mathbb{P}(R)$, say N_1 and N_2. Now $\mathbb{P}(R)$ is a Choquet simple by [**vnrr**, Theorem 17.5], and so we get from Lemma 4.3 functions $f_1, f_2 \in \mathrm{LAff}_\sigma(\mathbb{P}(R))^{++}$ such that $f_1(N_1) = f_2(N_2) = 1$ and $f_1 + f_2 = \infty$.

Therefore f_1 and f_2 are directly finite elements of M, but $f_1 + f_2 = \infty$, which is directly infinite. □

In [**Ku3**], Kutami showed that a unit-regular ring with s-comparability satisfies property (DF). By using this, he was able to completely characterize the directly finite projective modules over them. We will extend Kutami's results to the general case of regular rings with s-comparability. Note that directly finite regular rings with 2-comparability are not necessarily unit-regular by [**AOT**, Example 3.2], and therefore our extension is proper even in the directly finite case.

Although we will use some of the ideas of Kutami [**Ku3**], we will proceed in a self-contained manner, characterizing in the first place the directly finite projective modules over a regular ring with s-comparability.

Lemma 4.5. *Let R be a regular ring satisfying s-comparability for some $s \geq 1$. Let P and Q be finitely generated projective right R-modules.*

(a) *If P and Q are directly finite, then so is $P \oplus Q$.*

(b) *If I is a two-sided ideal of R, and P is directly finite, then so is P/PI.*

(c) *Assume that R has a minimal ideal I_0. Assume that P is directly finite, and let X be a directly finite countably generated projective module such that $X = XI_0$. Then $P \oplus X$*

is directly finite.

Proof. (a) Consider the ideal $J = \mathrm{tr}(P \oplus Q)$. Then, the arguments used in the proof of Theorem 2.6(c) allow us to assume that $R = J$. Note that either $R = \mathrm{tr}(P)$ or $R = \mathrm{tr}(Q)$ by comparability of ideals. We can assume $R = \mathrm{tr}(P)$. Then $\mathrm{End}(P_R)$ is directly finite and R is Morita equivalent to it. By [**AOT**, Corollary 4.7] R is stably finite. So $P \oplus Q$ is directly finite.

(b) By using the same argument as in (a), we can assume that R is directly finite. By [**AOT**, Corollary 4.7], R and all its factor rings are stably finite. Consequently, P/PI is directly finite.

(c) Let e_0 be a nonzero idempotent in I_0, and note that e_0Re_0 is a simple regular ring satisfying s-comparability. If e_0Re_0 is directly infinite, then the only directly finite projective R-module is 0. If e_0Re_0 is artinian, then X must be finitely generated, so $P \oplus X$ is directly finite by (a). So we can assume that e_0Re_0 is directly finite and nonartinian. Assume that $P \oplus X$ is directly infinite. Then there is a nonzero cyclic ideal Y such that $YI_0 = Y$ with $P \oplus X \oplus Y \lesssim P \oplus X$. Write $X = \oplus_{i=1}^{\infty} X_i$ for $X_i \in FP(R)$. Let D be the unique dimension function on I_0 such that $D(e_0R) = 1$. Since X is directly finite, we must have $D(X) = \sum D(X_i) < \infty$, so there is $n_0 \geq 1$ such that $D(X_{n_0+1} \oplus \cdots \oplus X_n) < D(Y)$ for all $n > n_0$. By [**AGPT**, Corollary 4.5], we obtain $X_{n_0+1} \oplus \cdots \oplus X_n \prec Y$ for all $n > n_0$. Since $P \oplus X \oplus Y \lesssim P \oplus X$, there exists $m \geq n_0$ such that $P \oplus Y \oplus X_1 \oplus \cdots \oplus X_{n_0} \lesssim P \oplus X_1 \oplus \cdots \oplus X_m$. Therefore

$$P \oplus X_1 \oplus \cdots \oplus X_m \prec P \oplus X_1 \oplus \cdots \oplus X_{n_0} \oplus Y \lesssim P \oplus X_1 \oplus \cdots \oplus X_m,$$

showing that $P \oplus X_1 \oplus \cdots \oplus X_m$ is directly infinite, in contradiction with (a). $\square$

Now we are ready to describe all the directly finite countably generated projective modules over a regular ring with s-comparability.

Theorem 4.6. *Let R be a regular ring satisfying s-comparability for some $s \geq 1$.*

(a) *Assume that R has a minimal ideal I_0. Then the directly finite countably generated projective modules are the modules of the form $P \oplus Q$, where P is a directly finite finitely generated projective module and Q is a countably generated directly finite projective module such that $Q = QI_0$.*

(b) *Assume that R does not have a minimal ideal. Then the directly finite countably generated projective modules which are not finitely generated are the modules of the form $P = \oplus_{i=1}^{\infty} P_i$, where P_i are directly finite finitely generated projective modules and $\mathrm{tr}(P_{i+1}) \subset \mathrm{tr}(P_i)$ for all i, and $\cap_{i=1}^{\infty} \mathrm{tr}(P_i) = 0$.*

Proof. Let P be a directly finite countably generated projective module, and let $P = \oplus_{i=1}^{\infty} P_i$, where $P_i \in FP(R)$. We assume that P is not finitely generated and $P_i \neq 0$ for all i. Let $\Gamma_1 = \{i \in \mathbb{N} \mid \mathrm{tr}(P_1) \subset \mathrm{tr}(P_i)\}$. By Theorem 2.6(a), $P_1 \prec P_i$ for all $i \in \Gamma_1$. If Γ_1 is infinite, then $\aleph_0 P_1 \lesssim^{\oplus} P$, and so P is directly infinite, a contradiction. So Γ_1 is finite, and collecting in the first position all P_i's with $i \in \Gamma_1$, we can assume that $\mathrm{tr}(P_j) \subseteq \mathrm{tr}(P_1)$ for all j. Applying the same argument to P_2 and the indexes ≥ 2, we can assume as well that $\mathrm{tr}(P_1) \supseteq \mathrm{tr}(P_2) \supseteq \mathrm{tr}(P_j)$ for all $j > 2$. Continuing in this way, we see that, without loss of generality, we can assume that the decomposition $P = \oplus_{i=1}^{\infty} P_i$ satisfies $\mathrm{tr}(P_i) \supseteq \mathrm{tr}(P_{i+1})$

for all i. Assume first that the sequence $\mathrm{tr}(P_1) \supseteq \mathrm{tr}(P_2) \supseteq \cdots$ stabilizes. Then there is n_0 such that $\mathrm{tr}(P_{n_0}) = \mathrm{tr}(P_n)$ for all $n \geq n_0$. Write $I_0 = \mathrm{tr}(P_{n_0})$, and note that I_0 is a nonzero principal ideal of R. If I_0 is not a minimal ideal there exists $a \in I_0$ such that $RaR \subset I_0$. By Theorem 2.6(a) we then have $aR \lesssim P_i$ for all i, and so P is directly infinite. So I_0 must be a minimal ideal of R. Write $Q = \oplus_{n=n_0}^{\infty} P_n$. Then Q is a directly finite countably generated projective module such that $Q = QI_0$, and $P = P' \oplus Q$, where $P' = P_1 \oplus \cdots \oplus P_{n_0-1}$ is a directly finite finitely generated projective module. So we showed that P is as in (a) if the chain $\mathrm{tr}(P_1) \supseteq \mathrm{tr}(P_2) \supseteq \cdots$ stabilizes. Assume now that that sequence does not stabilize. By a new arrangement of terms we can then assume that $\mathrm{tr}(P_i) \supset \mathrm{tr}(P_{i+1})$ for all i. Write $I = \cap_{i=1}^{\infty} \mathrm{tr}(P_i)$. If $I \neq 0$, then we get a contradiction as before. So $I = 0$ and P is as in (b) in this case.

It remains to prove that the modules in (a) and (b) are directly finite. Assume first that R has a minimal ideal I_0, and let $P \oplus Q$ be a module as in (a). Then the result follows from Lemma 4.5(c).

Finally assume that R does not have a minimal ideal, and let $P = \oplus_{i=1}^{\infty} P_i$, where P_i are directly finite finitely generated projective modules with $\mathrm{tr}(P_i) \supset \mathrm{tr}(P_{i+1})$ for all i, and $\cap_{i=1}^{\infty} \mathrm{tr}(P_i) = 0$. Assume that $P \oplus X \lesssim^{\oplus} P$ for some nonzero $X \in FP(R)$. Write $J = \mathrm{tr}(X)$. By comparability of ideals, there is n_0 such that $\mathrm{tr}(P_n) \subset J$ for all $n > n_0$ (otherwise, $0 \neq J \subseteq \cap_{i=1}^{\infty} \mathrm{tr}(P_i) = 0$). Let M be the unique maximal ideal of the principal ideal J. Then we get

$$(*) \qquad P/PM \oplus X/XM \lesssim^{\oplus} P/PM.$$

Now $P/PM = P_1/P_1M \oplus \cdots \oplus P_{n_0}/P_{n_0}M$ is directly finite by Lemma 4.5(a)(b), and $X/XM \neq 0$, so $(*)$ gives a contradiction. Therefore, all the modules in (b) are directly finite, as desired. $\square$

Corollary 4.7. *Let R be a regular ring satisfying s-comparability for some $s > 1$. Then R satisfies property (DF).*

Proof. Note that cases (a) and (b) in Theorem 4.6 are exclusive. In either case, taking into account Lemma 4.5(a) and Theorem 4.4, it is clear that the finite direct sums of directly finite countably generated projective modules are again directly finite. So the result follows from Proposition 4.2. $\square$

Remark 4.8.

(a) Let R be a regular ring with s-comparability for some $s > 1$. Since R satisfies property (DF) by Corollary 4.7, the proof of [**Ku3**, Proposition 4] applies to show that every non-countably generated projective R-module is directly infinite. So Theorem 4.6 describes in fact all the directly finite projective R-modules.

(b) Kutami gives in [**Ku3**] a classification of unit-regular rings satisfying s-comparability in three classes (A),(B),(C), according with the possible types of directly finite projective modules. As in Theorem 4.6, these types are reflected in the ideal structure of the ring, see [**Ku3**, Section 4]. A similar classification could be established by using Theorem 4.6 for a general regular ring satisfying s-comparability. So, for example, the regular rings with s-comparability such that every countably generated directly finite projective is finitely generated are those such that either $\mathrm{Soc}(R_R) \neq 0$, or there are no nonzero directly finite cyclic projectives, or there are neither minimal ideals nor sequences $\{I_n\}_{n=1}^{\infty}$ of ideals of R such that $\cap_{i=1}^{\infty} I_n = 0$.

References

Aex. P. Ara, *Extensions of Exchange Rings*, J. Algebra **197** (1997), 409-423.

AGOP. P. Ara, K.R. Goodearl, K.C. O'Meara and E. Pardo, *Separative cancellation for projective modules over exchange rings*, Israel J. Math. **105** (1998), 105-137.

AGPT. P. Ara, K.R. Goodearl, E. Pardo, and D.V. Tyukavkin, *K-theoretically simple von Neumann regular rings*, J. Algebra **174** (1995), 659-677.

AOT. P. Ara, K.C. O'Meara and D.V. Tyukavkin, *Cancellation of projective modules over regular rings with comparability*, J. Pure Appl. Algebra **107** (1996), 19-38.

APa. P.Ara and E. Pardo, *Refinement monoids with weak comparability and applications to regular rings and C^*-algebras*, Proc. Amer. Math. Soc. **124** (1996), 715-720.

AP. P. Ara and F. Perera, *Multipliers of von Neumann regular rings*, Preprint.

vnrr. K. R. Goodearl, *Von Neumann Regular Rings*, Pitman, London, 1979; Second Ed., Krieger, Malabar, Fl., 1991.

poag. K. R. Goodearl, *Partially Ordered Abelian Groups with Interpolation*, Math. Surveys and Monographs 20, Amer. Math. Soc., Providence, 1986.

gKth. K. R. Goodearl, *K_0 of multiplier algebras of C^*-algebras with real rank zero*, K-theory **10** (1996), 419-489.

HRS. J. Haefner, A. del Río and J.J. Simón, *Isomorphisms of row and column finite matrix rings*, Proc. Amer. Math. Soc. **125** (1997), 1651-1658.

Kad. J. Kado, *Projective modules over simple regular rings*, Osaka J. Math. **16** (1979), 405-412.

Kap. I. Kaplansky, *Projective modules*, Ann. Math. **68** (1958), 372-377.

Ku1. M. Kutami, *On projective modules over directly finite regular rings satisfying the comparability axiom*, Osaka J. Math. **22** (1985), 815-819.

Ku2. M. Kutami and K. Oshiro, *On projective modules over directly finite regular rings satisfying the comparability axiom II*, Osaka J. Math. **24** (1987), 465-473.

Ku3. M. Kutami, *On unit-regular rings satisfying s-comparability*, Osaka J. Math. **33** (1996), 983-995.

Lo. T.A. Loring, *Lifting solutions to Perturbing Problems in C^*-algebras*, Fields Institute Monographs **8**, Amer. Math. Soc., Providence, 1997.

Men. P. Menal, *On π-regular rings whose primitive factors are artinian*, J. Pure Applied Algebra **20** (1981), 71-78.

OM. K. C. O'Meara, *Simple regular rings satisfying weak comparability*, J. Algebra **141** (1991), 162–186.

Os. K. Oshiro, *Projective modules over von Neumann regular rings have the finite exchange property*, Osaka J. Math **20** (1983), 695-699.

tePa. E. Pardo, *Monoides de refinament i anells d'intercanvi*, Ph.D. Thesis, Universitat Autònoma de Barcelona, 1995.

Par. E. Pardo, *Comparability, separativity, and exchange rings*, Comm. in Algebra **24(9)** (1996), 2915-2929.

Per. F. Perera, *Ideal structure of multiplier algebras of simple C^*-algebras with real rank zero*, Preprint.

Ros. J. Rosenberg, *Algebraic K-Theory and Its Applications*, Grad. Texts in Math. 147, Springer-Verlag, New York, 1994.

Sto. J. Stock, *On rings whose projective modules have the exchange property*, J. Algebra **103** (1986), 437-453.

War. R. B. Warfield, Jr., *Exchange rings and decompositions of modules*, Math. Ann. **199** (1972), 31-36.

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