FAKE THREE CONNECTED COVERINGS OF LIE GROUPS

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1. Introduction. In [**1**] it was proved that for each prime p there are infinitely many fake 3-connected coverings of S^3. By "fake" we mean spaces with the same mod p cohomology than $S^3\langle 3\rangle$ (as algebras over the Steenrod algebra) but different p-completed homotopy type. After that work was completed one could wonder if the existence of such fake spaces was a general phenomenon and, in particular, if one could use the same methods to produce fake three connected coverings of other Lie groups beside S^3. In this paper we prove that the results of [**1**] cannot be extrapolated since, indeed, there is homotopy uniqueness up to p-completion for 3-connected coverings of several compact connected Lie groups and p-compact groups.

If p is a regular prime for the compact connected Lie group G then S^3 is a direct factor of G at the prime p and one can trivially construct infinitely many fake $G\langle 3\rangle$ out of the fake $S^3\langle 3\rangle$ constructed in [**1**]. If p is quasi-regular for G in the sense of [**10**] then G splits at p as a product of odd dimensional spheres and spaces $B_n(p)$ that are sphere bundles over spheres. Hence, in this more general situation fake $G\langle 3\rangle$ would arise if there are fake $B_3(p)\langle 3\rangle$. The main result of this paper shows that there are no such fakes.

Theorem 1. *Let $B(p)$ denote the S^3-bundle over S^{2p+1} classified by a generator of the p-component of $\pi_{2p}S^3$. Let X be such that $H^*(X;\mathbb{F}_p)\cong H^*(B(p)\langle 3\rangle;\mathbb{F}_p)$ as algebras over the Steenrod algebra. Then $\hat{X}_p \simeq B(p)\langle 3\rangle\hat{}_p$.*

Corollary 2. *Up to p-completion, there is only one space with the same mod p cohomology (as algebras over the Steenrod algebra) than $G\langle 3\rangle$ if G is i) $SU(3)$ for $p=2$; ii) $Sp(2)$ for $p=3$ or iii) G_2 for $p=5$.*

The proof of the main result is obtained by an analysis similar to the one done in [**1**] for the case of S^3. Starting with a space X with the same mod p cohomology than $B(p)\langle 3\rangle$ we construct an infinite tower

$$X \to X_1 \to X_2 \to X_3 \to \cdots$$

such that X_{i+1} is obtained from X_i essentially by dividing by an action of the group $B\mathbb{Z}/p$. At the far right end of the tower we get a space which up to p-completion coincides with $B(p)$ and so we get a map $X \to B(p)$ which induces isomorphism in mod p cohomology between X and $B(p)\langle 3\rangle$. At each stage we need to compute the cohomology of X_{i+1}

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out from the cohomology of X_i. Like in the case of $S^3\langle 3\rangle$ discussed in [**1**] there is an indeterminacy in the computation of $H^*(X_{i+1};\mathbb{F}_p)$ but, in contrast to what happens in that case, only one of the two algebras which are algebraically compatible with the Serre spectral sequence can be the cohomology algebra of a space. Hence, the key point in the proof and also the point that explains the opposite behavior of S^3 and $B(p)$ is the proof of the non-realizability of a certain algebra over the Steenrod algebra as the cohomology of a space. This proof is obtained by showing that, for this particular algebra, the Lannes T functor ([**9**]) is not compatible with higher Bocksteins.

During the present paper, mod p coefficients are assumed unless otherwise stated. Most of the proofs are written for the case of p odd but they can be translated to the case of $p = 2$ by using the standard identifications $Sq^1 = \beta$ and $Sq^{2i} = P^i$. In a few cases in which there is enough difference between the odd prime case and the case of $p = 2$, we provide separate arguments. We denote by $\mathcal{U}$ and $\mathcal{K}$ the categories of unstable modules and unstable algebras over the Steenrod algebra, respectively. When describing graded algebras, we tend to use subscripts to denote the degree of an element. In some cases, however, in order to simplify the notation, we may omit these subscripts.

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2. Uniqueness of $B(p)$. The homotopy uniqueness of $B(p)$ up to p-completion is a necessary step in the proof of theorem 1. We have the following result:

Proposition 3. *If $H^*(X) \cong H^*(B(p))$ as algebras over the Steenrod algebra then $X^\wedge_p \simeq B(p)^\wedge_p$.*

Proof. Recall that $H^*(B(p)) = E(x_3, y_{2p+1})$ with $P^1 x_3 = y_{2p+1}$. Since $H^1(X;\mathbb{F}_p) = 0$ the fundamental group of X is p-perfect and so ([**4**], VII.3) X is p-good, $\hat{X}_p$ is simply connected and $H^*(\hat{X}_p) \cong H^*(B(p))$. Hence, we can assume, without loss of generality, that X is p-complete and simply connected. By proposition 5.7 in [**1**] we know that the homotopy groups of X are finitely generated over $\hat{\mathbb{Z}}_p$ and also that $H^*(X;\hat{\mathbb{Z}}_p)$ is of finite type over $\hat{\mathbb{Z}}_p$. Now we can apply theorem 3.1 in [**3**] to conclude that $X \simeq W^\wedge_p$ where W is a CW-complex of finite type. Clearly, W can be chosen to be a three-cell complex

$$W = S^3 \cup_{\phi_1} e^{2p+1} \cup_{\phi_2} e^{2p+4}.$$

Since the 3-dimensional class is linked to the $(2p+1)$-dimensional class through a P^1, the map ϕ_1 is congruent modulo p to the generator of $\pi_{2p}(S^3)_{(p)}$ and the same is true in $B(p)$. Hence, one has the diagram:

$$\begin{tikzcd} S^{2p+3} \arrow[d] & \\ S^3 \cup_{\phi_1} e^{2p+1} \arrow[r, "i"] \arrow[d, "j"'] & B(p)_{(p)} \\ W & \end{tikzcd}$$

According to [**13**], the homotopy group $\pi_{2p+3}B(p)_{(p)}$ vanishes. Hence, we have a map $W \to B(p)_{(p)}$ inducing isomorphism in homology with coefficients in $\hat{\mathbb{Z}}_p$ (observe that $H^*(X)$ is generated as an algebra by x_3 and y_{2p+1}). This implies $X \simeq B(p)^{\widehat{}}_p$. $\square$

Remark. The above result will be used in the proof of theorem 1 and, actually, proposition 3 is a consequence of theorem 1. To see this, let L denote the localization functor with respect to the map $B\mathbb{Z}/p \to *$ as defined in [**5**]. Then a result of Neisendorfer ([**11**], also cf. lemma 9.9 in [**1**]) implies that $L(B(p)\langle 3\rangle^{\widehat{}}_p)$ and $B(p)$ are homotopy equivalent after p-completion. Hence, if X is a space with the same mod p cohomology than $B(p)$ then, by theorem 1 we have $B(p)\langle 3\rangle^{\widehat{}}_p \simeq X\langle 3\rangle^{\widehat{}}_p$. Applying now the functor L we get $X^{\widehat{}}_p \simeq B(p)^{\widehat{}}_p$.

3. The algebras K and L and the T functor. Let us denote by $\mathbf{L}$ the unstable algebra over the Steenrod algebra $\mathbf{L} = H^*(B(p)\langle 3\rangle)$. The structure of $\mathbf{L}$ is well known ([**13**]):

$$\mathbf{L} \cong \mathbb{F}_p[x_{2p^2}] \otimes E(y_{2p+1}, z_{2p^2+1})$$

(subscripts denote degrees) and the Steenrod algebra action is given by the formulas $\beta(x) = P^p(y) = z$. The action of the Steenrod algebra on $\mathbf{L}$ can be easily deduced from these formulas and the Adem relation. One obtains

$$P^1(x) = P^p(x) = \beta(y) = P^1(y) = \beta(z) = P^1(z) = P^p(z) = P^{p^2}(z) = 0.$$

The vanishing of all these Steenrod operations follows from dimensional reasons, except for the last two equalities which can be proved using the Adem relations

$$Sq^4Sq^4 = Sq^7Sq^1 + Sq^6Sq^2$$

$$Sq^8Sq^1 = Sq^4Sq^5 + Sq^9 + Sq^7Sq^2$$

$$P^{p^i}\beta = P^1\beta P^{p^i-1} + \beta P^{p^i}, \quad i > 0.$$

Let us define now $\mathbf{K}$ to be the graded algebra

$$\mathbf{K} = \mathbb{F}_p[x_2] \otimes E(y_3, z_{2p+1}).$$

We want to study actions of the Steenrod algebra on $\mathbf{K}$ which turn $\mathbf{K}$ into an unstable algebra over the Steenrod algebra. In particular, we are interested in actions of the Steenrod algebra on $\mathbf{K}$ subject to the conditions $\beta(y) = 0$ and $P^1(y) = z$. The following proposition gives a complete classification of such actions.

Proposition 4. *Assume* $\mathbf{K}$ *is an unstable algebra over the Steenrod algebra with* $\beta(y) = 0$ *and* $P^1(y) = z$*. Then the action of the Steenrod algebra on* $\mathbf{K}$ *is determined by* $\beta(x)$ *and* $P^p(z)$*. Moreover,* $P^p(z)$ *is either zero or* $x^{p(p-1)}z$*.*

Proof. It is obvious that the action of the Steenrod algebra on $\mathbf{K}$ is determined by unstability and the values of

$$\beta(x), \quad \beta(y), \quad \beta(z), \quad P^1(y), \quad P^1(z), \quad P^p(z).$$

Since $2P^2 = P^1P^1$ one obtains $P^1(z) = 0$ (use $Sq^2Sq^2 = Sq^3Sq^1$ if $p = 2$). The Adem relation $2P^1\beta P^1 = \beta P^1P^1 + P^1P^1\beta$ gives $\beta(z) = 0$ (use $Sq^1Sq^2 = Sq^3$ if $p = 2$).

By dimensional reasons we have $P^p(z) = \lambda x^{p(p-1)}z + \mu x^{p^2-1}y$. The relation $P^1P^p = P^{p+1}$ shows that $\mu = 0$. If $p = 2$ there is nothing more to prove. If p is odd and $P^p(z) = \lambda x^{p(p-1)}z$ then the Adem relation

$$P^pP^p = 2P^{2p} + P^{2p-1}P^1$$

shows that $\lambda = 0, 1$. $\square$

We introduce now the following family of unstable algebras over the Steenrod algebra. We define $\mathbf{K}_\lambda^{(r)}$, $r > 0$, $\lambda = 0, 1$ as

$$\mathbf{K}_\lambda^{(r)} = \mathbb{F}_p[x_2] \otimes E(y_3, z_{2p+1}),$$

with the action of the Steenrod algebra given by $P^1(y) = z$, $\beta(y) = \beta(z) = P^1(z) = 0$ and $P^p(z) = \lambda x^{p(p-1)}z$. We also assume that $\beta_{(r)}(x) = y$ where $\beta_{(r)}$ denotes the r-th order higher Bockstein. Since $\beta_{(r)}$ for $r > 1$ is not in the Steenrod algebra, it is clear that $\mathbf{K}_\lambda^{(r)}$ and $\mathbf{K}_\lambda^{(s)}$ for $r, s > 1$ are the same object of $\mathcal{K}$. However, it makes sense to ask if $\mathbf{K}_\lambda^{(r)}$ is the cohomology of some space and the notation $\mathbf{K}_\lambda^{(r)}$ will be a convenient one. We will also use the notation $\mathbf{K}'_\lambda$ to indicate the generic unstable algebra over the Steenrod algebra represented by any $\mathbf{K}_\lambda^{(r)}$ with $r > 1$.

Remark. One could make the notation $\mathbf{K}_\lambda^{(r)}$ more formal by considering these algebras as objects in a suitable category of unstable algebras over the Steenrod algebra "with higher Bocksteins". An object in this category $\mathcal{B}$ would be a spectral sequence (E_r, d_r) of graded algebras over $\mathbb{F}_p$ such that $E_1 \in \mathcal{K}$, $d_1 = \beta$, d_i are derivations and such that the usual formula for the higher Bockstein of x^p holds: If $x \in E_r^{2n}$ and $\beta_{(r)}(x) = y$ then

$$\beta_{(r+1)}\{x^p\} = \begin{cases} \{x^{p-1}y\} & \text{if } p > 2 \text{ or } p = 2,\ r > 1; \\ \{xy + Sq^{2n}y\} & \text{if } p = 2 \text{ and } r = 1. \end{cases}$$

In particular, the cohomology algebra of a space toghether with the Bockstein spectral sequence is an object in this category.

Let us denote by $B(p)\langle 3; p^r\rangle$ the homotopy fibre of the map $B(p) \to K(\mathbb{Z}, 3)$ of degree p^r. There is a fibration

$$B(p)\langle 3\rangle \longrightarrow B(p)\langle 3; p^r\rangle \longrightarrow B^2\mathbb{Z}/p^r.$$

It is an easy exercise to show that $H^*(B(p)\langle 3; p^r\rangle) \cong \mathbf{K}_0^{(r)}$. This shows that $\mathbf{K}_0^{(r)}$ is a genuine algebra over the Steenrod algebra. A key point in the proof of theorem 1 is to show that $\mathbf{K}_1^{(r)}$ is not the cohomology of any space (proposition 8). Although it is not strictly necessary for the proof of theorem 1 it seems appropiate to show that $\mathbf{K}_1^{(r)}$ is an unstable algebra over the Steenrod algebra, i. e. the Adem relations hold in $\mathbf{K}_1^{(r)}$. This can be proven in a straightforward way using the method of [**12**] suitably modified to include the case in which β acts non trivially. First one notices that $\mathbf{K}_1^{(r)}$ is an unstable $\bar{\mathcal{A}}$-algebra, where $\bar{\mathcal{A}}$ is the free associative algebra generated by β and P^i, $i > 0$. Then one observes that $\bar{\mathcal{A}}$ is a Hopf algebra and the diagonal $\bar{\psi}$ is such that if $r \in \bar{\mathcal{A}}$ is an Adem relation then

$$\bar{\psi}(r) \in \bar{\mathcal{A}} \otimes V + V \otimes \bar{\mathcal{A}}$$

where $V \subset \bar{\mathcal{A}}$ is the vector space spanned by the Adem relations and the elements βr where r is an Adem relation "of the first kind" (i. e. the ones without β). This shows that for an unstable $\bar{\mathcal{A}}$-algebra A to be an unstable $\mathcal{A}$-algebra it is enough to check the Adem relations on a set of algebra generators of A. Moreover, by unstability, only finitely many Adem relations can be non trivial on a given element and so, in general, to decide if a finitely generated algebra is an unstable algebra over the Steenrod algebra one just needs to check a finite number of Adem relations. In the case of the algebra $\mathbf{K}_1^{(r)}$ this checking is short and straightforward, the only significant Adem relation in this case beeing $P^pP^p = 2P^{2p} + P^{2p-1}P^1$. Details are left to the reader.

There are homomorphisms of unstable algebras over the Steenrod algebra

$$f : \mathbf{L} \to H^*(B\mathbb{Z}/p)$$

$$f : \mathbf{K}_\lambda^{(r)} \to H^*(B\mathbb{Z}/p)$$

which vanish on y and z and such that $f(x)$ is non trivial.

Let T denote the Lannes functor defined as left adjoint to $H^*(B\mathbb{Z}/p) \otimes -$ in the category $\mathcal{U}$ of unstable modules over the Steenrod algebra (see [**9**] for a full description of its properties.) When R is an unstable algebra over the Steenrod algebra then so is $T(R)$ and T becomes a functor in the category $\mathcal{K}$ of unstable algebras over the Steenrod algebra.

Let us write $H = H^*(B\mathbb{Z}/p)$. Given a $\mathcal{K}$-map $f : R \to H$, its adjoint restricts to a $\mathcal{K}$-map $T^0(R) \to \mathbb{F}_p$, where $T^0(R)$ is the subalgebra of $T(R)$ of all elements of degree zero. We define the connected component of $T(R)$ corresponding to f as:

$$T_f(R) = T(R) \otimes_{T^0(R)} \mathbb{F}_p.$$

Furthermore, T_f may be thought as a functor defined on the category $\mathcal{U}(R)$ of R-$\mathcal{U}$-modules and with values in the category of $T_f(R)$-$\mathcal{U}$-modules (cf. [**8**].) We can also consider $T_f(M)$ as an R-$\mathcal{U}$-module induced by the natural $\mathcal{K}$-map $\epsilon : R \to T_f(R)$ and then $\epsilon : M \to T_f(M)$ becomes a natural transformation of R-$\mathcal{U}$-modules. We recall from [**8**] that T_f is exact and commutes with suspensions, tensor products and direct limits.

We need to compute the value of T_f on the algebras $\mathbf{L}$ and $\mathbf{K}_\lambda^{(r)}$, where f denotes the homomorphisms that we have just described. The results are summarized in the following proposition:

Proposition 5. *(1) $\epsilon : K \to T_f(K)$ is an isomorphism for $K = \mathbf{K}_0^{(1)}$, $\mathbf{K}_0'$, $\mathbf{L}$.*

(2) $T_f(\mathbf{K}_1^{(1)}) \cong \mathbb{F}_p[x_2] \otimes E(y_3, w_1)$ with $\beta(x_2) = y_3$ and $P^1(y_3) = x_2^p w_1$. $x_2 \in \mathbf{K}_1^{(1)}$ is sent to $x_2 \in T_f(\mathbf{K}_1^{(1)})$ by ϵ. A similar result (with $\beta(x_2) = 0$) holds for $\mathbf{K}_1'$.

Proof. Consider first the case of $\mathbf{K}_0^{(1)}$. We work in the category $\mathcal{U}(\mathbf{K}_0^{(1)})$. There is an exact sequence in this category

$$0 \to z\mathbf{K}_0^{(1)} \to \mathbf{K}_0^{(1)} \to B \to 0$$

where $B \cong \mathbb{F}_p[x_2] \otimes E(y_3)$ is the algebra denoted by $\mathbf{B}_{0,1}$ in [**1**]. The exactness of T_f yields a commutative diagram of $\mathbf{K}_0^{(1)}$-$\mathcal{U}$-modules with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & z\mathbf{K}_0^{(1)} \arrow[r] \arrow[d, "\epsilon_1"'] & \mathbf{K}_0^{(1)} \arrow[r] \arrow[d, "\epsilon_2"'] & B \arrow[r] \arrow[d, "\epsilon_3"'] & 0 \\
0 \arrow[r] & T_f(z\mathbf{K}_0^{(1)}) \arrow[r] & T_f(\mathbf{K}_0^{(1)}) \arrow[r] & T_f(B) \arrow[r] & 0
\end{tikzcd}$$

Theorem 3.1 in [**1**] gives $B \cong T_h(B)$ where $h : B \to H$ is the obvious map. By lemma 3.2 in [**1**] we also have $B \cong T_f(B)$ because $f = h \circ g$ with $g : \mathbf{K}_0^{(1)} \to B$ and the hypothesis of lemma 3.2 in [**1**] is trivially satisfied. Hence, ϵ_3 is an isomorphism.

On the other side, since the Steenrod algebra acts trivially on $z \in \mathbf{K}_0^{(1)}$ there is an isomorphism $z\mathbf{K}_0^{(1)} \cong \Sigma^{2p+1}B$. Since T_f commutes with suspensions, ϵ_1 is an isomorphism and so is ϵ_2. The case of $\mathbf{K}'_0$ is analogous.

Let now M denote the algebra $M = \mathbb{F}_p[x_{2p^2}] \otimes E(y_{2p+1})$ with the Steenrod algebra acting trivially on y_{2p+1}. There is an exact sequence in $\mathcal{U}(\mathbf{L})$:

$$0 \to z\mathbf{L} \to \mathbf{L} \to M \to 0.$$

Notice that $z\mathbf{L} \cong \Sigma^{2p^2+1}M$. Since T_f commutes with suspensions, to show that $\epsilon : \mathbf{L} \to T_f\mathbf{L}$ is an isomorphism it suffices to prove that $\epsilon : M \to T_fM$ is an isomorphism. We can work now in the category $\mathcal{U}(\mathbb{F}_p[x_{2p^2}])$ and observe that $T_fM \cong T_gM$ where g is the non trivial map $g : \mathbb{F}_p[x_{2p^2}] \to H^*(B\mathbb{Z}/p)$ ([**1**], 3.2). We consider the exact sequence

$$0 \to yM \to M \to \mathbb{F}_p[x_{2^2}] \to 0$$

and notice that $yM \cong \Sigma^{2p+1}\mathbb{F}_p[x_{2p^2}]$ and $T_g\mathbb{F}_p[x_{2p^2}] \cong \mathbb{F}_p[x_{2p^2}]$. This concludes the analysis for the algebra $\mathbf{L}$.

For $\mathbf{K}_1^{(1)}$ we start in the same way as with $\mathbf{K}_0^{(1)}$ by considering the diagram of $\mathbf{K}_1^{(1)}$-$\mathcal{U}$-modules

$$\begin{tikzcd}
0 \arrow[r] & z\mathbf{K}_1^{(1)} \arrow[r] \arrow[d, "\epsilon_1"'] & \mathbf{K}_1^{(1)} \arrow[r] \arrow[d, "\epsilon_2"'] & B \arrow[r] \arrow[d, "\epsilon_3"'] & 0 \\
0 \arrow[r] & T_f(z\mathbf{K}_1^{(1)}) \arrow[r] & T_f(\mathbf{K}_1^{(1)}) \arrow[r] & T_f(B) \arrow[r] & 0
\end{tikzcd} \tag{1}$$

ϵ_3 is an isomorphism and we can observe that $z\mathbf{K}_1^{(1)} \cong \Sigma x^p B$. To compute $T_f(x^pB)$ we consider the exact sequence

$$0 \to x^pB \to B \to B/x^pB \to 0$$

Multiplication by x^p annihilates the right hand module. Hence, by proposition 2.3 in [**8**], $T_f(B/x^pB) = 0$ and so $T_f(x^pB) \cong B$ and the map $x^pB \to T_f(x^pB) \cong B$ is the natural inclusion. If we input now this fact into diagram (1) we easily get the structure of $T_f\mathbf{K}_1^{(1)}$. The case of $\mathbf{K}'_1$ is analogous. $\square$

4. Spectral sequence computations. In this section we compute the mod p cohomology of a space which is obtained from a space X by dividing by an action of the abelian group $B\mathbb{Z}/p$, in the cases in which the cohomology of X is of the form either $\mathbf{L}$ or $\mathbf{K}_\lambda^{(r)}$. We use the notation

$$P^{\Delta_j} = P^{p^j} \cdots P^p P^1, \quad j \geq 0,$$

$$Sq^{\Delta_j} = Sq^{2^j} \cdots Sq^2 Sq^1, \quad j \geq 0.$$

Proposition 6. *Let $B\mathbb{Z}/p \xrightarrow{i} X \xrightarrow{k} Y$ be a principal fibration with $H^*(X) \cong \mathbf{K}_0^{(r)}$ and $i^*(x_2) \neq 0$. Then $H^*(Y) \cong \mathbf{K}_\lambda^{(r+1)}$ for some $\lambda \in \{0,1\}$ and $k^*(x_2) = 0$, $k^*(y_3) = y_3$ and $k^*(z_{2p+1}) = z_{2p+1}$.*

Proof. Let us consider the Serre spectral sequence of the fibration $X \to Y \to B^2\mathbb{Z}/p$. Recall that

$$H^*(B^2\mathbb{Z}/p) = \begin{cases} \mathbb{F}_p[\iota, \beta P^1 \beta\iota, \ldots, \beta P^{\Delta_j}\beta\iota, \ldots] \otimes E(\beta\iota, P^1\beta\iota, \ldots, P^{\Delta_j}\beta\iota, \ldots), & p > 2 \\ \mathbb{F}_2[\iota, Sq^1\iota, \ldots, Sq^{\Delta_j}\iota, \ldots], & p = 2. \end{cases}$$

Observe that $x_2 \in H^*(X)$ has to be transgressive to $\beta\iota$ modulo units. Then, since the transgression commutes with the Steenrod operations, there are differentials

$$d(x^{p^j}) = \begin{cases} P^{\Delta_{j-1}}\beta\iota, & j \geq 1, \quad p > 2 \\ Sq^{\Delta_{j-1}}\iota, & j \geq 1, \quad p = 2 \end{cases}$$

In the case of p odd, Kudo's transgression theorem implies that there are differentials

$$d(x^{p^j(p-1)} P^{\Delta_{j-1}}\beta\iota) = \beta P^{\Delta_j}\beta\iota.$$

The spectral sequence with coefficients in $\mathbb{Z}_{(p)}$ shows that y_3 has to survive. Using this, one can compute all differentials in the spectral sequence and conclude that

$$E_\infty = \mathbb{F}_p[\iota_2] \otimes E(y_3, z_{2p+1}).$$

The spectral sequence with coefficients in $\mathbb{Z}_{(p)}$ yields also that $\beta_{(r+1)}$ has to be non trivial on the two dimensional class of Y. From here one concludes

$$H^*(Y) \cong \mathbb{F}_p[x_2] \otimes E(y_3, z_{2p+1})$$

and the class z_{2p+1} can be chosen such that $P^1(y_3) = z_{2p+1}$. In the case of $p = 2$ one needs to prove that y_3 and z_{2p+1} are indeed exterior generators. This follows easily by the Bockstein spectral sequence. The conclusion follows from proposition 4. $\square$

The next spectral sequence that we have to study needs a more involved analysis.

Proposition 7. *Let $B\mathbb{Z}/p \xrightarrow{i} X \xrightarrow{k} Y$ be a principal fibration with $H^*(X) \cong \mathbf{L}$ and $i^*(x_{2p^2}) \neq 0$. Then $H^*(Y) \cong \mathbf{K}_\lambda^{(1)}$ for some $\lambda \in \{0,1\}$ and k^* is trivial.*

Proof. Let us consider the Serre spectral sequence of $X \longrightarrow Y \xrightarrow{l} B^2\mathbb{Z}/p$. Obviously, $y_{2p+1} \in H^*(Y)$ has to be transgressive and our first goal is to compute the transgression of y_{2p+1}. By dimensional reasons,

$$\tau(y_{2p+1}) = \begin{cases} \delta\beta P^1\beta\iota + \mu\iota^{p+1}, & p > 2 \\ \delta(Sq^1\iota)^2 + \mu\iota^3, & p = 2 \end{cases}$$

If we notice now that $\beta(y_{2p+1}) = 0$ and that the Steenrod operations commute with the transgression, we easily get $\mu = 0$. We want to prove that $\delta \neq 0$. Assume that $\delta = 0$ and so y_{2p+1} survives to E_∞. In this case, the first differential which may be non trivial appears in total degree $2p^2$. Hence, we know $H^*(Y)$ up to dimension $2p^2 - 1$ and it turns out to be generated by

$$x_2 = l^*(\iota), \quad \beta x_2, \quad P^1\beta x_2, \quad \beta P^1\beta x_2, \quad \bar{y}.$$

We input now this information on $H^*(Y)$ into the Serre spectral sequence of the fibration $B\mathbb{Z}/p \to X \to Y$. Obviously, $u \in H^1(B\mathbb{Z}/p)$ has to be transgressive to $x_2 \in H^2(Y)$ (up to units). Then, $v \in H^2(B\mathbb{Z}/p)$ transgresses to βx_2, v^p transgresses to $P^1\beta x_2$, v^{p^2} transgresses to $P^pP^1\beta x_2$ and, in the case of p odd, Kudo's formula determines differentials on $v^{p-1}\beta x_2$ and $v^{p(p-1)}P^1\beta x_2$. In this form, one is able to fully determine the differentials in this spectral sequence up to total degree $2p^2 - 1$. Notice that $v^{p^2} \in H^*(B\mathbb{Z}/p)$ is in the image of $H^*(X)$, hence it has to survive and so we get that $P^pP^1\beta x_2 \in H^*(Y)$ has to be decomposable in terms of x_2, βx_2, $P^1\beta x_2$, $\beta P^1\beta x_2$ and $\bar{y}$. Hence, the same is true for $\beta P^pP^1\beta x_2 \in H^*(Y)$.

Let us return now to the Serre spectral sequence of $X \to Y \to B^2\mathbb{Z}/p$. Since, as we have just proved, $\beta P^pP^1\beta x_2$ has to be decomposable in $H^*(Y)$, there should be a differential in this spectral sequence starting at total degree $2p^2 + 1$. hence, z_{2p^2+1} has to support a non-trivial differential. This is a contradiction since $z_{2p+1} = P^p y_{2p+1}$ and y_{2p+1} is asumed to be transgressive to zero. This contradiction proves that $y_{2p+1} \in H^*(X)$ cannot survive and, indeed, up to units we have

$$\tau(y_{2p+1}) = \begin{cases} \beta P^1\beta\iota, & p > 2 \\ (Sq^1\iota)^2, & p = 2 \end{cases}$$

with zero indeterminacy. From here we get that $z_{2p^2+1} \in H^*(X)$ is also transgressive and

$$\tau(z_{2p^2+1}) = \begin{cases} \beta P^pP^1\beta\iota, & p > 2 \\ (Sq^2Sq^1\iota)^2, & p = 2 \end{cases}$$

modulo the indeterminacy. Let us study now the differentials of the spectral sequence of $X \to Y \to B^2\mathbb{Z}/p$ when applied to $x_{2p^2} \in H^*(X)$. Notice that this element cannot survive because z does not. If $p \neq 2$, the y_{2p+1}-row has disappeared when the first non trivial differential on x_{2p^2} appears because $\tau(y_{2p+1})$ is not a zero divisor in $H^*(B^2\mathbb{Z}/p)$ and so x_{2p^2} is transgressive. By naturality and up to units we have

$$\tau(x_{2p^2}) = P^p P^1 \beta\iota + d \tag{1}$$

If $p = 2$ the analysis needs to be slightly different but one also concludes that x_{2p^2} is transgressive and formula (1) holds.

Now, the knowledge about the transgressions of x_{2p^2}, y_{2p+1} and z_{2p^2+1} as well as the commutativity of the transgression with the Steenrod operations and Kudo's transgression formula, allow us to fully determine the differentials in the spectral sequence of the fibration $X \to Y \to B^2\mathbb{Z}/p$. The conclusion is that

$$H^*(B^2\mathbb{Z}/p) \to H^*(Y)$$

is an epimorphism and $H^*(Y)$ is generated by the images of ι, $\beta\iota$ and $P^1\beta\iota$. In the case of $p = 2$, $(Sq^1\iota)^2$ and $(Sq^2Sq^1\iota)^2$ are the transgressions of y_5 and z_9, respectively. Hence, we have, in any case

$$H^*(Y) \cong \mathbb{F}_p[x_2] \otimes E(\beta(x_2), P^1\beta(x_2))$$

and so, by proposition 4, $H^*(Y) \cong \mathbf{K}_\lambda^{(1)}$ for some $\lambda \in \{0, 1\}$. □

5. Non-realizability of $\mathbf{K}_1^{(r)}$. In this section we will prove that $\mathbf{K}_1^{(r)}$ cannot be the mod p cohomology of any space. Roughly speaking, this non-realizability result will follow from the fact that the homomorphism $\mathbf{K}_1^{(r)} \to T_f\mathbf{K}_1^{(r)}$ is not compatible with the Bockstein spectral sequence.

Proposition 8. *There is no space X such that $H^*(X) \cong \mathbf{K}_1^{(r)}$ for any $r > 0$.*

Proof. Let X be a space such that $H^*(X) \cong \mathbf{K}_1^{(r)}$. By the same argument as in proposition 3 we can assume, without loss of generality, that X is p-complete and simply connected. Let $f : \mathbf{K}_1^{(r)} \to H^*(B\mathbb{Z}/p)$ be the non trivial homomorphism considered before. Then by [**9**; 3.1.1] there is a map $\phi : B\mathbb{Z}/p \to X$ inducing f in mod p cohomology. By proposition 5 we have $T_f\mathbf{K}_1^{(r)} \cong \mathbb{F}_p[x_2] \otimes E(y_3, w_1)$.

Let $\{P_sX\}_{s>0}$ denote the Postnikov tower of X. Then, the main theorem in [6] implies that

$$T_f\mathbf{K}_1^{(r)} \cong \varinjlim H^*(\mathrm{map}(B\mathbb{Z}/p, P_sX)_{\phi_s}).$$

For a given s, consider the map induced in cohomology by the evaluation map

$$e^* : H^*(P_sX) \to H^*(\mathrm{map}(B\mathbb{Z}/p, P_sX)_{\phi_s}).$$

In the limit when $s \to \infty$ this is the homomorphism $\mathbf{K}_1^{(r)} \to \mathbb{F}_p[x_2] \otimes E(y_3, w_1)$ given by $x \mapsto x$, $y \mapsto y$, $z \mapsto x^p w$. It is clear that in any argument which involves only finitely many cohomology classes and finitely many primary and secondary operations, by taking s large enough we can substitute $H^*(P_sX)$ by $\mathbf{K}_1^{(r)}$ and $H^*(\mathrm{map}(B\mathbb{Z}/p, P_sX)_\phi)$ by $\mathbb{F}_p[x_2] \otimes E(y_3, w_1)$.

Let us analyze now the Bockstein spectral sequence of $H^*(X) \cong \mathbf{K}_1^{(r)}$ up to degree $2p+3$. We have $\beta_{(r)}(x) = y$ and so E_{r+1} only contains x^p, $x^{p-1}y$ and z up to that degree. If p is odd or $p = 2$ and $r > 1$ it is well known that $\beta_{(r+1)}(x^p) = x^{p-1}y$. If $p = 2$ and $r = 1$ then $\beta_{(2)}(x^2) = xy + z$. In any case, z is a permanent cycle.

In the Bockstein spectral sequence of $\mathrm{map}(B\mathbb{Z}/p, P_sX)_\phi$ for s big enough we have by naturality $\beta_i(w) = 0$ for $i < N$ and N arbitrarily large and also $\beta_{(r)}(x) = y$. Hence,

$$\beta_{(r+1)}(x^p w) = x^{p-1}yw \neq 0$$

and this is a contradiction since z maps to $x^p w$ by the evaluation map. $\square$

Remark. Let $\mathcal{B}$ bet the category of unstable algebras over the Steenrod algebra "with higher Bocksteins" discussed in section 3 and recall that the cohomology algebra of a space is an object in this category. There is a forgetful functor from $\mathcal{B}$ to $\mathcal{K}$. Then the above proof shows that the T functor cannot be lifted to $\mathcal{B}$.

Remark. The above proof is a further example of the usefulness of the T functor to deal with the classical problem of realizability of unstable algebras over the Steenrod algebra as cohomology rings. Other examples that we would like to mention here are the celebrated theorem by Dwyer-Miller-Wilkerson on the non-realizability of inseparable subalgebras of $H^*(BT^n)$ for p odd ([**7**]) and also the proof of the Cooke conjecture ([**1**], [**2**]) on the non-realizability of algebras of the form $\mathbb{F}_p[x] \otimes E(\beta(x))$.

6. Proof of theorem 1. Like in proposition 3, we can assume, without loss of generality, that X is p-complete and simply connected. Let $f : \mathbf{L} \to H^*(B\mathbb{Z}/p)$ be the non trivial homomorphism considered before. Then by [**9**; 3.1.1] there is a map $\phi : B\mathbb{Z}/p \to X$ inducing f in mod p cohomology. By proposition 5 we have $T_f\mathbf{L} \cong \mathbf{L}$. Then, [**9**; 3.2.1] shows that

$$H^*(\mathrm{map}(B\mathbb{Z}/p, X)_\phi) \cong \mathbf{L}$$

where $\mathrm{map}(B\mathbb{Z}/p, X)_\phi$ is the space of all maps $B\mathbb{Z}/p \to X$ homotopic to ϕ. Hence, up to p-completion, $\mathrm{map}(B\mathbb{Z}/p, X)_\phi$ is homotopy equivalent to X. Observe now that $B\mathbb{Z}/p$ is a connected abelian simplicial group and the action of $B\mathbb{Z}/p$ on itself by right translations induces an action of $B\mathbb{Z}/p$ on the space $\mathrm{map}(B\mathbb{Z}/p, X)_\phi$. If X_1 is the homotopy quotient of $\mathrm{map}(B\mathbb{Z}/p, X)_\phi$ by this action, we have a fibration

$$B\mathbb{Z}/p \xrightarrow{\phi} X \longrightarrow X_1.$$

By proposition 7 we have $H^*(X_1) \cong \mathbf{K}_\lambda^{(1)}$ for some $\lambda \in \{0, 1\}$. But proposition 8 shows that $\mathbf{K}_1^{(1)}$ cannot be the cohomology of any space. Hence, $H^*(X_1) \cong \mathbf{K}_0^{(1)}$. Consider now the homomorphism $f : \mathbf{K}_0^{(1)} \to H^*(B\mathbb{Z}/p)$. If we substitute X_1 by its p-completion we can find a map $\phi : B\mathbb{Z}/p \to X_1$ inducing f in cohomology. Let us consider the mapping space $\mathrm{map}(B\mathbb{Z}/p, X_1)_\phi$. According to [**9**; 3.2.1] the cohomology of this mapping space is isomorphic to $T_f\mathbf{K}_0^{(1)} \cong \mathbf{K}_0^{(1)}$ (proposition 5). Hence, the evaluation map gives a homotopy equivalence between the p-completion of this mapping space and X_1. In the same way as before, this allows us to construct a fibration

$$B\mathbb{Z}/p \xrightarrow{\phi} X_1 \longrightarrow X_2$$

and proposition 6 plus the non realizability of $\mathbf{K}_1^{(r)}$ (proposition 8) gives $H^*(X_2) \cong \mathbf{K}_0^{(2)}$. It is clear that this process can be repeated again and again producing a sequence of spaces and maps

$$X \to X_1 \to X_2 \to X_3 \to \cdots$$

with $H^*(X_r) \cong \mathbf{K}_0^{(r)}$, $r > 0$. Let X_∞ be the homotopy colimit of this sequence of spaces. The cohomology of X_∞ can be calculated by the classical short exact sequence

$$1 \to \varprojlim{}^1 H^{*+1}(X_j) \to H^*(X_\infty) \to \varprojlim H^*(X_j) \to 1 \ .$$

The first term vanishes because all the groups are finite, and

$$\varprojlim H^*(X_j) \cong H^*(B(p)).$$

Hence, by proposition 3 $(X_\infty)\hat{}_p \simeq B(p)\hat{}_p$. Let F be the homotopy fibre of the map $X_1 \to X_\infty$. An elementary spectral sequence argument shows that $H^*(F) \cong H^*(BS^1)$. Hence, $F \simeq (BS^1)\hat{}_p = K(\hat{\mathbb{Z}}_p, 2)$ and the fibration is principal because $B(p)\hat{}_p$ is simply connected. Then X_1 is the homotopy fibre of a map $B(p)\hat{}_p \to K(\hat{\mathbb{Z}}_p, 3)$. The cohomology of X_1 forces this map to be of degree $p\alpha$ where α is a unit in $\hat{\mathbb{Z}}_p$. Hence, $X_1 \simeq B(p)\langle 3; p\rangle\hat{}_p$. Now X is the fibre of the map $B(p)\langle 3; p\rangle\hat{}_p \to B^2\mathbb{Z}/p$ classified by the two dimensional class x_2. Hence, $X \simeq B(p)\langle 3\rangle\hat{}_p$ and the proof is complete. □

It is obvious from the above proof that the same arguments show the homotopy uniqueness of the spaces $B(p)\langle 3; p^r\rangle$:

Theorem 9. *Let X be such that $H^*(X) \cong \mathbf{K}_0^{(r)}$. Then $\hat{X}_p \simeq B(p)\langle 3; p^r\rangle\hat{}_p$.* □

References

[**1**] J. Aguadé, C. Broto and D. Notbohm, *Homotopy Classification of Spaces With Interesting Cohomology and a Conjecture of Cooke, Part I*, Topology **33** (1994), 455–492.

[**2**] J. Aguadé, C. Broto and D. Notbohm, *A mod 2 analogue of a conjecture of Cooke*, (submitted).

[**3**] V. Belfi and C. Wilkerson, *Some Examples in the Theory of P-completions*, Indiana U. Math. J. **25** (1976), 565–576.

[**4**] A. Bousfield and D. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Math. 304, Springer Verlag, 1972.

[**5**] E. Dror Farjoun, *Homotopy localization and v_1-periodic spaces*, Proceedings of the 1990 Barcelona Conference on Algebraic Topology, Lecture Notes in Math. 1509, Springer Verlag, 1992.

[6] E. Dror Farjoun and J. Smith, *A geometric interpretation of Lannes' functor T*, Astérisque **191** (1990), 87–95.

[**7**] W.G.Dwyer, H. Miller and C.W. Wilkerson, *Homotopical Uniqueness of classifying spaces*, Topology **31** (1992), 29–45.

[**8**] W.G. Dwyer and C.W. Wilkerson, *Smith theory and the functor T*, Comm. Math. Helv. **661** (1991), 1–17.

[**9**] J. Lannes, *Sur les espaces fonctionnels dont la source est le classifiant d'un p-groupe abélien élémentaire*, Publ. Math. IHES **75** (1992), 135–244.

[**10**] M. Mimura and H. Toda, *Cohomology operations and the homotopy of compact Lie groups-I*, Topology **9** (1970), 317–336.

[**11**] J.A. Neisendorfer, *Localization and Connected Covers of Finite Complexes*, to appear in Contemp. Math.

[**12**] N.E. Steenrod, *Polynomial algebras over the algebra of cohomology operations*, Proceedings, Neuchâtel 1970, Lecture Notes in Math. 196, Springer Verlag, 1971.

[**13**] H. Toda, *On homotopy groups of S^3-bundles over spheres*, J. Math. Kioto Univ. **2-2** (1963), 193–207.

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