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THE TORSION INDEX OF A p-COMPACT GROUP

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Abstract. We extend the theory of torsion indices of compact connected Lie groups to p-compact groups and compute these indices in all cases.

1. Introduction and statement of results

The torsion index of a compact connected Lie group was defined by Grothendieck in 1958 ([10]) and has been investigated by several authors ([14], [6], [15], etc.). Recently, the computation of the torsion indices of all simply connected compact Lie groups has been completed (see [16]). Since we are going to work at a single prime p, instead of the torsion index of a Lie group G, we want to consider its p-primary part $t_p(G)$. We summarize the properties of $t_p(G)$ which are relevant to the present work in the following proposition ($\mathbb{Z}_p$ denotes the ring of p-adic integers).

Theorem 1.1. *Let p be a prime and let G be a compact connected Lie group with a maximal torus T and corresponding Weyl group W. The positive integer $t_p(G)$ has the following properties:*

- (TI1) *If A is a finite abelian p-subgroup of G, then A has a subgroup of index dividing $t_p(G)$ which is contained in a conjugate of T.*
- (TI2) *$t_p(G)$ kills the kernel and the cokernel of the homomorphism*
$$H^*(BG;\mathbb{Z}_p) \to H^*(BT;\mathbb{Z}_p)^W.$$
- (TI3) *$H^*(G/T;\mathbb{Z}_p)$ is torsion free and concentrated in even degrees $\leq N = \dim(G) - \operatorname{rank}(G)$, with $H^N(G/T;\mathbb{Z}_p) \cong \mathbb{Z}_p$. Then, $t_p(G)$ is the order of the cokernel of $H^N(BT;\mathbb{Z}_p) \to H^N(G/T;\mathbb{Z}_p)$.*
- (TI4) *If p is not a torsion prime for G, then $t_p(G) = 1$.*

Notice that the property (TI3) can be taken as a *definition* of the (p-primary) torsion index $t_p(G)$. The other properties are well known and can be found in [15], which provides proofs or references for all of them. Actually, the properties above are usually stated using $H^*(-;\mathbb{Z})$ and $t(G) = \prod_p t_p(G)$ instead of $H^*(-;\mathbb{Z}_p)$ and $t_p(G)$, but it is easy to see that both formulations are indeed equivalent. For property (TI2) one should notice that $H^*(BT;\mathbb{Z}_p)^W = H^*(BT;\mathbb{Z})^W \otimes \mathbb{Z}_p$. This follows from exactness of $- \otimes \mathbb{Z}_p$ and the fact that the elements invariant under W can be viewed as the kernel of the homomorphism $\bigoplus_{g\in W}(1-g)$.

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The purpose of this paper is to extend the theorem above to connected p-compact groups ([8]) and to compute the torsion indices in all cases. We prove:

Theorem 1.2. *Let p be a prime and let X be a connected p-compact group with maximal torus T and corresponding Weyl group W. There is an integer $t_p(X)$ such that:*

(1) *The properties* (TI1), (TI2), (TI3), (TI4) *in Theorem* 1.1 *hold after replacing G with X.*
(2) *If X is exotic, then $t_p(X) = 1$ for p odd and $t_2(X) = 2$.*

Here we use the work *exotic* with the same meaning as in [1]: A p-compact group X is exotic if the associated pseudoreflection representation of the Weyl group of X over the p-adic field is irreducible and does not come from a reflection group over $\mathbb{Z}$.

Section 2 deals with the (easier) odd prime case, and we show that if we define $t_p(X) = 1$ for any exotic X, then properties (TI1), (TI2), (TI3), (TI4) hold true. The hardest part consists of computing the torsion index of the only exotic 2-compact group, which we (following [12]) denote G_3 (other authors denote it as $DI(4)$). We need a comprehensive review of the cohomology of G_3 and BG_3 (section 3) and some computations on the cohomology of the exotic homogeneous space $G_3/\operatorname{Spin}(7)$ (section 4) before we can prove that $t_2(G_3) = 2$. Finally, we prove Theorem 1.2 in section 6.

2. The odd prime case

The classification theorem for p-compact groups ([2]) tells us that any connected p-compact group X splits uniquely as a product $X \cong G^{\wedge}_p \times X_1$, where G is a compact connected Lie group and X_1 is a product of exotic p-compact groups. Notice that the splitting is as p-compact groups and not just as spaces. This splitting implies that it is enough to prove Theorem 1.2 for each exotic p-compact group, since it is already known to be true for the (p-completions of) compact connected Lie groups. Let us discuss this in some more detail. If Theorem 1.2 holds for the p-compact groups X_1 and X_2, let $X = X_1 \times X_2$ and let us define $t_p(X) = t_p(X_1)\, t_p(X_2)$. We need to check that properties (TI1) to (TI4) hold for X if they hold for X_1 and X_2. (TI4) is trivial and (TI3) is straightforward. To prove (TI2) let us observe that the kernel of $\gamma : H^*(BX; \mathbb{Z}_p) \to H^*(BT; \mathbb{Z}_p)^W$ is equal to the torsion elements in $H^*(BX; \mathbb{Z}_p)$. If X is of Lie type, this is well known (cf. [9]). If X is exotic and $p = 2$ (i.e. $X = G_3$), then this is assertion 4 in [12]; and if p is odd, this is proven in [1]. Then, it is clear that $t_p(X_1)\, t_p(X_2)$ kills the kernel of γ. It is obvious that $t_p(X_1)\, t_p(X_2)$ kills the cokernel of γ as well. Finally, (TI1) follows easily since we can use the theory of kernels of homomorphisms between p-compact groups which is developed in [8], section 7.

Let us assume now that p is odd and let X be an exotic p-compact group. These objects are very well understood. In particular, they satisfy the following properties (see [1]). Let T and W denote a maximal torus of X and the corresponding Weyl group, respectively. Then:

(1) X is simply connected and center free and $H^*(X; \mathbb{Z}_p)$ is torsion free.
(2) The natural map $BT \to BX$ induces an isomorphism

$$H^*(BX; \mathbb{Z}_p) \cong H^*(BT; \mathbb{Z}_p)^W.$$

In particular, $H^*(BX;\mathbb{Z}_p)$ is concentrated in even degrees.

(3) $H^*(X/T;\mathbb{Z}_p)$ is a free $\mathbb{Z}_p$-module concentrated in even degrees. Moreover (see [13], th. 7.5.1) $H^*(X/T;\mathbb{Z}_p)\otimes\mathbb{Q}$ is a Poincaré duality algebra with fundamental class in degree $\dim(X)-\operatorname{rank}(X)$. Actually, as a W-module, $H^*(X/T;\mathbb{Z}_p)\otimes\mathbb{Q}$ coincides with the regular representation of W.

We also need another property of p-compact groups (which holds also for $p=2$) that follows from the work in [5].

(4) If X is any p-compact group such that $H^*(BX;\mathbb{F}_p)$ is concentrated in even degrees, then any finite abelian p-subgroup of X is conjugated to a subgroup of the maximal torus of X. In particular, this holds for any product of exotic p-compact groups for p odd.

Theorem 1.2 for p odd follows immediately from all these properties of p-compact groups. □

3. The 2-compact group G_3 and its maximal torus

In this section we recollect several properties of G_3 that we need in the forthcoming sections. We state these properties without proof because either they can be found in the papers [7], [12], [4], [11] or they follow from straightforward computations that are left to the reader.

As is well known, G_3 is an exotic connected 2-compact group of rank three whose Weyl group W is the reflection group number 24 in the Shephard-Todd list of finite complex reflection groups. Its existence was established by Dwyer and Wilkerson in [7]. We remind the reader that some authors call this 2-compact group $DI(4)$, but we follow the notation used in [12]. As an abstract group, W is isomorphic to $\mathbb{Z}/2\mathbb{Z}\times GL_3(\mathbb{F}_2)$ and for a maximal torus T of G_3, there is a basis $\{\epsilon_1,\epsilon_2,\epsilon_3\}$ of $H^2(BT;\mathbb{Z}_2)$ such that the action of W on $H^*(BT;\mathbb{Z}_2)$ is given by the pseudoreflections

$$s_1=\begin{pmatrix}-1 & -\bar\alpha & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\quad s_2=\begin{pmatrix}1 & 0 & 0\\ -\alpha & -1 & 1\\ 0 & 0 & 1\end{pmatrix},\quad s_3=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 1 & 1 & -1\end{pmatrix},$$

where $\alpha,\bar\alpha\in\mathbb{Z}_2$ are the roots of x^2-x+2 chosen in such a way that α is odd and $\bar\alpha$ is even.

G_3 has $\mathrm{Spin}(7)$ as a 2-compact subgroup of maximal rank. This means that there is a map $\phi: B\,\mathrm{Spin}(7)^\wedge_2\to BG_3$ whose homotopical fibre is $\mathbb{F}_2$-finite. It is natural to denote this fibre by $G_3/\mathrm{Spin}(7)$. The restriction of ϕ to a maximal torus of $\mathrm{Spin}(7)$ is a maximal torus of G_3.

There is a subgroup $V\subset\mathrm{Spin}(7)$ (explicitly described in [7]) which is an elementary abelian 2-group of rank four and such that the homomorphisms

$$H^*(BG_3;\mathbb{F}_2)\xrightarrow{\phi^*}H^*(B\,\mathrm{Spin}(7);\mathbb{F}_2)\xrightarrow{k^*}H^*(BV;\mathbb{F}_2)\cong\mathbb{F}_2[V^*]$$

are monomorphisms (k^* is induced by the inclusion $V\subset\mathrm{Spin}(7)$). Moreover, the image of $(\phi k)^*$ coincides with the rank four Dickson algebra which is the algebra of invariants of $H^*(BV;\mathbb{F}_2)$ under the action of the full linear group $GL(V^*)$, and the image of k^* coincides with the algebra of invariants $H^*(BV;\mathbb{F}_2)^H$ where $H\subset GL(V^*)$ can be described, in some appropriate basis of V^*, as the set of matrices with first row equal to $(1,0,0,0)$. These algebras of invariants are well known (also as algebras over the Steenrod algebra) and we have isomorphisms (subscripts denote degrees)

$$H^*(BG_3;\mathbb{F}_2) \cong \mathbb{F}_2[c_8, c_{12}, c_{14}, c_{15}],$$
$$H^*(B\operatorname{Spin}(7);\mathbb{F}_2) \cong \mathbb{F}_2[d_4, d_6, d_7, d_8],$$

where the generators c_i and d_i can be explicitly described. In particular, we can see that ϕ^* is given by $\phi^*(c_8) = d_4^2 + d_8$, $\phi^*(c_{12}) = d_6^2 + d_4d_8$, $\phi^*(c_{14}) = d_7^2 + d_6d_8$, $\phi^*(c_{15}) = d_7d_8$. Sq^1 vanishes on d_4, d_7, d_8, while $Sq^1(d_6) = d_7$.

As was said before, a maximal torus T of $\operatorname{Spin}(7)$ is also a maximal torus of G_3. We have maps

$$BT_2^\wedge \xrightarrow{i} B\operatorname{Spin}(7)_2^\wedge \xrightarrow{\phi} BG_3$$

and we can view the Weyl group W_1 of $\operatorname{Spin}(7)$ as a subgroup of W, namely $W_1 = \langle s_1, s_2, s_1s_3s_2s_1s_2s_3s_1\rangle$. It is known that the homomorphism

$$i^* : H^*(B\operatorname{Spin}(7);\mathbb{Z}_2) \to H^*(BT;\mathbb{Z}_2)^{W_1}$$

is surjective and its kernel coincides with the ideal of torsion elements. The integral invariants of W_1 are computed in [4]. They turn out to form a polynomial algebra on generators of degrees 4, 8, 12:

$$H^*(BT;\mathbb{Z}_2)^{W_1} \cong \mathbb{Z}_2[u_4, u_8, u_{12}].$$

Choosing an appropriate basis $\{x_1, x_2, A\}$ of $H^2(BT;\mathbb{Z}_2)$, these generators are

$$\begin{aligned} u_4 &= (1/2)(x_1^2 + x_2^2 + x_3^2),\\ u_8 &= (1/16)(x_1^4 + x_2^4 + x_3^4 - 2x_1^2x_2^2 - 2x_1^2x_3^2 - 2x_2^2x_3^2),\\ u_{12} &= x_1^2x_2^2x_3^2, \end{aligned}$$

where we have used the notation $x_3 = 2A - x_1 - x_2$, and one can check that in spite of the denominators, these polynomials belong to $\mathbb{Z}_2[x_1, x_2, A]$.

The generators u_4, u_8 and u_{12} have a rather simple form as polynomials on x_1, x_2, A, but this basis of $H^2(BT;\mathbb{Z}_2)$ does not coincide with the basis $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ that we have used to describe the action of W on $H^*(BT;\mathbb{Z}_2)$. The matrix that expresses $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ in terms of $\{x_1, x_2, A\}$ is

$$\begin{pmatrix} 0 & -\bar\alpha/2 & -(1+\alpha)/2\\ 1 & 0 & -(1+\alpha)/2\\ 0 & \bar\alpha & \alpha \end{pmatrix} \in GL_3(\mathbb{Z}_2).$$

Using this matrix we can express the generators u_4, u_8, u_{12} as polynomials in ϵ_1, ϵ_2, ϵ_3 and so we have an explicit description of the homomorphism

$$\mathbb{Z}_2[u_4, u_8, u_{12}] = H^*(B\operatorname{Spin}(7);\mathbb{Z}_2)/\text{Torsion} \to H^*(BT;\mathbb{Z}_2) = \mathbb{Z}_2[\epsilon_1, \epsilon_2, \epsilon_3].$$

Finally, we want to use this to describe the homomorphism

$$\mathbb{F}_2[d_4, d_6, d_7, d_8] = H^*(B\operatorname{Spin}(7);\mathbb{F}_2) \xrightarrow{i^*} H^*(BT;\mathbb{F}_2) = \mathbb{F}_2[\epsilon_1, \epsilon_2, \epsilon_3].$$

In the Bockstein spectral sequence for $B\operatorname{Spin}(7)$ we have $E_2 = E_\infty = \mathbb{F}_2[\bar d_4, \bar d_8, \bar d_6^2]$, and the surjection

$$j : H^*(B\operatorname{Spin}(7);\mathbb{Z}_2)/\text{Torsion} \to E_\infty$$

is given by $j(u_4) = \bar{d}_4$, $j(u_8) = \bar{d}_8$, $j(u_{12}) = \bar{d}_6^2$. From this it is straightforward to perform the computations that yield

$$\begin{aligned}
i^*(d_4) &= \epsilon_1^2 + \epsilon_1\epsilon_2 + \epsilon_2^2,\\
i^*(d_6) &= Sq^2 i^*(d_4) = \epsilon_1^2\epsilon_2 + \epsilon_1\epsilon_2^2,\\
i^*(d_7) &= 0,\\
i^*(d_8) &= \epsilon_1\epsilon_2\epsilon_3(\epsilon_1+\epsilon_2+\epsilon_3) + \epsilon_3^2(\epsilon_1+\epsilon_2+\epsilon_3)^2.
\end{aligned}$$

4. The exotic homogeneous space $G_3/\operatorname{Spin}(7)$

In this section we want to investigate the cohomology of the exotic homogeneous space $G_3/\operatorname{Spin}(7)$. The computations presented here are probably known to experts, but it may be worthwhile to work them out here in some detail.

Let us consider the fibration $G_3/\operatorname{Spin}(7) \xrightarrow{\rho} B\operatorname{Spin}(7)^\wedge_2 \to BG_3$ and let $V \subset \operatorname{Spin}(7)$ denote the elementary abelian 2-group of rank 4 considered in the preceding section. To simplify the notation, let us write $S = H^*(BV;\mathbb{F}_2)$. Then, we have $H^*(B\operatorname{Spin}(7);\mathbb{F}_2) = S^H$ and $H^*(BG_3;\mathbb{F}_2) = S^G$ for $G = GL_4(\mathbb{F}_2)$.

The computation of $H^*(G_3/\operatorname{Spin}(7);\mathbb{F}_2)$ is best worked out with the Eilenberg-Moore spectral sequence

$$\operatorname{Tor}_{H^*(BG_3;\mathbb{F}_2)}(H^*(B\operatorname{Spin}(7);\mathbb{F}_2),\mathbb{F}_2) \quad\Rightarrow\quad H^*(G_3/\operatorname{Spin}(7);\mathbb{F}_2).$$

Here the key observation is that $H^*(B\operatorname{Spin}(7);\mathbb{F}_2) = S^H$ is a free module over $H^*(BG_3;\mathbb{F}_2) = S^G$ because of the following classic argument. S is an integral extension of S^G; hence S^H is also an integral extension of S^G and, since S^H is a finitely generated algebra, we obtain that S^H is a finitely generated S^G-module. But both S^H and S^G are polynomial algebras, and we can apply [3], Chap. V, 5.5, or [13], 6.7.1, to conclude that S^H is S^G-free.

Hence the Eilenberg-Moore spectral sequence collapses to an isomorphism

$$H^*(G_3/\operatorname{Spin}(7);\mathbb{F}_2) \cong \mathbb{F}_2[\bar{d}_4,\bar{d}_6,\bar{d}_7]/(\bar{d}_6{}^2 + \bar{d}_4{}^3, \bar{d}_7{}^2 + \bar{d}_4{}^2\bar{d}_6, \bar{d}_4{}^2\bar{d}_7),$$

where $\bar{d}_4$, $\bar{d}_6$, $\bar{d}_7$ are the images of $d_4, d_6, d_7 \in H^*(B\operatorname{Spin}(7);\mathbb{F}_2)$, respectively. It is rather easy to completely work out the algebra structure of $H^*(G_3/\operatorname{Spin}(7);\mathbb{F}_2)$. We obtain the following:

(1) The Poincaré series of $H^*(G_3/\operatorname{Spin}(7);\mathbb{F}_2)$ is

$$1 + t^4 + t^6 + t^7 + t^8 + t^{10} + t^{11} + t^{12} + t^{13} + t^{14} + t^{16} + t^{17} + t^{18} + t^{20} + t^{24}$$

and the Euler characteristic is $7 = [W : H]$.

(2) An additive basis for $H^*(G_3/\operatorname{Spin}(7);\mathbb{F}_2)$ is given by

$$\left\{\bar{d}_4{}^i,\ i = 0,\ldots,6,\ \bar{d}_6,\ \bar{d}_7,\ \bar{d}_4\bar{d}_6,\ \bar{d}_4\bar{d}_7,\ \bar{d}_6\bar{d}_7,\ \bar{d}_4{}^2\bar{d}_6,\ \bar{d}_4\bar{d}_6\bar{d}_7,\ \bar{d}_4{}^3\bar{d}_6\right\}.$$

(3) $H^*(G_3/\operatorname{Spin}(7);\mathbb{F}_2)$ is a Poincaré duality algebra with top class $\bar{d}_4{}^6$ (see [13], 6.5).

(4) The Bockstein spectral sequence of $H^*(G_3/\operatorname{Spin}(7);\mathbb{F}_2)$ collapses after the second term; i.e. $H^*(G_3/\operatorname{Spin}(7);\mathbb{Z}_2)$ has only torsion of order 2. We have

$$H^*(G_3/\operatorname{Spin}(7);\mathbb{Z}_2)/\text{Torsion} \cong \mathbb{Z}_2[\bar{a}]/\bar{a}^7$$

and

$$H^*(G_3/\operatorname{Spin}(7);\mathbb{Z}_2) \cong \mathbb{Z}_2[\bar{a},\bar{c}]/(\bar{a}^7,\ \bar{c}^3,\ \bar{a}^2\bar{c},\ 2\bar{c}).$$

In particular, the top class in $H^*(G_3/\operatorname{Spin}(7);\mathbb{Z}_2)$ is ${\bar{d}_4}^6$ in dimension 24, and it is in the image of

$$\phi^* : H^*(B\operatorname{Spin}(7);\mathbb{Z}_2) \to H^*(G_3/\operatorname{Spin}(7);\mathbb{Z}_2).$$

5. The torsion index of G_3

To compute the torsion index of the 2-compact group G_3 we need a lemma on Poincaré duality in fibrations. I'm grateful to Aniceto Murillo for some helpful conversations on this subject. For this lemma we use the following notation. Let $\mathcal{O}$ denote the ring of integers or the ring of p-adic integers. Cohomology is taken with coefficients in $\mathcal{O}$, and we assume that all spaces are of finite type over $\mathcal{O}$. We say that $\eta \in H^n(X)$ is an *orientation class* if $H^i(X) = 0$ for $i > n$, $H^n(X) \cong \mathcal{O}$, and η is a generator of $H^n(X)$.

Lemma 5.1. *Let $F \xrightarrow{j} E \xrightarrow{\pi} B$ be a fibration of 1-connected spaces and assume that $\eta^F \in H^m(F)$ and $\eta^B \in H^n(B)$ are orientation classes. Assume $\alpha \in H^m(E)$ is such that $j^*(\alpha) = \lambda\eta^F$ for some $\lambda \neq 0$. Then there is an orientation class η^E for E such that $\alpha \cdot \pi^*(\eta^B) = \lambda\eta^E$.*

Proof. This follows easily from the cohomology spectral sequence of the fibration $F \xrightarrow{j} E \xrightarrow{\pi} B$. First of all, it is clear that $H^i(E) = 0$ for $i > n + m$ while $H^{n+m}(E) = E_\infty^{n,m} = E_2^{n,m} \cong \mathcal{O}$. Recall that the cohomology spectral sequence is multiplicative in the sense that (up to some signs which would not play any role here) the product in E_2 induced by the products in $H^*(B)$ and $H^*(F)$ yields a product in each E_r, $2 \leq r \leq \infty$, in such a way that the product in E_∞ is compatible with the product in $H^*(E)$.

At the E_2 level we have that $\eta^E := \eta^F \cdot \eta^B$ is a generator of $E_2^{n,m} = E_\infty^{n,m} = H^{n+m}(E)$. The hypothesis $j^*(\alpha) = \lambda\eta^F$, $\lambda \neq 0$ implies that α has filtration zero in $H^m(E)$ and its image in $E_\infty^{0,m}$ is $\lambda\eta^F$. Then, $\lambda\eta^E = (\lambda\eta^F) \cdot [\eta^B]$ holds in E_∞ where $[\eta^B]$ denotes the image of η^B in $E_\infty^{n,0}$. Since $E_\infty^{i,m+n-i} = 0$ for $i \neq n$, we deduce $\lambda\eta^E = \alpha \cdot \pi^*(\eta^B)$, as desired. $\square$

Now we can proceed to the computation of the torsion index of G_3 or, to be more precise, to the computation of the order of the cokernel of $k^* : H^{42}(BT;\mathbb{Z}_2) \to H^{42}(G_3/T;\mathbb{Z}_2)$. We consider the diagram

$$\begin{tikzcd}
(\operatorname{Spin}(7)/T)^\wedge_2 \arrow[r, "j"] & G_3/T \arrow[r, "\pi"] \arrow[d, "k"] & G_3/\operatorname{Spin}(7) \arrow[d, "\phi"] \\
 & (BT)^\wedge_2 \arrow[r, "i"] & (B\operatorname{Spin}(7))^\wedge_2
\end{tikzcd}$$

$\operatorname{Spin}(7)/T$ is a compact orientable differentiable manifold of dimension 18, and we can choose an orientation class $\eta \in H^{18}(\operatorname{Spin}(7)/T;\mathbb{Z}_2)$. The torsion indices of the Lie groups $\operatorname{Spin}(n)$ have been computed by Totaro for all values of n ([15]), and it turns out that the torsion index of $\operatorname{Spin}(7)$ is equal to 2. This means that there is $\omega \in H^*(BT;\mathbb{Z})$ such that $f^*(\omega) = 2\eta$ for the natural map $f : \operatorname{Spin}(7)/T \to BT$.

The computations in the preceding section show that there is an orientation class $\rho \in H^{24}(G_3/\operatorname{Spin}(7);\mathbb{Z}_2)$ which is in the image of ϕ^*. Let $\rho = \phi^*(\gamma)$. We can now apply the lemma above to the fibration $\operatorname{Spin}(7)/T \to G_3/T \to G_3/\operatorname{Spin}(7)$ with $\alpha = k^*(\omega)$ and deduce that there is an orientation class $\theta \in H^{42}(G_3/T;\mathbb{Z}_2)$ such that $k^*(\omega \cdot i^*(\gamma)) = 2\,\theta$. This implies that the torsion index of G_3 divides 2.

Next, we prove that the torsion index of G_3 cannot be equal to 1. It is enough to prove that the homomorphism $H^{42}(BT;\mathbb{F}_2) \to H^{42}(G_3/T;\mathbb{F}_2)$ is equal to zero. Let us consider the $\mathbb{F}_2$-spectral sequence of the fibration $G_3 \to G_3/T \to BT_2^\wedge$. We have that

$$H^*(G_3;\mathbb{F}_2) \cong \mathbb{F}_2[x_7]/x_7^4 \otimes E(x_{11}, x_{13}),$$
$$Sq^4(x_7) = x_{11},\ Sq^2(x_{11}) = x_{13},\ Sq^1(y_{13}) = x_7^2.$$

Hence, the generators $x_7, x_{11}, x_{13}, x_7^2$ are transgressive to $c_8, c_{12}, c_{14}, 0$, respectively. Here we denote by c_8, c_{12}, c_{14} the images in $H^*(BT;\mathbb{F}_2)$ of the generators $c_8, c_{12}, c_{14} \in H^*(BG_3;\mathbb{F}_2)$. Recall that in section 3 we have computed these elements as explicit polynomials in some basis $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ of $H^2(BT;\mathbb{F}_2)$.

In the E_2-term of the spectral sequence of $G_3 \to G_3/T \to BT_2^\wedge$, let us consider the row containing x_7^2. All elements in this row are permanent cycles, and the only boundaries are the elements of the form $x_7^2 q$ with q in the ideal of $\mathbb{F}_2[\epsilon_1, \epsilon_2, \epsilon_3]$ generated by c_8, c_{12}, c_{14}. If we compute the quotient algebra $\mathbb{F}_2[\epsilon_1, \epsilon_2, \epsilon_3]/(c_8, c_{12}, c_{14})$ (using any choice of computer algebra software), we see that it is a graded algebra with Poincaré series equal to

$$1 + 3t^2 + 6t^4 + 10t^6 + 14t^8 + 18t^{10} + 21t^{12} + 22t^{14} \\ + 21t^{16} + 18t^{18} + 14t^{20} + 10t^{22} + 6t^{24} + 3t^{26} + t^{28},$$

and so in particular there is an element $q \in H^{28}(BT;\mathbb{F}_2)$ which does not belong to the ideal (c_8, c_{12}, c_{14}). Hence, the element $x_7^2 q$ in the E_2-term of the spectral sequence survives to a nontrivial element in $H^{42}(G_3/T;\mathbb{F}_2)$ which cannot be in the image of $H^*(BT;\mathbb{F}_2)$. This finishes the proof of

Theorem 5.2. *The cokernel of $H^{42}(BT;\mathbb{Z}_2) \to H^{42}(G_3/T;\mathbb{Z}_2)$ has order two.*

6. Proof of Theorem 1.2

In section 2 we saw that it is enough to prove Theorem 1.2 for each exotic p-compact group, and we also saw that Theorem 1.2 is true for all odd primes. Since it is known ([2]) that the only exotic 2-compact group is G_3, the only thing that remains to be proved is that G_3 satisfies the properties (TI1) to (TI4) with $t_2(G_3) = 2$.

(TI4) is void, and (TI3) is just Theorem 5.2 plus some facts about G_3/T which were proven in [2]. In [12] it is proven that the torsion elements in $H^*(BG_3;\mathbb{Z}_2)$ are of order two and the homomorphism $H^*(BG_3;\mathbb{Z}_2) \to H^*(BT;\mathbb{Z}_2)^W$ is surjective. This implies immediately that (TI2) holds. Let A be a nontrivial finite abelian 2-subgroup of G_3 and let E be a subgroup of A of order 2. Then, A factors through the centralizer of E in G_3 which is Spin(7). Since Spin(7) has 2-torsion index equal to 2, we deduce that A has a subgroup of index at most 2 which is included in a maximal torus of G_3. So, we have (TI1), and the proof is complete. □

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