Homotopy properties of rank two Kac-Moody groups

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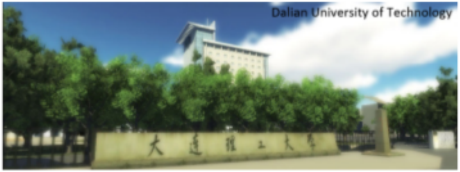

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Motivation

Aim

1. Do homotopy theory with Kac-Moody groups: understand the classifying space.
2. Try to generalize well known results for compact Lie groups: homotopy decompositions, mapping spaces.

Finite groups
$\Downarrow$
Compact Lie groups
$\Downarrow$
Kac-Moody groups.

Good properties:
- Maximal torus T (finite rank)
- $W := N_K(T)/T$ } $\Rightarrow$ reflection group infinite

Rank 1: S^3

Definition

$ab \leq 3$

$\Updownarrow$

compact Lie groups

Consider a 2×2 generalized Cartan matrix:

$$\begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$$

with $ab > 4$: then $K(a, b)$, the associated Kac-Moody group, is rank 2 and infinite dimensional.
The Weyl group $W(a, b) \cong \mathbb{Z}/2 * \mathbb{Z}/2$ and the action on the Lie algebra of the rank two maximal torus is induced by the matrices:

$$w_1 = \begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix} \text{ and } w_2 = \begin{pmatrix} 1 & 0 \\ a & -1 \end{pmatrix}.$$

$\langle w_1, w_2 \rangle \cong \mathbb{Z}$

Main properties

Theorem (Kitchloo)

If K is an infinite dimensional Kac-Moody group, then

$$BK = \operatorname*{hocolim}_{P_I \text{ fin.dim.}} BP_I .$$

P

class. spaces of compact Lie groups.

$A = \begin{pmatrix} \boxed{\ } & \\ & \end{pmatrix}$ n, n pos. def

$P_\emptyset = T$

$P_{\{1\}}, P_{\{2\}}$

$$\begin{tikzcd} BT = BP_\emptyset \arrow[r] \arrow[d] & BP_{\{1\}} \arrow[d] \\ BP_{\{2\}} \arrow[r] & BK(a,b) \end{tikzcd}$$

$\leftarrow$ compact Lie groups of rank 2 and $W \cong \mathbb{Z}/2$

In the rank two case, this reduces to the following diagrams:

$$\underbrace{BS^3 \times BS^1}_{} \xleftarrow{\begin{pmatrix} \frac{-a}{2} & 1 \\ 1 & 0 \end{pmatrix}} BT \xrightarrow{\begin{pmatrix} 1 & \frac{-b}{2} \\ 0 & 1 \end{pmatrix}} BS^3 \times BS^1 \qquad a \equiv b \equiv 0 \mod (2)$$

$$BU(2) \xleftarrow{\begin{pmatrix} \frac{1-a}{2} & 1 \\ \frac{1+a}{2} & -1 \end{pmatrix}} BT \xrightarrow{\begin{pmatrix} 1 & \frac{-b}{2} \\ 0 & 1 \end{pmatrix}} BS^3 \times BS^1 \qquad \underline{a \not\equiv b \equiv 0} \mod (2)$$

$$BU(2) \xleftarrow{\begin{pmatrix} \frac{1-a}{2} & 1 \\ \frac{1+a)}{2} & -1 \end{pmatrix}} BT \xrightarrow{\begin{pmatrix} 1 & \frac{1-b}{2} \\ -1 & \frac{1+b}{2} \end{pmatrix}} BU(2) \qquad a \equiv b \equiv 1 \mod (2)$$

Cohomology

rank 2

Theorem (Kitchloo)

There is a class $q\colon BK \to K(\mathbb{Z},4)$ inducing a rational equivalence.

$BT \longrightarrow BK \xrightarrow{q} K(\mathbb{Z},4)$

$H^*(BT;\mathbb{Z}) = \mathbb{Z}[u,v]$

$q \equiv \frac{1}{\gcd(a,b)} \left(a u^2 - ab uv + b v^2 \right)$

The mod p cohomology $H^*(BK;\mathbb{F}_p) \cong \mathbb{F}_p[x_4, y_{2k}] \otimes E[z_{2k+1}]$ with a $\beta_r(y_{2k}) = z_{2k+1}$, where subscripts are the degrees and are explained in the following table:

k	r	Conditions	p
2	$\max_n\{2^n \mid \gcd(a,b)\}$	$a \equiv b \equiv 0 \mod (2)$	
3	$\max_n\{2^n \mid ab-1\}$	$a \equiv b \equiv 1 \mod (2)$	2
4	$\max_n\{2^n \mid ab-2\}$	$a \not\equiv b \equiv 0 \mod (2)$	
$\lvert W_p\rvert/2$	$\min_n\{\lvert W_{p^n}\rvert < \lvert W_{p^{n+1}}\rvert\}$		$p>2$

where W_{p^n} is the mod p^n reduction of $W \subset \mathrm{GL}_2(\mathbb{Z})$.

Strategy and tools

1. Consider $BK_p^\wedge$ and study the maps $\overbrace{[X, BK_p^\wedge]}^{\forall p}$ for each prime p. X will be
   - a $B\pi$, the classifying space of a finite p-group,
   - b BT, the classifying space of a torus and
   - c BK', the classifying space of a Kac-Moody group K'.
2. If possible, in the last two cases, obtain a map $[X, BK]$ (Sullivan arithmetic square).

Sullivan arithmetic square

If K is a rank two Kac-Moody group we have the following pullback:

$\prod(f_p) \Leftarrow (f_p^*)(q)$ doesn't depend on p.

$$
\begin{tikzcd}
X \arrow[dr, dashed] \arrow[drr, bend left, "\prod(f_p)"] \arrow[ddr] & & \\
BT \arrow[r] & BK \arrow[r] \arrow[d] & \prod_p BK^\wedge_p \arrow[d] \\
K(\mathbb{Q},4) \simeq & BK_{\mathbb{Q}} \arrow[r] & (\prod_p BK^\wedge_p)_{\mathbb{Q}}
\end{tikzcd}
$$

| Image

p

Lemma

Let X be a space such that $H^3(X;\mathbb{Q}) = 0$. Then:

- *The map $l\colon [X, BK] \to \prod_p [X, BK^\wedge_p]$ is injective,*
- *the image of l are families $\{f_p\colon X \to BK^\wedge_p\}$ such that $\exists q \in H^4(X;\mathbb{Q})$ with $f_p^*(q \otimes \hat{\mathbb{Q}}_p) = x \otimes \hat{\mathbb{Q}}_p$.*

Maps from BT to BK

Lemma

Let T be a rank 2 torus and T_{p^∞} the p-torsion subgroup.

- *The map $l\colon [BT, BK] \to \prod_p [BT_{p^\infty}, BK^\wedge_p]$ is injective,*
- *the image of l are families $\{f_p\colon BT_{p^\infty} \to BK^\wedge_p\}$ such that $f_p^*(q)$ lies in $H^4(BT;\mathbb{Z}) \subset H^4(BT_{p^\infty};\mathbb{Z})$ and is independent of p.*

Theorem (Aguadé-R)

In general, the map $Hom(T, K) \to [BT, BK]$ is not surjective.

q a quadratic form with non-trivial genus: $\exists\, q'$ q. form s.t.

$q \cong_{\mathbb{Z}_p} q'$

$a = 13$

$b = 39$

$\boxed{q \cong_{\mathbb{Q}} q'}$

$q \not\cong_{\mathbb{Z}} q'$

$f_p\colon BT_p \to BT_p$ { naturally will have the same image

$f\colon BT \to BK$

$\mathbb{Z}\text{-eq} \to BT \xrightarrow{B\rho} BK$

Maps from finite p-groups to $BL^\wedge_p$

Theorem (Broto-Kitchloo)

If L is a Kac-Moody group and π is a finite p-group, then there are *homotopy equivalences*

$$\coprod_{\rho \in \mathrm{Rep}(\pi, L)} (BC_L(\rho))^\wedge_p \to \mathrm{Map}(B\pi, BL^\wedge_p),$$

$$\left(\operatorname*{hocolim}_{P_I \text{ Lie type}} \mathrm{Map}(B\pi, BP_I{}^\wedge_p) \right)^\wedge_p \to \mathrm{Map}(B\pi, BL^\wedge_p),$$

where $C_L(\rho)$ means the centralizer in L of $\rho(\pi)$.

Lemma

Let $f\colon BK \to BK^\wedge_p$. There is a homomorphism $\rho\colon T_{p^\infty} \to K$ such that $f|_{BT_{p^\infty}} \simeq B\rho$. If $\rho \neq 1$ then ρ has finite kernel.

Groups with the same classifying space

Lemma

$K \cong K'$ if and only if $\{a, b\}=\{a', b'\}$.

Theorem

$BK \simeq BK'$ if and only if

1. $ab = a'b'$ *and* $\gcd(a, b) = \gcd(a', b')$.
2. *One can order a', b' in such a way that aa' is a square in $\mathbb{Z}$ and ab' is a square in $\hat{\mathbb{Z}}_p$ for all primes such that $\nu_p(a) \neq \nu_p(a')$.*

At some primes: $BK(a,b)^\wedge_p \longrightarrow BK(a',b')^\wedge_p$

other primes. $BK(a,b)^\wedge_p \longrightarrow BK(b',a')^\wedge_p$

glue all together to a map

$BK \longrightarrow BK'$

#

(Integral) Adams maps

Definition

1. An **Adams map** ψ^λ is a map extending the homomorphism of the torus induced by $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$
2. A **twisted Adams map** $\psi^{\lambda,\mu}$ is a map extending the homomorphism of the torus induced by $\begin{pmatrix} 0 & \lambda \\ \mu & 0 \end{pmatrix}$

Theorem

1. *There is an Adams map $\psi^\lambda \colon BK \to BK$ if and only if $\lambda = 0$ or $\lambda \equiv 1 \mod (2)$.*
2. *There is a non-trivial twisted Adams map $\psi^{\lambda,\mu} \colon BK \to BK$ if and only if $\lambda \equiv \mu \equiv 1 \mod (2)$ and $a\lambda = b\mu$.*

Generic Adams maps

We can also construct ψ^λ, $\psi^{\lambda,\mu}\colon BK_p^\wedge \to BK_p^\wedge$, with $\lambda, \mu \in \mathbb{Z}_p^\wedge$ (the only restriction is that $\lambda \equiv \mu \equiv 1$ when $p = 2$).

And, if they are rationally compatible, we can glue all together to define a generic Adams map: such a map will be codified by

$$\underbrace{\{(\varepsilon_p, \lambda_p)\}} \in \prod_p (\{0,1\} \times \hat{\mathbb{Z}}_p)$$

$\varepsilon_p = 0 \Rightarrow \begin{pmatrix} \lambda_p & 0 \\ 0 & \lambda_p \end{pmatrix}$

$\varepsilon_p = 1 \Rightarrow \begin{pmatrix} 0 & \lambda_p \\ \mu_p & 0 \end{pmatrix}$

$\mu_p = \dfrac{\lambda_p \cdot a}{b}$

Theorem

Let $f\colon BK \to BK$ be a map. Then f is a generic Adams map. , f!

Kac-Moody groups over finite fields

Tits defined, in a functorial way, Kac-Moody groups over fields k.
Fixed a Cartan matrix, lets denote $K(k)$ the corresponding Kac-Moody group.
Till now, we have been talking about $K(\mathbb{C})$.

Consider now $\mathbb{F}_q$ a finite field of characteristic different to p, and $K(\mathbb{F}_q)$ the Tits construction of the rank two Kac-Moody group corresponding to the generalized Cartan matrix

$$\begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$$

Cohomology of rank two $BK(\mathbb{F}_q)$

$H^*(BK(\mathbb{F}_q, \mathbb{F}_p)$

Theorem (Aguadé-R)

Let p such that $p \nmid ab(ab-4)$, then:

- *If $q \not\equiv \pm 1$ mod p, then* $\boxed{H^*(BK(\mathbb{F}_q), \mathbb{F}_p) \cong \mathbb{F}_p}$ $\tilde{H}(BK(\mathbb{F}_q); \mathbb{F}_p) = 0$
- *If $q \equiv -1$ mod p, then* $H^*(BK(\mathbb{F}_q), \mathbb{F}_p) \cong (\mathbb{F}_p[x_4] \otimes E(y_3)) \oplus (\mathbb{F}_p[x_4'] \otimes E(y_3'))$.
- *If $q \equiv 1$ mod p, then $H^*(BK(\mathbb{F}_q), \mathbb{F}_p)$ is an $\mathbb{F}_p[x_4, x_{2m}] \otimes E(y_3, y_{2m-1})$-module with generators 1, α_3, α_4, J_{2m}, J_{2m+1} subject to some relations.*

Kac-Moody groups over finite fields and fixed points

For G a compact connected Lie group, Friedlander proved that if p, ℓ are different primes, $q = \ell^r$ and $G(\mathbb{F}_q)$ the Chevalley group over $\mathbb{F}_q$ of type G. Then $\underbrace{BG(\mathbb{F}_q)^\wedge_p}_{} \simeq \underline{\underline{BG^{h\psi^q}}}$, where $BG^{h\psi^q}$ is defined as the pullback:

$$\begin{tikzcd}
\boxed{B\overset{K}{\cancel{G}}^{h\psi^q}} \arrow[r] \arrow[d] & B\overset{K}{\cancel{G}}{}^\wedge_p \arrow[d, "\Delta \leftarrow \text{diagonal}"] \\
B\overset{K}{\cancel{G}}{}^\wedge_p \arrow[r, "{(1,\psi^q)}", "\uparrow \text{Adams operation}"'] & B\overset{K}{\cancel{G}}{}^\wedge_p \times B\overset{K}{\cancel{G}}{}^\wedge_p
\end{tikzcd}$$

Kac-Moody groups over finite fields and fixed points

Theorem (Aguadé-R, Foley)

In general $BK(\mathbb{F}_q)^\wedge_p \not\simeq BK^{h\psi^q}$.

<u>Proof:</u>

a, b, p, q s.t.
$p \nmid ab(ab-4)$
$q \not\equiv \pm 1 \ (p)$
$q^{k} \equiv 1 \ (p)$

$\tilde{H}^*(BK(\mathbb{F}_q)^\wedge_p ; \mathbb{F}_p) = 0$

$$\begin{tikzcd} K \arrow[r, equal] \arrow[d] & K \arrow[d] \\ BK^{h\psi^q} \arrow[r] \arrow[d] & BK \arrow[d, "\Delta"] \\ BK \arrow[r, "1\times\psi^q"'] & \underbrace{BK \times BK}_{\text{Serre s.s.}} \end{tikzcd}$$

these cohomologies are known.

$\boxed{x_{2k-1}}$

$x_{2k-1} \in \tilde{H}^*(BK^{h\psi^q}; \mathbb{F}_p)$

$\neq 0$

Serre s.s.

$\Longleftarrow$

∎

Thank you for your attention!

谢谢