On exotic examples of p-local finite groups

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Workshop on Algebraic Topology and Its Applications

Dalian

Introduction

Definition

We say that a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ is exotic if there is no finite group G and an inclusion $S \subset G$ such that S is a Sylow p-subgroup in G and $(S, \mathcal{F}, \mathcal{L}) = (S, \mathcal{F}_S(G), \mathcal{L}_S(G))$.

From Remi's talk:

$\mathbb{Z}/p \rtimes \mathbb{Z}/p-1$

Fusion:

$S = \mathbb{Z}/p$

$\mathrm{Hom}_{\mathcal{F}}(\mathbb{Z}/p, \mathbb{Z}/p) = \mathbb{Z}/p-1$

$\mathbb{Z}/p \rtimes \mathbb{Z}/p-1$

Solomon's examples

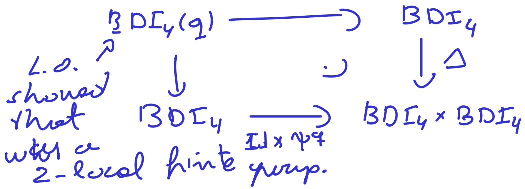
In the classification of finite simple groups Ron Solomon [8] considered the problem of classifying all finite simple groups whose Sylow 2-subgroups are isomorphic to those of the Conway group Co_3.

In the process he had to consider groups G where all involutions are conjugated and that for any involution $x \in G$ there are subgroups $K \lhd H \lhd C_G(x)$ such that $C_G(x)/H$ has odd order and $H/K \cong Spin_7(q)$.

Solomon showed that such a group G cannot exist: the 2-fusion was consistent, but when checking the p-fusion of G, for p the odd prime of the characteristic of q, there are incompatibilities.

Solomon, Benson, Levi-Oliver's example

Later on, David Benson constructed in [1] some spaces X which have the properties of the classifying space of the non-existing Solomon's group.



Family of examples for p=2
Smaller Sylow: 2^{10}

Finally, Ran Levi and Bob Oliver proved in [5] that there were 2-local finite groups $BSol(q)$ with the structure studied by Solomon and with the homotopy type of Benson's spaces.

Broto-Levi-Oliver's examples

In [3] Carles Broto, Ran Levi and Bob Oliver constructed the first exotic examples. The idea was to consider the structure of a nice p-group S, as a Sylow p-subgroup of a nice G and add some morphisms ϕ_i to some subgroups of S in a way that $(S, \langle \mathcal{F}_S(G), \{\phi_i\}_i \rangle)$ keeps being saturated.

$(\mathbb{Z}/p^\ell)^p \rtimes \Sigma_p$ ← permuting components

$\cup$

$\{(a_1, \dots, a_p) \mid \sum a_i \equiv 0 \ (p^\ell)\} \cong (\mathbb{Z}/p^\ell)^{p-1} \rtimes \Sigma_p$

$0 \longrightarrow (\mathbb{Z}/p^\ell)^{p-1} \longrightarrow G \longrightarrow \Sigma_p \times C_{p-1}$ ← cyclic of order $p-1$

Broto-Levi-Oliver's examples

$$0 \longrightarrow (\mathbb{Z}/p^\ell)^{p-1} \longrightarrow G \longrightarrow \Sigma_p \times C_{p-1}$$

$$Z(G) = \langle \xi \rangle \cong \mathbb{Z}/p \qquad \mathbb{Z}/p = \langle s \rangle \subseteq \Sigma_p$$

$p \geq 5$ } New family of examples

$$V_0 = \langle \xi, s \rangle \cong \mathbb{Z}/p \times \mathbb{Z}/p \subseteq GL_2(p)$$

construct the saturated fusion system.

$$\Rightarrow \langle G, \mathbb{Z}/p \times \mathbb{Z}/p \rtimes GL_2(p) \rangle \Leftarrow$$ this is exotic.

<u>Problem</u>: obtain something saturated

Exotics?

How can one prove that a given $(S, \mathcal{F}, \mathcal{L})$ is exotic?

In general, there is no answer, but if S and $\mathcal{F}$ are nice, then, if there exists G such that realizes $(S, \mathcal{F}, \mathcal{L})$, then there exists $\tilde{G}$, a finite almost-simple group such that $\tilde{G}$ realizes $(S, \mathcal{F}, \mathcal{L})$.

"nice" means

S contains no proper strongly closed subgroups and
S does not factor as a product of two or more subgroups which are permuted transitively by $Aut_{\mathcal{F}}(S)$.

$P \leq S$ is strongly closed if $\forall x \in P$ and $\varphi \in Hom_{\mathcal{F}}$, then $\varphi(x) \in P$.

Strategy

Consider a small p-group S and classify all the saturated fusion systems over S.

Problem

We have to specify $Hom_{\mathcal{F}}(P, Q)$ for all $P, Q \leq S$.

$S = (\mathbb{Z}/p^e)^{p-1} \rtimes \mathbb{Z}/p$ ← too many subgroups.

Tool [Puig,BLO]

Theorem (Alperin's thm for sfs)

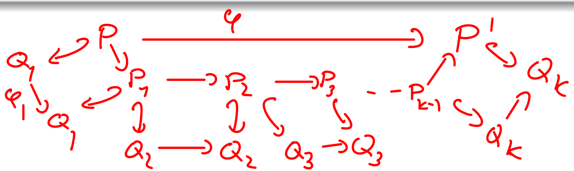
Let $(S, \mathcal{F})$ a saturated fusion system. Then for each morphism $\varphi \in Iso_{\mathcal{F}}(P, P')$ in $\mathcal{F}$, there exist sequences of subgroups of S:

$$P = P_0, \ldots, P_k = P' \quad \text{and} \quad Q_1, \ldots, Q_k,$$

and elements $\varphi_i \in Aut_{\mathcal{F}}(Q_i)$ such that

$C_S(Q_i) \leq Q_i$

1. *Q_i is fully $\mathcal{F}$-normalized, $\mathcal{F}$-radical, and $\mathcal{F}$-centric for each i;*
2. *$P_{i-1}, P_i \leq Q_i$ and $\varphi_i(P_{i-1}) = P_i$ for each i; and*
3. *$\varphi = \varphi_k \circ \cdots \circ \varphi_1$.*



every map is the restriction of comp. of automorphisms of p-radical, fully F-norm, F-centric subgroups.

Implications of Alperin's thm

To describe a saturated fusion system $(S, \mathcal{F})$ we just have to give $Aut_{\mathcal{F}}(P)$ for some $P \leq S$.

$\mathcal{F}$-centric, $\mathcal{F}$-reduced, fully $\mathcal{F}$-normalized

Example

If S is an abelian group, $(S, \mathcal{F}, \mathcal{L})$ is controlled by $Aut_{\mathcal{F}}(S)$.

$P \lneq S$
only $\mathcal{F}$-centric

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & Q \\ \cap & & \cap \\ S & \xrightarrow[\bar{\varphi}]{} & S \end{array}$$

$(S, \mathcal{F}, \mathcal{L})$ is controlled by the normalizer fusion system.

If S is a p-group s.t. $\forall (S, \mathcal{F})$ s.f.s. $(S, \mathcal{F}) = (S, N_{\mathcal{F}}(S)) \Rightarrow S$ is a Swan group

resistant.

R-Viruel examples [7]

By the previous argument we need S a non-abelian p-group, so the smaller ones are p_+^{1+2} and p_-^{1+2} ← resistant group

p

$\langle a, b, c \mid a^p = b^p = c^p = 1, [a,b] = c, [a,c] = [b,c] = 1 \rangle$

To give a s.f.s. we need to tell which are the radical subgroups, (and centric)

In p_+^{1+2}, the possible proper $\mathcal{F}$-centric are the elementary rank two p-subgroups, and there are exactly $p+1$: $V_i = \langle ab^i, c \rangle$ for $i = 1, \ldots, p-1$ and $V_p = \langle b, c \rangle$.

$\cong \mathbb{Z}/p \times \mathbb{Z}/p$

$P \leq p_+^{1+2}$ is $\mathcal{F}$-centric $\Rightarrow \langle c \rangle \subset P \leq S$

$\# p$, order p^2, $\# p^3$

V_i $\mathcal{F}$-radical if $Out_{\mathcal{F}}(V_i)$ doesn't contain normal p-sub.

R-Viruel examples (II)

By the saturation hypothesis $p \nmid \# Out_{\mathcal{F}}(p_+^{1+2})$. $Out_{\mathcal{F}}(p_+^{1+2}) \leq Out(p_+^{1+2}) = GL_2(p)$.

Main trick

If V_i is $\mathcal{F}$-radical $\iff SL_2(p) \leq Out_{\mathcal{F}}(V_i)$. This implies $-Id \in Out_{\mathcal{F}}(V_i)$. Then, we can see that $N_{-Id} = p_+^{1+2}$, so there is a $\phi_i \in Out_{\mathcal{F}}(S)$.

Property

For any $p \geq 17$ and any subgroup $H \leq GL_2(p)$ containing more than 2 ϕ_i then $p | \# H$.

R-Viruel examples (III)

So we cannot have more than 2 proper $\mathcal{F}$-radical subgroups except for $p \in 3, 5, 7, 11, 13$.

How many Vi are F-radical?

s.f.s. $\sharp\mathcal{F}^{ec}$-rad > 2 and $p \in \{3, 5\}$

$Out_{\mathcal{F}}(3^{1+2}_{+})$	$\sharp\mathcal{F}^{ec}$-rad	$Aut_{\mathcal{F}}(V)$	Group
D_8	$2+2$	$GL_2(3)$	${}^2F_4(2)'$
SD_{16}	4	$GL_2(3)$	J_4

Table : $p = 3$ $S = 3^3$

$Out_{\mathcal{F}}(5^{1+2}_{+})$	$\sharp\mathcal{F}^{ec}$-rad	$Aut_{\mathcal{F}}(V)$	Group
$4S_4$	6	$GL_2(5)$	Th

Table : $p = 5$

R-Viruel examples (IV)

s.f.s. $\sharp\mathcal{F}^{ec}$-rad > 2 and $p = 7$

$Out_{\mathcal{F}}(7^{1+2}_{+})$	$\sharp\mathcal{F}^{ec}$-rad	$Aut_{\mathcal{F}}(V)$	Group
$S_3 \times 3$	3	$SL_2(7)$	He
$S_3 \times 6$	3	$SL_2(7):2$	$He:2$
$S_3 \times 6$	$3+3$	$SL_2(p):2$	Fi'_{24}
$6^2:2$	6	$SL_2(7):2$	Fi_{24}
$6^2:2$	$6+2$	$SL_2(7):2, GL_2(7)$	Exotic
$D_8 \times 3$	$2+2$	$SL_2(7):2$	$O'N$
$D_{16} \times 3$	4	$SL_2(7):2$	$O'N:2$
$D_{16} \times 3$	$4+4$	$SL_2(7):2$	EXOTIC
$SD_{32} \times 3$	8	$SL_2(7):2$	Exotic

R-Viruel examples (V)

s.f.s. $\sharp\mathcal{F}^{ec}$-rad > 2 and $p \neq 3$

$Out_{\mathcal{F}}(13_{+}^{1+2})$	$\sharp\mathcal{F}^{ec}$-rad	$Aut_{\mathcal{F}}(V)$	Group
$3 \times 4S_4$	6	$SL_2(13).4$	M

Díaz-R-Viruel examples (I) [4]

This is the same game as in R-Viruel examples, but considering all p-rank two p-groups S for odd p.

In the classification of those S, we saw that the more interesting where p_+^{1+2} and the ones which fit in the following split extension:

$$\mathbb{Z}/3^k \times \mathbb{Z}/3^k \to S \to \mathbb{Z}/3$$

3-Sylow

$$\mathbb{Z}/3^k \times \mathbb{Z}/3^k \to G \to GL_2(3)$$

$Z(S) = \langle \zeta \rangle$

$V_0 = \langle \zeta, s \rangle$

$V_1 = \langle \zeta, s s_1 \rangle$

$V_{-1} = \langle \zeta, s s_1^{-1} \rangle$

$E_i = N_S(V_i) \cong 3_+^{1+2}$

Only possible $\mathcal{F}$-radical, $\mathcal{F}$-centric are:

$\gamma_1, V_0, V_1, V_{-1}, E_0, E_1, E_{-1}$

V_i is $\mathcal{F}$-radical $\Rightarrow$ E_i cannot be $\mathcal{F}$-radical

Díaz-R-Viruel examples (II)

<table>
<tr><th>W</th><th>V_0</th><th>V_1</th><th>V_{-1}</th><th>E_0</th><th>E_1</th><th>E_{-1}</th><th>γ_1</th><th>Exotic?</th></tr>
<tr><td rowspan="3">$\langle\omega\rangle$</td><td rowspan="3"></td><td rowspan="3"></td><td rowspan="3"></td><td>★</td><td></td><td></td><td></td><td>Yes</td></tr>
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Keep it!

Subsystems of saturated fusion systems

Broto-Castellana-Grodal-Levi-Oliver [2] studied the saturated fusion subsystems (and extensions) of p-local finite groups in two special cases:

- s.f. subsystems of p-power index and
- s.f. subsystems of index prime to p.

is known!

To classify all the subsystems of index prime to p in $(S, \mathcal{F}, \mathcal{L})$ we need to control $O_*^{p'}(\mathcal{F})$, the smallest subcategory of $\mathcal{F}$ containing all p-power automorphims in $\mathcal{F}$.

Reduction

$O_*^{p'}(\mathcal{F})$ is generated by p-power automorphims of $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups.

Normal fusion subsystems of the grassmannians [6]

Consider $GL_n(q)$, invertible $n \times n$ matrices over $\mathbb{F}_q$. $p \nmid q$
Alperin-Fong's work gives us the radical subgroups.

Example

Consider q a prime power, p a prime such that $p \nmid q$, e, the order of q modulo p, i.e. $p | (q^e - 1)$ and l such that $p^l | (q^e - 1)$ and $p^{l+1} \nmid (q^e - 1)$.
Consider $(S, \mathcal{F}, \mathcal{L})$ the p-lfg corresponding to $GL_n(q)$, $n \geq ep$.
For each divisor r of e, there is a s.f.subsystem $(S, \mathcal{F}_r, \mathcal{L}_r)$, and if $r \geq 2$, it is exotic.

$p = 5$, $q = 7$ } $e = 4$

$GL_{20}(\mathbb{F}_7)$

$r \in \{1, 2, 4\} \Rightarrow (S, \mathcal{F}_r, \mathcal{L}_r) \Leftarrow$ s.f. subsystem

$r \geq 2 \Rightarrow$ exotic

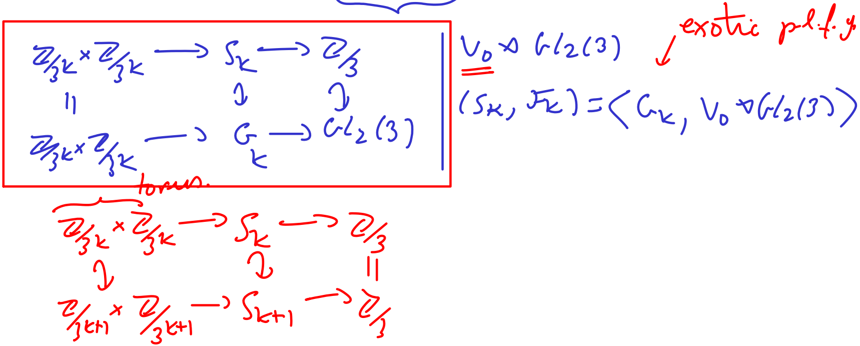
Normal exotic subsystems of f.s.

p-local compact groups

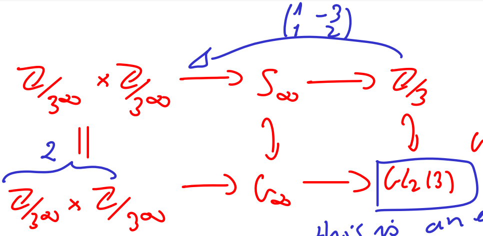
Consider the split extension of Diaz-R-Viruel examples:

$$\mathbb{Z}/3^k \times \mathbb{Z}/3^k \longrightarrow S_k \longrightarrow \mathbb{Z}/3$$

and the exotic p-local finite group which has, as proper $\mathcal{F}$-radical subgroups $\mathbb{Z}/3^k \times \mathbb{Z}/3^k$ and $\langle Z(S_k), s\rangle$ =: V_0



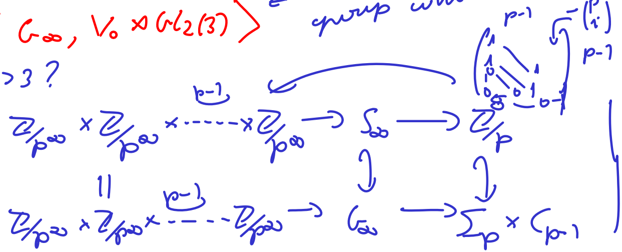
p-local compact groups



V_0 is constant ↻ $GL_2(3)$

$\langle G_\infty, V_0 \rtimes GL_2(3) \rangle$ ← this is an example of p-local compact group which is not a p-compact group

If $p>3$?



this will give a family of exotic p-local compact groups. $p \geq 5$.

Don't have examples for $p=2$.

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Thank you for your attention!

谢谢