

# COHOMOLOGICAL UNIQUENESS OF SOME $p$ -GROUPS

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**ABSTRACT.** In this paper we consider classifying spaces of a family of  $p$ -groups and we prove that mod  $p$  cohomology enriched with Bockstein spectral sequences determines their homotopy type among  $p$ -completed CW-complexes. We end with some applications to group theory.

## 1. INTRODUCTION

Let  $p$  be a prime number. A naive way of describing Bousfield-Kan  $p$ -completion functor [1] is to say that it does transform mod  $p$  cohomology isomorphisms into actual homotopy equivalences. It is then therefore natural to think that the homotopy type of a  $p$ -complete space  $X$  should be characterized in some sense by its mod  $p$  cohomology ring  $H^*X$ . Classifying spaces of finite  $p$ -groups provide nice examples of  $p$ -complete spaces. Then the following question arises: given a finite  $p$ -group  $P$ , and a  $p$ -complete space  $X$  such that  $H^*X \cong H^*BP$ , is then  $X \simeq BP$ ?

One would like to give a positive answer to the question above, but the very first step towards that positive answer is to understand, or to give the appropriate meaning to, the isomorphism  $H^*X \cong H^*BP$ .

It is well known that there are infinitely many examples of non isomorphic finite  $p$ -groups (hence infinitely many examples of non homotopic  $p$ -complete spaces) having isomorphic mod  $p$  cohomology rings, even as unstable algebras (see [3] for a general proof of this fact in the case of  $p = 2$ ). This is not surprising since  $p$ -completion does not invert abstract mod  $p$  cohomology isomorphisms, but just those which are induced by continuous maps, and these compare unstable algebras plus secondary operations.

In this direction, Broto-Levi [2] suggested that mod  $p$  cohomology rings of finite  $p$ -groups should be considered objects in the category  $\mathcal{K}_\beta$  of unstable algebras endowed with Bockstein spectral sequences (see Section 2 for precise definitions). Here we follow that line and consider the family of groups studied by I. Leary in [6], proving:

**Theorem 1.1.** *Let  $p$  be an odd prime and define the finite  $p$ -group*

$$P(p, n) = \langle A, B, C \mid A^p = B^p = C^{p^{n-2}} = [A, C] = [B, C] = 1, [A, B] = C^{p^{n-3}} \rangle.$$

*Given  $X$  a  $p$ -complete CW-complex:*

(a) *If  $n = 3, 4$  and  $H^*(X) \cong H^*(BP(p, n))$  as unstable algebras, then  $X \simeq BP(p, n)$ .*

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(b) If  $n \geq 5$  and  $H_\beta^*(X) \cong H_\beta^*(BP(p, n))$  as objects in  $\mathcal{K}_\beta$ , then  $X \simeq BP(p, n)$ .

*Proof.* Statement (a) is proved in Corollary 4.6,  $n = 3$ , and Corollary 4.7.(a),  $n = 4$ . Statement (b) is proved in Corollary 4.7.(b).  $\square$

Besides of its own topological interest, the result above and the techniques developed in its proof may be appealing from a group theoretical point of view. First, since the classifying space of a finite  $p$ -group is a  $p$ -complete CW-complex, Theorem 1.1 provides a cohomological characterization of  $P(p, n)$ :

**Theorem 1.2.** *Let  $p$  be an odd prime and  $G$  be a finite  $p$ -group. Then  $G \cong P(p, n)$  if and only if  $H_\beta^*(BG) \cong H_\beta^*(BP(p, n))$ .*

Second, the ideas in the proof of Theorem 1.1 can be used to obtain a cohomological characterization of  $P(p, n)$  as a complement for some  $N \trianglelefteq G$ . This characterization can be seen as a generalization of Tate's cohomological criteria of  $p$ -nilpotency [7]:

**Theorem 1.3.** *Let  $p$  be an odd prime and  $P(p, n) \leq G$  be a group. Then  $P(p, n)$  is a complement for some  $N \trianglelefteq G$  if and only if one of the following holds:*

- (a)  $n = 3, 4$  and there exists  $\psi: H^*(BP(p, n)) \rightarrow H^*(BG)$  as unstable algebras such that  $(\text{res} \circ \psi)|_{H_\beta^1(BP(p, n))}$  is the identity.
- (b)  $n \geq 5$  there exists  $\psi: H_\beta^*(BP(p, n)) \rightarrow H_\beta^*(BG)$  in  $\mathcal{K}_\beta$  such that  $(\text{res} \circ \psi)|_{H_\beta^1(BP(p, n))}$  is the identity.

*Proof.* If  $P(p, n)$  is a complement for some  $N \trianglelefteq G$ , then the induced projection  $G \xrightarrow{\pi} G/N \cong P(p, n)$  gives rise to a map between classifying spaces  $BG \xrightarrow{B\pi} BP(p, n)$  that provides the desired cohomological morphism  $\psi = B\pi^*$ .

The converse is proved in Proposition 5.1, in case  $n = 3$ , and Proposition 5.2 in case  $n > 3$ .  $\square$

**Organization of the paper:** In Section 2 we introduce the notation we use along paper. In Section 3 the group  $P(p, n)$  is defined and the mod  $p$  cohomology ring of its classifying space is described. In Section 4, we explore endomorphisms of the mod  $p$  cohomology ring of  $BP(p, n)$ , and we conclude that mod  $p$  cohomology determines the homotopy type of  $BP(p, n)$ . Finally, in Section 5 we apply the ideas developed in the previous section to the group theoretical framework.

## 2. DEFINITIONS AND NOTATION

We follow the notation and conventions in [2, Section 2]. As our study is done for a fixed odd prime  $p$ , we just recall the definitions in this case.

All the spaces considered here have the homotopy type of a  $p$ -complete CW-complex. Unless otherwise stated  $H^*(X)$  refers to the cohomology of the space  $X$  with trivial coefficients in  $\mathbb{F}_p$ .

**Definition 2.1.** Let  $p$  be an odd prime and  $K$  be an unstable algebra. A *Bss* (Bockstein spectral sequence) for  $K$  is a spectral sequence of differential graded algebras  $\{E_i(K), \beta_i\}_{i=1}^\infty$  where the differentials have degree one and such that

- (a)  $E_1(K) = K$  and  $\beta_1 = \beta$  is the primary Bockstein operator.  
 (b) If  $x \in E_i(K)^{\text{even}}$  and  $x^p \neq 0$  in  $E_{i+1}(K)$ ,  $i \geq 1$ , then  $\beta_{i+1}(x^p) = x^{p-1}\beta_i(x)$ .

We work in the category  $\mathcal{K}_\beta$ , whose objects are pairs  $(K; \{E_i(K), \beta_i\}_{i=1}^\infty)$ , where  $K$  is an unstable algebra and  $\{E_i(K); \beta_i\}_{i=1}^\infty$  is a Bss for  $K$ . A morphism  $f: K \rightarrow K'$  in  $\mathcal{K}_\beta$  is a family of morphisms  $\{f_i\}_{i=1}^\infty$ , where  $f_1: K \rightarrow K'$  is a morphism of  $\mathcal{A}_p$ -algebras and for each  $i \geq 2$ ,  $f_i: E_i(K) \rightarrow E_i(K')$  is a morphism of differential graded algebras, which as morphism of graded algebras, is induced by  $f_{i-1}$ .

The mod  $p$  cohomology of a space  $X$  is an object of  $\mathcal{K}_\beta$  that is denoted by  $H_\beta^*(X)$ .

**Definition 2.2.** We say that two spaces  $X$  and  $Y$  are *comparable* if  $H_\beta^*(X)$  and  $H_\beta^*(Y)$  are isomorphic objects in the category  $\mathcal{K}_\beta$ . We say that  $X$  is *determined by cohomology* if given a space  $Y$  comparable to  $X$ , there is a homotopy equivalence  $X \simeq Y$ .

**Definition 2.3.** Let  $K_\beta$  be an object in  $\mathcal{K}_\beta$ . Let  $K$  be the underlying unstable algebra over  $\mathcal{A}_p$ . We say that  $K_\beta$  is *weakly generated by*  $x_1, \dots, x_n$  if any endomorphism  $f$  of  $K_\beta$  such that the restriction of  $f$  to the vector subspace of  $K$  generated by  $x_1, \dots, x_n$  is an isomorphism, is an isomorphism in  $\mathcal{K}_\beta$ .

### 3. THE COHOMOLOGY OF SOME $p$ -GROUPS

In this section, the  $p$ -group  $P(p, n)$ ,  $p$  an odd prime, and  $n \geq 3$ , is introduced, and in what follows the notation in [6] is used.

The group

$$P(p, n) = \langle A, B, C \mid A^p = B^p = C^{p^{n-2}} = [A, C] = [B, C] = 1, [A, B] = C^{p^{n-3}} \rangle \quad (1)$$

have order  $p^n$  and fits in a central extension:

$$0 \longrightarrow \mathbb{Z}/p^{n-2} \longrightarrow P(p, n) \longrightarrow \mathbb{Z}/p \times \mathbb{Z}/p \longrightarrow 0. \quad (2)$$

The cohomology of  $P(p, n)$  is calculated in [6]:

**Theorem 3.1** ([6, Proposition 3, Theorem 7, Proposition 8]).  $H^*(BP(3, 3))$  is generated by elements  $y, y', x, x', Y, Y', X, X', z$  with

$$\deg(y) = \deg(y') = 1, \deg(x) = \deg(x') = \deg(Y) = \deg(Y') = 2,$$

$$\deg(X) = \deg(X') = 3 \text{ and } \deg(z) = 6$$

subject to the following relations:

$$\begin{aligned} yy' &= 0, xy' = x'y, yY = y'Y' = xy', yY' = y'Y, \\ YY' &= xx', Y^2 = xY', Y'^2 = x'Y, \\ yX &= xY - xx', y'X' = x'Y' - xx', \\ Xy' &= x'Y - xY', X'y = xY' - x'Y, \\ XY &= x'X, X'Y' = xX', XY' = -X'Y, xX' = -x'X, \\ XX' &= 0, x(xY' + x'Y) = -xx'^2, x'(xY' + x'Y) = -x'x^2, \\ x^3y' - x'^3y &= 0, x^3x' - x'^3x = 0, \\ x^3Y' + x'^3Y &= -x^2x'^2 \text{ and } x^3X' + x'^3X = 0. \end{aligned}$$

Moreover, the action of the mod 3 Steenrod algebra is determined by:

$$\beta(y) = x, \beta(y') = x', \beta(Y) = X, \beta(Y') = X',$$

$$\mathcal{P}^1(X) = x^2X + zy, \mathcal{P}^1(X') = x'^2X' - zy', \text{ and } \mathcal{P}^1(z) = zc_2,$$

where  $c_2 = xY' - x'Y - x^2 - x'^2$ .

**Theorem 3.2** ([6, Proposition 3, Theorem 6, Proposition 8]). *For an odd prime  $p \geq 5$ , the cohomology  $H^*(BP(p, 3))$  is generated by elements  $y, y', x, x', Y, Y', X, X', d_4, \dots, d_p, c_4, \dots, c_{p-1}$  and  $z$  with*

$$\deg(y) = \deg(y') = 1, \deg(x) = \deg(x') = \deg(Y) = \deg(Y') = 2,$$

$$\deg(X) = \deg(X') = 3, \deg(d_i) = 2i - 1, \deg(c_i) = 2i, \text{ and } \deg(z) = 2p$$

subject to the following relations:

$$yy' = 0, xy' = x'y, yY = y'Y' = 0, yY' = y'Y,$$

$$Y^2 = Y'^2 = YY' = 0, yX = xY, y'X' = x'Y',$$

$$Xy' = 2xY' + x'Y, X'y = 2x'Y + xY'$$

$$XY = X'Y' = 0, XY' = -X'Y, xX' = -x'X,$$

$$x(xY' + x'Y) = x'(xY' + x'Y) = 0,$$

$$x^py' - x'^py = 0, x^px' = x'^px = 0,$$

$$x^pY' + x'^pY = 0, x^pX' + x'^pX = 0,$$

$$c_iy = \begin{cases} 0 & \text{for } i < p-1 \\ -x^{p-1}y & \text{for } i = p-1, \end{cases} \quad c_iy' = \begin{cases} 0 & \text{for } i < p-1 \\ -x'^{p-1}y' & \text{for } i = p-1, \end{cases}$$

$$c_ix = \begin{cases} 0 & \text{for } i < p-1 \\ -x^p & \text{for } i = p-1, \end{cases} \quad c_ix' = \begin{cases} 0 & \text{for } i < p-1 \\ -x'^p & \text{for } i = p-1, \end{cases}$$

$$c_iY = \begin{cases} 0 & \text{for } i < p-1 \\ -x^{p-1}Y & \text{for } i = p-1, \end{cases} \quad c_iY' = \begin{cases} 0 & \text{for } i < p-1 \\ -x'^{p-1}Y' & \text{for } i = p-1, \end{cases}$$

$$c_iX = \begin{cases} 0 & \text{for } i < p-1 \\ -x^{p-1}X & \text{for } i = p-1, \end{cases} \quad c_iX' = \begin{cases} 0 & \text{for } i < p-1 \\ -x'^{p-1}X' & \text{for } i = p-1, \end{cases}$$

$$c_ic_j = \begin{cases} 0 & \text{for } i+j < 2p-2 \\ x^{2p-2} + x'^{2p-2} - x^{p-1}x'^{p-1} & \text{for } i=j=p-1, \end{cases}$$

$$d_iy = \begin{cases} 0 & \text{for } i < p \\ -x^{p-1}Y & \text{for } i = p, \end{cases} \quad d_iy' = \begin{cases} 0 & \text{for } i < p \\ -x'^{p-1}Y' & \text{for } i = p, \end{cases}$$

$$d_ix = \begin{cases} 0 & \text{for } i < p-1 \\ -x^{p-1}y & \text{for } i = p-1 \\ x^{p-1}X & \text{for } i = p, \end{cases} \quad d_ix' = \begin{cases} 0 & \text{for } i < p-1 \\ -x'^{p-1}y' & \text{for } i = p-1 \\ -x'^{p-1}X' & \text{for } i = p, \end{cases}$$

$$d_iY = 0, d_iY' = 0,$$

$$\begin{aligned}
d_i X &= \begin{cases} 0 & \text{for } i \neq p-1 \\ -x^{p-1}Y & \text{for } i = p-1, \end{cases} & d_i X' &= \begin{cases} 0 & \text{for } i \neq p-1 \\ -x'^{p-1}Y' & \text{for } i = p-1, \end{cases} \\
d_i d_j &= \begin{cases} 0 & \text{for } i < p \text{ or } j < p-1 \\ x^{2p-3}Y + x'^{2p-3}Y' + x^{p-1}x'^{p-2}Y' & \text{for } i = p \text{ and } j = p-1, \end{cases} \\
d_i c_j &= \begin{cases} 0 & \text{for } i < p-1 \text{ or } j < p-1 \\ -x^{2p-3}y + x'^{2p-3}y' - x^{p-1}x'^{p-2}y' & \text{for } i = j = p-1 \\ -x^{2p-3}X + x'^{2p-3}X' - x^{p-1}x'^{p-2}X' & \text{for } i = p, j = p-1. \end{cases}
\end{aligned}$$

Moreover, the action of the mod  $p$  Steenrod algebra is determined by:

$$\begin{aligned}
\beta(y) &= x, \beta(y') = x', \beta(Y) = X, \beta(Y') = X', \\
\beta(d_i) &= \begin{cases} c_i & \text{for } i < p \\ 0 & \text{for } i = p. \end{cases} \\
\mathcal{P}^1(X) &= x^{p-1}X + zy, \mathcal{P}^1(X') = x'^{p-1}X' - zy', \\
\mathcal{P}^1(c_i) &= \begin{cases} izc_{i-1} & \text{if } 2 \leq i < p-1 \\ -zc_{p-2} + x^{2p-2} + x'^{2p-2} - x^{p-1}x'^{p-1} & \text{if } i = p-1, \end{cases} \\
\mathcal{P}^1(z) &= zc_{p-1},
\end{aligned}$$

where  $c_2$  and  $c_3$  are non zero multiples of  $xY' + x'Y$  and  $XX'$  respectively.

**Remark 3.3.** As stated in [6, p. 71] one can verify that in the cohomology ring  $H^*(BP(p, 3))$ ,  $p \geq 5$ , any product of the generators  $y, y', x, x', Y, Y', X, X'$  in degree greater than 6 may be expressed in the form

$$\begin{aligned}
&f_1 + f_2Y + f_3Y' && \text{for even total degree} \\
&f_1y + f_2y' + f_3X + f_4X' && \text{for odd total degree,}
\end{aligned}$$

where each  $f_i$  is a polynomial in  $x$  and  $x'$ . So, for  $n \leq p$ , any element  $u \in H^{2n-1}(BP(p, 3))$  can be expressed as:

$$u = ad_n + f_1y + f_2y' + f_3X + f_4X',$$

where  $a \in \mathbb{F}_p$  and each  $f_i$  is a polynomial in  $x$  and  $x'$ .

**Theorem 3.4** ([6, Theorem 4]). *For  $n \geq 4$ ,  $H^*(BP(p, n))$  is generated by elements  $u, y, y', x, x', c_2, c_3, \dots, c_{p-1}, z$  with*

$$\deg(u) = \deg(y) = \deg(y') = 1, \deg(x) = \deg(x') = 2, \deg(c_i) = 2i, \deg(z) = 2p,$$

subject to the following relations:

$$\begin{aligned}
xy' &= x'y, x^p y' = x'^p y, x^p x' = x'^p x, \\
c_i y &= \begin{cases} 0 & \text{for } i < p-1 \\ -x^{p-1}y & \text{for } i = p-1, \end{cases} & c_i y' &= \begin{cases} 0 & \text{for } i < p-1 \\ -x'^{p-1}y' & \text{for } i = p-1, \end{cases} \\
c_i x &= \begin{cases} 0 & \text{for } i < p-1 \\ -x^p & \text{for } i = p-1, \end{cases} & c_i x' &= \begin{cases} 0 & \text{for } i < p-1 \\ -x'^p & \text{for } i = p-1, \end{cases}
\end{aligned}$$

$$c_i c_j = \begin{cases} 0 & \text{for } i + j < 2p - 2 \\ x^{2p-2} + x'^{2p-2} - x^{p-1} x'^{p-1} & \text{for } i = j = p - 1. \end{cases}$$

Moreover, we have the following operations of the mod  $p$  Steenrod algebra:

$$\beta(y) = x, \beta(y') = x', \beta(u) = \begin{cases} 0 & \text{for } n > 4 \\ y'y & \text{for } n = 4, \end{cases}$$

and

$$\mathcal{P}^1(z) = zc_{p-1}, \mathcal{P}^1(c_i) = \begin{cases} izc_{i-1} & \text{for } i < p - 1 \\ -zc_{p-2} + x^{2p-2} + x'^{2p-2} - x^{p-1} x'^{p-1} & \text{for } i = p - 1, \end{cases}$$

where  $c_1 = y'y$ .

**Remark 3.5.** If we look at  $H_\beta^*(BP(p, n))$  for  $n \geq 4$  as an object in  $\mathcal{K}_\beta$  we have that there is a Bockstein operator  $\beta_{n-3}(u) = yy'$  [6, p. 66]. From this and the cohomology of  $BP(p, n)$  we can deduce that for  $n = 3$  and  $n = 4$  the study of the cohomological uniqueness of these spaces can be done in  $\mathcal{K}$ , the category of unstable algebras, instead of working in  $\mathcal{K}_\beta$ .

**Remark 3.6.** Consider the groups  $B\mathbb{Z}/p^i \times B\mathbb{Z}/p \times B\mathbb{Z}/p$  and fix the following notation for the cohomology:

$$H^*(B\mathbb{Z}/p^i \times B\mathbb{Z}/p \times B\mathbb{Z}/p; \mathbb{F}_p) = E(u_i, y, y') \otimes \mathbb{F}_p[v_i, x, x']$$

where generators are sorted as components.

There is a tower of extensions:

$$\begin{aligned} BP(p, n) &\xrightarrow{\pi_{n-3}} B\mathbb{Z}/p^{n-3} \times B\mathbb{Z}/p \times B\mathbb{Z}/p \xrightarrow{\pi_{n-4}} \\ &B\mathbb{Z}/p^{n-4} \times B\mathbb{Z}/p \times B\mathbb{Z}/p \longrightarrow \cdots \quad \cdots \xrightarrow{\pi_1} B\mathbb{Z}/p \times B\mathbb{Z}/p \times B\mathbb{Z}/p \end{aligned}$$

where each extension  $\pi_i$  for  $i < n - 3$ :

$$0 \longrightarrow B\mathbb{Z}/p \longrightarrow B\mathbb{Z}/p^i \times B\mathbb{Z}/p \times B\mathbb{Z}/p \xrightarrow{\pi_i} B\mathbb{Z}/p^{i-1} \times B\mathbb{Z}/p \times B\mathbb{Z}/p \longrightarrow 0$$

is classified by  $\beta_{i-1}(u_i)$  and

$$0 \longrightarrow B\mathbb{Z}/p \rightarrow BP(p, n) \xrightarrow{\pi_{n-3}} B\mathbb{Z}/p^{n-3} \times B\mathbb{Z}/p \times B\mathbb{Z}/p \longrightarrow 0$$

is classified by  $\beta_{n-3}(u_{n-3}) - yy'$ .

#### 4. COHOMOLOGICAL UNIQUENESS

Let  $p$  be an odd prime,  $n \geq 3$  and  $P(p, n)$  be the group defined in Equation (1). In this section we prove that the homotopy type of the classifying space of  $P(p, n)$  is determined by its cohomology (Definition 2.2). The very first step toward that result is to study the behaviour of some endomorphisms of the mod  $p$  cohomology ring of  $BP(p, n)$ .

First we consider the case  $n \leq 4$ . In this case we do not need to use higher Bocksteins and it is enough to consider the structure of unstable algebra.

**Theorem 4.1.** *Let  $\varphi: H^*(BP(3, 3)) \rightarrow H^*(BP(3, 3))$  be a homomorphism of  $\mathcal{A}_3$ -algebras which restricts to the identity in  $H^1$ . Then  $\varphi$  is an isomorphism.*

*Proof.* In this proof we follow the notation in Theorem 3.1 for generators and relations in cohomology.

By hypothesis  $\varphi(y) = y$  and  $\varphi(y') = y'$ . Now, since  $\beta(y) = x$  and  $\beta(y') = x'$ , then  $\varphi(x) = \varphi(\beta(y)) = \beta(\varphi(y)) = \beta(y) = x$  and analogously  $\varphi(x') = x'$ . Moreover, by dimensional reasons,

$$\varphi(Y) = aY + bY' + cx + dx'$$

for some  $a, b, c, d \in \mathbb{F}_3$ . Because  $yY = xy'$  we obtain

$$xy' = \varphi(xy') = \varphi(yY) = y\varphi(Y) = ayY + byY' + cyx + dyx'$$

and regrouping terms

$$xy' = (a + d)xy' + byY' + cyx.$$

From here we obtain  $a + d = 1$  and  $b = c = 0$ , and  $\varphi(Y) = aY + dx'$  with  $a + d = 1$ . Analogously  $\varphi(Y') = bY' + cx$  with  $b, c \in \mathbb{F}_3$  and  $b + c = 1$ . Now, as  $Y^2 = xY'$ , we have

$$\begin{aligned} \varphi(Y)^2 &= x\varphi(Y') \\ a^2Y^2 + d^2x'^2 + 2adYx' &= bxY' + cx^2. \end{aligned}$$

This implies that  $c = d = 0$  and  $a^2 = a = b = 1$ . So  $\varphi(Y) = Y$  and  $\varphi(Y') = Y'$ , and applying Bockstein again  $\varphi(X) = X$  and  $\varphi(X') = X'$  too. So  $\varphi$  is the identity up to dimension five and it remains to check where does it map  $z$ .

Using the first Steenrod power of  $X$

$$\begin{aligned} \varphi(\mathcal{P}^1(X)) &= \mathcal{P}^1(\varphi(X)) \\ \varphi(x^2X + zy) &= \mathcal{P}^1(X) \\ x^2X + \varphi(z)y &= x^2X + zy \\ \varphi(z)y &= zy. \end{aligned}$$

Thus  $\varphi(z) = z + \alpha$  where  $\alpha y = 0$  and  $\alpha \in \langle y, y', x, x', Y, Y', X, X' \rangle$ . So  $\varphi(\alpha) = \alpha$ ,  $z = \varphi(z - \alpha)$  and  $\varphi$  is an epimorphism. In fact, because  $H^*(BP(3, 3))$  is a finite dimensional  $\mathbb{F}_3$ -vector space in each dimension,  $\varphi$  is an isomorphism dimension-wise, and thus  $\varphi$  is an isomorphism.  $\square$

**Theorem 4.2.** *Let  $p \geq 5$  be a prime. If  $\varphi: H^*(BP(p, 3)) \rightarrow H^*(BP(p, 3))$  is a homomorphism in  $\mathcal{K}$  that restricts to the identity in  $H^1$ , then  $\varphi$  is an isomorphism.*

*Proof.* Consider the notation of generators and relations in  $H_\beta^*(BP(p, 3))$  given in Theorem 3.2. We calculate the image under  $\varphi$  of every generator in  $H^*(BP(p, 3))$ .

As  $\varphi$  is the identity on  $y$  and  $y'$ , applying Bockstein operations we get that  $\varphi(x) = x$  and  $\varphi(x') = x'$ .

As  $Y$  is in degree 2, there exist coefficients  $a, b, c, d$  such that

$$\varphi(Y) = ax + bx' + cY + dY'.$$

Using the relation  $Y^2 = 0$ , we get  $\varphi(Y)^2 = 0$ , which implies that  $a = b = 0$ , and so  $\varphi(Y) = cY + dY'$ . The relation  $yY = 0$  implies  $0 = y\varphi(Y) = dyY'$ , so  $d = 0$ , getting that there is  $c$  such that  $\varphi(Y) = cY$ . Using the same arguments, there is  $d$  such that  $\varphi(Y') = dY'$ .

According to Remark 3.3, there are  $a_n \in \mathbb{F}_p$  and  $f_{n,i}$  polynomials in  $x$  and  $x'$  such that for  $4 \leq n \leq p$

$$\varphi(d_n) = a_n d_n + f_{n,1}y + f_{n,2}y' + f_{n,3}X + f_{n,4}X',$$

and applying the Bockstein operation, we get that for  $4 \leq n \leq p-1$ :

$$\varphi(c_n) = a_n c_n + f_{n,1}x + f_{n,2}x'.$$

The relation  $c_{p-1}x = -x^p$  gives rise to the following equalities:

$$\begin{aligned} -x^p &= \varphi(-x^p) = \varphi(c_{p-1}x) = \varphi(c_{p-1})\varphi(x) = \varphi(c_{p-1})x = \\ &= a_{p-1}c_{p-1}x + f_{p-1,1}x^2 + f_{p-1,2}xx' = -a_{p-1}x^p + f_{p-1,1}x^2 + f_{p-1,2}xx', \end{aligned}$$

so  $(a_{p-1} - 1)x^p = f_{p-1,1}x^2 + f_{p-1,2}xx'$ , and as there are no relations involving  $x$  and  $x'$  till degree  $2p+2$ , we can simplify:

$$f_{p-1,1}x + f_{p-1,2}x' = (a_{p-1} - 1)x^{p-1}. \quad (3)$$

Doing the same computations using the relation  $c_{p-1}x' = -x'^p$  we get

$$f_{p-1,1}x + f_{p-1,2}x' = (a_{p-1} - 1)x'^{p-1}. \quad (4)$$

Comparing now (3) and (4), and using again that there is no relation between  $x$  and  $x'$  till degree  $2p+2$ , we get  $a_{p-1} = 1$ ,  $\varphi(c_{p-1}) = c_{p-1}$ .

Now we see that  $\varphi(c_n) = a_n c_n$ , for  $4 \leq n < p-1$ : using the relation  $c_n x = 0$  and applying  $\varphi$  we get  $f_{n,1}x + f_{n,2}x' = 0$ , so

$$\varphi(c_n) = a_n c_n. \quad (5)$$

In order to calculate  $\varphi(z)$ , we apply  $\varphi$  to the equality:

$$\mathcal{P}^1(c_{p-1}) = -zc_{p-2} + x^{2p-2} + x'^{2p-2} - x^{p-1}x'^{p-1}.$$

Since  $\varphi(c_{p-1}) = c_{p-1}$ ,  $\varphi(x) = x$ , and  $\varphi(x') = x'$ , we get

$$zc_{p-2} = \varphi(z)a_{p-2}c_{p-2}. \quad (6)$$

As the generator  $z$  does not appear in any relation, Equation (6) implies that  $a_{p-2} \neq 0$  and  $\varphi(z) = a_{p-2}^{-1}z + g$ , where  $g$  is an expression not involving  $z$ , and such that  $gc_{p-2} = 0$ .

We use that we know that  $a_{p-2} \neq 0$  to check that  $a_n \neq 0$  for  $4 \leq n < p-2$  with an induction argument: assume  $\varphi(c_n) = a_n c_n$  with  $a_n \neq 0$  and  $5 \leq n \leq p-2$ , and compute  $\varphi(c_{n-1})$ :

$$nz c_{n-1} = \mathcal{P}^1(c_n) = \mathcal{P}^1(\varphi(a_n^{-1}c_n)) = \varphi(a_n^{-1}\mathcal{P}^1(c_n)) = a_n^{-1}n\varphi(z)a_{n-1}c_{n-1}.$$

This implies  $zc_{n-1} = a_n^{-1}a_{n-1}\varphi(z)c_{n-1}$ , and this can only happens if  $a_{n-1} \neq 0$  and  $\varphi(z) = a_n a_{n-1}^{-1}z + g$  ( $g$  not involving  $z$ ).

From the expression  $c_3 = \mu X X'$  we deduce that  $\varphi(c_3) = a_3 c_3$  with  $a_3 = cd$ , where  $c$  and  $d$  were introduced at the beginning of the proof and are such that  $\varphi(Y) = cY$  and  $\varphi(Y') = dY'$ . The argument above for  $\mathcal{P}^1(c_4)$  shows that  $a_3$  is non-zero neither. Hence  $c$ ,  $d$  and  $a_n$  for all  $n \in \{3, \dots, p-1\}$  are non-zero.

Let us check now that the coefficients  $c$  and  $d$  are equal: recall that  $c_2$  was defined as  $\lambda(xY' + x'Y)$  with  $\lambda$  non-zero. Then, applying  $\mathcal{P}^1$  to  $c_3$  we get:

$$\begin{aligned} 3zc_2 &= \mathcal{P}^1(c_3) = \mathcal{P}^1(\varphi(a_3^{-1}c_3)) = a_3^{-1}\varphi(\mathcal{P}^1(c_3)) = a_3^{-1}(\varphi(3zc_2)) = \\ &= a_3^{-1}3\varphi(z)\varphi(c_2) = a_3^{-1}3\varphi(z)\lambda(dxY' + cx'Y), \end{aligned}$$

which implies  $\lambda z(xY' + x'Y) = a_3^{-1}\lambda\varphi(z)(dxY' + cx'Y)$  and can be simplified to:

$$zxY' + zx'Y = da_3^{-1}\varphi(z)xY' + ca_3^{-1}\varphi(z)x'Y. \quad (7)$$

Again, as  $z$  does not appear in any relation, Equation (7) can be true only if  $c = d$ . In particular,  $\varphi(c_2) = a_2c_2$  with  $a_2 = c\lambda \neq 0$ .

Now we can assume that all the coefficients  $a_n$  for  $2 \leq n \leq p-1$  and  $c$  and  $d$  are equal to 1: as all are different to zero, and  $r^{p-1} = 1$  if  $r \in \mathbb{F}_p \setminus \{0\}$ ,  $\varphi^{p-1}$  is the identity in  $Y$ ,  $Y'$  and  $c_n$ . Use now that  $\varphi$  is an isomorphism if and only if  $\varphi^{p-1}$  is so. Therefore at this point we have that:

$$\varphi(y) = y, \varphi(y') = y', \varphi(x) = x, \varphi(x') = x',$$

$$\varphi(Y) = Y, \varphi(Y') = Y', \varphi(X) = X, \varphi(X') = X',$$

$$\varphi(c_i) = c_i \text{ for } 2 \leq i \leq p-1, \varphi(d_i) = d_i + g_i \text{ for } 4 \leq i \leq p-1, \text{ and } f(z) = z + g$$

where  $g$  and all  $g_i$  are expressions in  $x, x', y, y', X, X', Y$  and  $Y'$ . This implies that all generators but  $d_p$  are in the image of  $\varphi$ .

The image of  $d_p$ , as it is in odd degree greater than 6, must be:

$$\varphi(d_p) = a_p d_p + f_{p,1}y + f_{p,2}y' + f_{p,3}X + f_{p,4}X'$$

with  $a_p \in \mathbb{F}_p$ , and  $f_{p,i}$  polynomials in  $x$  and  $x'$ . As  $\beta(d_p) = 0$ , the Bockstein operation on  $\varphi(d_p)$  must vanish, and this means:

$$0 = \beta(\varphi(d_p)) = f_{p,1}x + f_{p,2}x'.$$

So this is a polynomial in  $x, x'$  which must be zero. As there are not relations involving just  $x$  and  $x'$  in this degree, we deduce that there exist  $f_p$  a polynomial in  $x$  and  $x'$  such that  $f_{p,1} = f_p x'$  and  $f_{p,2} = f_p x$ . This implies that (recall  $xy' = x'y$ ),

$$f_{p,1}y + f_{p,2}y' = f_p(x'y - xy') = 0,$$

and then

$$\varphi(d_p) = a_p d_p + f_{p,3}X + f_{p,4}X'.$$

As any expression on  $x, x', X$  and  $X'$  is in the image, we have only to check that  $a_p \neq 0$ . To do that we assume that  $a_p = 0$ , getting a contradiction.

If  $\varphi(d_p) = f_{p,3}X + f_{p,4}X'$ , using the relation  $d_p Y = 0$  we get  $(f_{p,3}X + f_{p,4}X')Y = 0$ . Applying that  $XY = 0$  this means that  $f_{p,4}X'Y = 0$ . Recall now that  $f_{p,4}$  is a polynomial on  $x, x'$  of degree  $2p-4$ . Looking at the relations under this degree,  $f_{p,4}X'Y = 0$  means that  $f_{p,4}$  must be zero. The same argument can be applied to the relation  $d_p Y' = 0$ , obtaining that  $f_{p,3} = 0$ . So at this point we have that  $\varphi(d_p) = 0$ .

Use now that  $d_p x = x^{p-1}X \neq 0$ , and applying  $\varphi$  to both sides of this equality we get the following contradiction:

$$0 = \varphi(d_p x) = \varphi(x^{p-1}X) = x^{p-1}X \neq 0.$$

Therefore  $a_p \neq 0$  and  $\varphi$  is an isomorphism.  $\square$

We now consider the case of  $n > 3$ . Here the use of Bockstein operators is needed.

**Theorem 4.3.** *Let  $p$  be an odd prime and consider the notation of the generators and relations in  $H_\beta^*(BP(p, n))$  as in Theorem 3.4.*

- (a) *If  $\varphi: H^*(BP(p, 4)) \rightarrow H^*(BP(p, 4))$  is a homomorphism of unstable algebras that fixes  $y$  and  $y'$ , then  $\varphi$  is an isomorphism.*
- (b) *If  $n \geq 5$  and  $\varphi: H_\beta^*(BP(p, n)) \rightarrow H_\beta^*(BP(p, n))$  is a homomorphism in  $\mathcal{K}_\beta$  which fixes  $y$  and  $y'$ . Then  $\varphi$  is an isomorphism.*

*Proof.* We prove both results at the same time. Just observe that the Bockstein used in the proof is  $\beta_{n-3}$ , which is part of the mod  $p$  Steenrod algebra when  $n = 4$ .

Starting from  $\varphi(y) = y$  and  $\varphi(y') = y'$  and using the Bockstein operator we reach  $\varphi(x) = x$  and  $\varphi(x') = x'$ . On the other hand there exist  $a, b, c \in \mathbb{F}_p$  such that  $\varphi(u) = au + by + cy'$ . From Remark 3.5 we deduce that  $\beta_{n-3}(u) = y'y$ , so, as the morphism is in  $\mathcal{K}_\beta$ ,

$$\varphi(\beta_{n-3}(u)) = \beta_{n-3}(\varphi(u)) \Rightarrow y'y = \begin{cases} ay'y + bx + cx' & \text{for } n = 4 \\ ay'y & \text{for } n > 4. \end{cases}$$

We obtain then that  $a = 1$  and  $b = c = 0$  for  $n = 4$ , and that  $a = 1$  for  $n > 4$ . Hence,  $u = \varphi(u) - by - cy' = \varphi(u - by - cy')$  and  $\langle u, y, y', x, x' \rangle \leq \text{Im } \varphi$ .

Now consider the generator  $c_{p-1}$ . We can write  $\varphi(c_{p-1}) = a_{p-1}c_{p-1} + bx^{p-1} + g_{p-1}$  with  $a_{p-1}, b \in \mathbb{F}_p$  and  $g_{p-1}$  not containing multiples of the monomials  $c_{p-1}$  and  $x^{p-1}$ . Applying  $\varphi$  to the equation  $c_{p-1}x' = -x'^p$  we obtain  $-x'^p = a_{p-1}c_{p-1}x' + bx^{p-1}x' + g_{p-1}x' = -a_{p-1}x'^p + bx^{p-1}x' + g_{p-1}x'$ . The only equation at degree  $p$  involving  $x'^p$  is  $c_{p-1}x' = -x'^p$  (notice that a multiple of the equation  $xy' = x'y$  in degree 3 does not involve  $x'^p$  and that this may occur just for  $p = 3$ ). As  $g_{p-1}$  does not contain neither  $c_{p-1}$  nor  $x^{p-1}$  we deduce that  $a_{p-1} = 1$ ,  $b = 0$ . Hence  $\varphi(c_{p-1}) = c_{p-1} + g_{p-1}$ .

Next we deal with  $c_{p-2}$  of degree  $2(p-2)$  and  $z$  of degree  $2p$ . Their images are  $\varphi(c_{p-2}) = a_{p-2}c_{p-2} + g_{p-2}$  and  $\varphi(z) = a_z z + g_z$ , with  $a_{p-2}, a_z \in \mathbb{F}_p$ , and  $g_{p-2}$  and  $g_z$  not involving the monomials  $c_{p-2}$  and  $z$  respectively. Write the Steenrod power  $\mathcal{P}^1(c_{p-1}) = -zc_{p-2} + x^{2p-2} + x'^{2p-2} - x^{p-1}x'^{p-1}$  as  $\mathcal{P}^1(c_{p-1}) = -zc_{p-2} + f$ , with  $f = x^{2p-2} + x'^{2p-2} - x^{p-1}x'^{p-1}$ . Applying  $\varphi$  we get:

$$\begin{aligned} \varphi(\mathcal{P}^1(c_{p-1})) &= \mathcal{P}^1(\varphi(c_{p-1})) \\ \varphi(-zc_{p-2} + f) &= \mathcal{P}^1(c_{p-1} + g_{p-1}) \\ -(a_z z + g_z)(a_{p-2}c_{p-2} + g_{p-2}) + f &= -zc_{p-2} + f + \mathcal{P}^1(g_{p-1}) \\ -a_z a_{p-2} z c_{p-2} - a_z z g_{p-2} - a_{p-2} g_z c_{p-2} - g_z g_{p-2} &= -zc_{p-2} + \mathcal{P}^1(g_{p-1}). \end{aligned}$$

Notice that there is no relation involving the generator  $z$  and the equations involving  $c_{p-2}$  are  $c_{p-2}y = c_{p-2}y' = c_{p-2}x = c_{p-2}x' = c_{p-2}c_j = 0$  for  $j < p$ . Also, the monomial  $zc_{p-2}$  cannot appear in  $zg_{p-2}$ ,  $g_z c_{p-2}$ , and  $g_z g_{p-2}$ . Finally,  $\mathcal{P}^1(g_{p-1})$  does not involve  $zc_{p-2}$  as  $g_{p-1}$  does not involve  $c_{p-1}$  and the action of  $\mathcal{P}^1$  on  $u, y, y', x, x'$  is determined by the axioms. Hence,  $a_z a_{p-2} = 1$  and both  $a_z$  and  $a_{p-2}$  are non-zero.

For the rest of the generators  $c_i$  for  $i = 2, 3, \dots, p-3$  we can write  $\varphi(c_i) = a_i c_i + g_i$ , with  $a_i \in \mathbb{F}_p$  and  $g_i$  not involving  $c_i$ . The Steenrod power  $\mathcal{P}^1(c_{i+1}) = (i+1)zc_i$  provides then

$$\begin{aligned}\varphi(\mathcal{P}^1(c_{i+1})) &= \mathcal{P}^1(\varphi(c_{i+1})) \\ \varphi((i+1)zc_i) &= \mathcal{P}^1(a_{i+1}c_{i+1} + g_{i+1}) \\ (i+1)(a_z z + g_z)(a_i c_i + g_i) &= (i+1)a_{i+1}zc_i + \mathcal{P}^1(g_{i+1}) \\ (i+1)(a_z a_i z c_i + a_z z g_i + a_i a_z c_i + a_z g_i) &= (i+1)a_{i+1}zc_i + \mathcal{P}^1(g_{i+1}).\end{aligned}$$

Notice again that there is no relation involving the generator  $z$  and the relations involving  $c_i$  are  $c_i y = c_i y' = c_i x = c_i x' = c_i c_j = 0$  for  $j < 2p-2-i$ . Also, the monomial  $zc_i$  cannot appear in  $zg_i$ ,  $g_z c_i$ , and  $g_z g_i$ . Moreover,  $\mathcal{P}^1(g_{i+1})$  does not involve  $zc_i$  as  $g_{i+1}$  does not involve  $c_{i+1}$ . We deduce that  $(i+1)a_z a_i = (i+1)a_{i+1}$ . As  $a_z \neq 0$  and  $a_{p-2} \neq 0$ , an inductive argument shows that  $a_i \neq 0$  for  $i = 2, 3, \dots, p-3$ , and hence for all  $i = 2, 3, \dots, p-1$ .

To finish we show that all the generators  $c_2, c_3, \dots, c_{p-1}, z$  are in the image of  $\varphi$ . We start with  $c_2 = \frac{1}{\alpha_2}(\varphi(c_2) - g_2)$ . As  $g_2 \in \langle u, x, x', y, y' \rangle \leq \text{Im } \varphi$  then  $c_2$  is also in the image of  $\varphi$ . An inductive argument shows that  $c_i = \frac{1}{\alpha_i}(\varphi(c_i) - g_i)$  is in the image of  $\varphi$  as  $g_i$  belongs to  $\langle u, x, x', y, y', c_2, c_3, \dots, c_{i-1} \rangle$ . This argument also applies to show that  $z \in \text{Im } \varphi$ .

Hence,  $\varphi$  is an epimorphism. Because  $H_\beta^*(BP(p, n))$  is finite in each dimension  $\varphi$  is an isomorphism.  $\square$

Then, the following corollary is straightforward:

**Corollary 4.4.**  $H_\beta^*(BP(p, n))$  for odd  $p$  and  $n \geq 3$  is weakly generated (Definition 2.3) by  $y$ , and  $y'$ .

*Proof.* Let  $\varphi$  be an endomorphism of  $H_\beta^*(BP(p, n))$  which is an isomorphism on  $\langle y, y' \rangle$ . Using the outer automorphism group of  $P(p, n)$  which is described in [4, Lemma A.5] there is a morphism  $f: BP(p, n) \rightarrow BP(p, n)$  such that the composition  $f^* \circ \varphi$  fixes  $y$  and  $y'$ . Use now Theorems 4.1, 4.2 and 4.3 to get the result.  $\square$

Notice that for any finite  $p$ -group there is a natural isomorphism  $H^1 P \cong P/\Phi(P)$ , where  $\Phi(P)$  stands for the Frattini subgroup of  $P$  [5, p. 173]. Therefore, Theorems 4.1, 4.2 and 4.3 can be seen as cohomological counterpart of the following group theoretical result.

**Proposition 4.5.** Let  $P$  be a finite  $p$ -group and  $f: P \rightarrow P$  a group morphism such that the induced morphism at the level of Frattini quotients  $\tilde{f}: P/\Phi(P) \rightarrow P/\Phi(P)$  is an isomorphism. Then  $f$  is an isomorphism.

*Proof.* Let  $n$  be such that  $P/\Phi(P) = (\mathbb{Z}/p)^n$  [5, Theorem 5.1.3]. Assume  $f$  is not an isomorphism, then  $f(P) \leq H < P$  for some maximal subgroup  $H < P$ , and therefore  $\tilde{f}(P/\Phi(P)) < H/\Phi(P) = (\mathbb{Z}/p)^{n-1} < P/\Phi(P)$ , that is,  $\tilde{f}$  is not an isomorphism.  $\square$

Now, we apply the results above to obtain the cohomology uniqueness of the classifying space  $BP(p, n)$ . We split this result into two corollaries because the structure of  $P(p, 3)$  is essentially different from that of  $P(p, n)$ ,  $n > 4$ .

**Corollary 4.6.** Let  $p$  be an odd prime and  $X$  be a  $p$ -complete space such that  $H^*(X) \cong H^*(BP(p, 3))$  as unstable algebras. Then  $X \simeq BP(p, 3)$ .

*Proof.* We begin with the central extension

$$0 \longrightarrow \mathbb{Z}/p \longrightarrow P(p, 3) \xrightarrow{\pi} \mathbb{Z}/p \times \mathbb{Z}/p \longrightarrow 0$$

classified by  $yy' \in H^2(B\mathbb{Z}/p \times B\mathbb{Z}/p)$ , which gives rise to the principal fibration

$$BP(p, 3) \xrightarrow{B\pi} B\mathbb{Z}/p \times B\mathbb{Z}/p \xrightarrow{yy'} B^2\mathbb{Z}/p.$$

Consider the map  $\pi_X: X \rightarrow B\mathbb{Z}/p \times B\mathbb{Z}/p$  that classifies the classes  $y, y' \in H^1(X)$ . Then the composite

$$X \xrightarrow{\pi_X} B\mathbb{Z}/p \times B\mathbb{Z}/p \xrightarrow{yy'} B^2\mathbb{Z}/p$$

is null-homotopic because of Theorems 3.1 and 3.2, and so  $\pi_X$  lifts to  $\varphi: X \rightarrow BP(p, 3)$ , giving the commutative diagram

$$\begin{array}{ccc} & & BP(p, 3) \\ & \nearrow \varphi & \downarrow B\pi \\ X & \xrightarrow{\pi_X} & B\mathbb{Z}/p \times B\mathbb{Z}/p \end{array}$$

which implies that  $\varphi^*$  fixes  $y$  and  $y'$ . Now apply Theorems 4.1 and 4.2 to  $\varphi^*$ .  $\square$

**Corollary 4.7.** *Let  $p$  be an odd prime and  $X$  be a  $p$ -complete space.*

- (a) *If  $H^*(X) \cong H^*(BP(p, 4))$  as unstable algebras then  $X \simeq BP(p, 4)$ .*
- (b) *If  $n \geq 5$  and  $H_\beta^*(X) \cong H_\beta^*(BP(p, n))$  as objects in  $\mathcal{K}_\beta$ . Then  $X \simeq BP(p, n)$ .*

*Proof.* Consider the central extension and notations as in Remark 3.6.

If  $n = 4$  we have

$$0 \longrightarrow \mathbb{Z}/p \longrightarrow P(p, 4) \xrightarrow{\pi_1} \mathbb{Z}/p \times \mathbb{Z}/p \times \mathbb{Z}/p \longrightarrow 0$$

Let now  $\pi_{1,X}$  be the map  $\pi_{1,X}: X \rightarrow B\mathbb{Z}/p \times B\mathbb{Z}/p \times B\mathbb{Z}/p$  that classifies the classes  $u, y, y' \in H^1(X)$ .

The composite

$$X \xrightarrow{\pi_{1,X}} B\mathbb{Z}/p \times B\mathbb{Z}/p \times B\mathbb{Z}/p \xrightarrow{yy' - \beta(u)} B^2\mathbb{Z}/p$$

is null-homotopic because of Remark 3.6, and so  $\pi_{1,X}$  lifts to  $\varphi: X \rightarrow BP(p, 4)$ , giving the commutative diagram

$$\begin{array}{ccc} & & BP(p, 4) \\ & \nearrow \varphi & \downarrow B\pi_1 \\ X & \xrightarrow{\pi_{1,X}} & B\mathbb{Z}/p \times B\mathbb{Z}/p \times B\mathbb{Z}/p \end{array}$$

This implies that  $\varphi^*$  fixes  $y$  and  $y'$  and the result is a consequence of Theorem 4.3.

If  $n > 4$ , we must consider before the extensions:

$$0 \longrightarrow \mathbb{Z}/p \longrightarrow \mathbb{Z}/p^i \times \mathbb{Z}/p \times \mathbb{Z}/p \xrightarrow{\pi_{i-1}} \mathbb{Z}/p^{i-1} \times \mathbb{Z}/p \times \mathbb{Z}/p \longrightarrow 0,$$

classified by  $\beta_{i-1}(u) \in H_\beta^*(B\mathbb{Z}/p^{i-1} \times B\mathbb{Z}/p \times B\mathbb{Z}/p)$ .

Let  $\pi_{1,X}$  be now the map  $\pi_{1,X}: X \rightarrow B\mathbb{Z}/p \times B\mathbb{Z}/p \times \mathbb{Z}/p$  that classifies the classes  $u, y, y' \in H^1(X)$ . As  $\beta_1(u) = 0 \in H_\beta^*(X)$  the map  $\pi_{1,X}$  extends to a map  $\pi_{2,X}$  having the following commutative diagram:

$$\begin{array}{ccc} & B\mathbb{Z}/p^2 \times B\mathbb{Z}/p \times B\mathbb{Z}/p & \\ \nearrow \pi_{2,X} & \downarrow B\pi_1 & \\ X \xrightarrow{\pi_{1,X}} & B\mathbb{Z}/p \times B\mathbb{Z}/p \times B\mathbb{Z}/p & \end{array}$$

Using the same argument we can proceed extending the map till  $B\pi_{n-3}: B\mathbb{Z}/p^{n-3} \times B\mathbb{Z}/p \times B\mathbb{Z}/p$ . To do the last step we use again Remark 3.6 and that  $\beta_{n-3}(u) = yy' \in H_\beta^*(X)$ , and obtain a map  $\varphi$  which gives the commutative diagram:

$$\begin{array}{ccc} & BP(p, n) & \\ \nearrow \varphi & \downarrow B\pi_{n-2} \circ \dots \circ B\pi_1 & \\ X \xrightarrow{\pi_{1,X}} & B\mathbb{Z}/p \times B\mathbb{Z}/p \times B\mathbb{Z}/p & \end{array}$$

which implies again that  $\varphi^*$  fixes  $y$  and  $y'$ , so apply Theorem 4.3.  $\square$

## 5. SOME APPLICATIONS TO GROUP THEORY

The techniques used along the proof of the last two corollaries in the previous section can be used to obtain a cohomological characterization of  $P(p, n)$  as a complement for some  $N \trianglelefteq G$ , for a super group  $P(p, n) \leq G$ . Recall that given a group  $G$ , and a normal subgroup  $N \trianglelefteq G$ ,  $K \leq G$  is a complement for  $N$  if  $G = NK$  and  $N \cap K = 1$ , that is, if  $G = N \rtimes K$ .

Again, we consider the case  $n = 3$  separately.

**Proposition 5.1.** *Let  $p$  be an odd prime and  $G$  be a finite group such that  $P(p, 3) \leq G$ . Assume also that there exists  $\psi: H^*(BP(p, 3)) \rightarrow H^*(BG)$  as unstable algebras such that  $(\text{res} \circ \psi)|_{H_\beta^1(BP(p, 3))}$  is the identity. Then  $P(p, 3)$  is a complement for some  $N \trianglelefteq G$ .*

*Proof.* As it was announced, we work along the lines in the proof of Corollary 4.6. We begin by considering the map  $B\pi_G: BG \rightarrow B\mathbb{Z}/p \times B\mathbb{Z}/p$  that classifies the classes  $\psi(y), \psi(y') \in H^1(BG)$ . Then  $B\pi_G^*(yy') = B\pi_G^*(y)B\pi_G^*(y') = \psi(y)\psi(y') = \psi(yy') = \psi(0) = 0$  (Theorems 3.1 and 3.2), and the composite

$$BG \xrightarrow{B\pi_G} B\mathbb{Z}/p \times B\mathbb{Z}/p \xrightarrow{yy'} B^2\mathbb{Z}/p$$

is null-homotopic. Therefore  $B\pi_G$  lifts to  $B\phi: BG \rightarrow BP(p, 3)$ , giving the commutative diagram

$$\begin{array}{ccccc} & & BP(p, 3) & & \\ & \nearrow B\phi & \downarrow B\pi & & \\ BP(p, 3) & \xrightarrow{\text{res}} & BG & \xrightarrow{B\pi_G} & B\mathbb{Z}/p \times B\mathbb{Z}/p \end{array}$$

which implies that  $B\phi^*(y) = \psi(y)$  and  $B\phi^*(y') = \psi(y')$ , and

$$(\text{res} \circ B\phi)(y) = (\text{res}^* \circ \psi)(y) = y \text{ and } (\text{res} \circ B\phi)^*(y') = (\text{res}^* \circ \psi)(y') = y'.$$

Now, applying Theorems 4.1 and 4.2, or Proposition 4.5 we obtain that  $\phi|_{P(p,3)}$  is an automorphism of  $P(p,3)$ , that is,  $P(p,3)$  is a complement for  $N = \text{Ker } \phi \trianglelefteq G$ .  $\square$

We now proceed with the case  $n > 3$ .

**Proposition 5.2.** *Let  $p$  be an odd prime and  $G$  be a finite group such that  $P(p,n) \leq G$ .*

- (a) *If  $n = 4$  and there exists  $\psi: H^*(BP(p,4)) \rightarrow H^*(BG)$  as unstable algebras such that  $(\text{res} \circ \psi)|_{H^1_\beta(BP(p,4n))}$  is the identity, then  $P(p,4)$  is a complement for some  $N \trianglelefteq G$*
- (b) *If  $n \geq 5$  and there exists  $\psi: H^*_\beta(BP(p,n)) \rightarrow H^*_\beta(BG)$  in  $\mathcal{K}_\beta$  such that  $(\text{res} \circ \psi)|_{H^1_\beta(BP(p,n))}$  is the identity, then  $P(p,n)$  is a complement for some  $N \trianglelefteq G$*

*Proof.* We now follow along the lines of the proof of Corollary 4.7.

If  $n = 4$  we have

$$0 \longrightarrow \mathbb{Z}/p \longrightarrow P(p,4) \xrightarrow{\pi_1} \mathbb{Z}/p \times \mathbb{Z}/p \times \mathbb{Z}/p \longrightarrow 0$$

Let now  $B\pi_{1,G}$  be the map  $B\pi_{1,G}: BG \rightarrow B\mathbb{Z}/p \times B\mathbb{Z}/p \times B\mathbb{Z}/p$  that classifies the classes  $\psi(u), \psi(y), \psi(y') \in H^1(BG)$ .

The composite

$$BG \xrightarrow{\pi_{1,G}} B\mathbb{Z}/p \times B\mathbb{Z}/p \times B\mathbb{Z}/p \xrightarrow{yy' - \beta(u)} B^2\mathbb{Z}/p$$

is null-homotopic because of Remark 3.6, and

$$\begin{aligned} B\pi_{1,G}^*(yy' - \beta(u)) &= B\pi_{1,G}^*(y)B\pi_{1,G}^*(y') - B\pi_{1,G}^*(\beta(u)) = \\ &= \psi(y)\psi(y') - \psi(\beta(u)) = \psi(yy' - \beta(u)) = \psi(0) = 0. \end{aligned} \quad (8)$$

Therefore  $B\pi_{1,G}$  lifts to  $B\phi: BG \rightarrow BP(p,4)$ , giving the commutative diagram

$$\begin{array}{ccccc} & & & & BP(p,4) \\ & & & \nearrow B\phi & \downarrow B\pi_1 \\ BP(p,3) & \xrightarrow{\text{res}} & BG & \xrightarrow{B\pi_{1,G}} & B\mathbb{Z}/p \times B\mathbb{Z}/p \times B\mathbb{Z}/p. \end{array}$$

This implies that  $B\phi^*(y) = \psi(y)$ ,  $B\phi^*(y') = \psi(y')$ , and  $B\phi^*(u) = \psi(u)$ , and

$$(\text{res} \circ B\phi)(y) = (\text{res}^* \circ \psi)(y) = y,$$

$$(\text{res} \circ B\phi)^*(y') = (\text{res}^* \circ \psi)(y) = y', \text{ and}$$

$$(\text{res} \circ B\phi)(u) = (\text{res}^* \circ \psi)(u) = u.$$

Now, according to either Proposition 4.5 or Theorem 4.3,  $\phi|_{P(p,4)}$  is an automorphism of  $P(p,4)$ , that is,  $P(p,4)$  is a complement for  $N = \text{Ker } \phi \trianglelefteq G$ .

If  $n > 4$ , we must consider before the extensions:

$$0 \longrightarrow \mathbb{Z}/p \longrightarrow \mathbb{Z}/p^i \times \mathbb{Z}/p \times \mathbb{Z}/p \xrightarrow{\pi_{i-1}} \mathbb{Z}/p^{i-1} \times \mathbb{Z}/p \times \mathbb{Z}/p \longrightarrow 0,$$

classified by  $\beta_{i-1}(u) \in H^*_\beta(\mathbb{Z}/p^{i-1} \times \mathbb{Z}/p \times \mathbb{Z}/p)$ .

Let  $B\pi_{1,G}: BG \rightarrow B\mathbb{Z}/p \times B\mathbb{Z}/p \times \mathbb{Z}/p$  be the map that classifies the classes  $\psi(u)$ ,  $\psi(y)$ ,  $\psi(y') \in H^1(BG)$ . As  $\beta_1(\psi(u)) = 0 \in H_\beta^*(BG)$  the map  $B\pi_{1,G}$  extends to a map  $B\pi_{2,G}$  having the following commutative diagram:

$$\begin{array}{ccc} & B\mathbb{Z}/p^2 \times B\mathbb{Z}/p \times B\mathbb{Z}/p & \\ & \nearrow B\pi_{2,G} \quad \downarrow B\pi_1 & \\ BG & \xrightarrow{B\pi_{1,G}} B\mathbb{Z}/p \times B\mathbb{Z}/p \times B\mathbb{Z}/p & \end{array}$$

Using the same argument we can proceed extending the map till  $B\pi_{n-3}: B\mathbb{Z}/p^{n-3} \times B\mathbb{Z}/p \times B\mathbb{Z}/p$ . To do the last step we use again Remark 3.6 and that  $\beta_{n-3}(\psi(u)) = \psi(y)\psi(y') \in H_\beta^*(BG)$ , and obtain a map  $B\phi$  which gives the commutative diagram:

$$\begin{array}{ccc} & BP(p, n) & \\ & \nearrow B\phi \quad \downarrow B\pi_{n-2} \circ \dots \circ B\pi_1 & \\ BG & \xrightarrow{B\pi_{1,G}} B\mathbb{Z}/p \times B\mathbb{Z}/p \times B\mathbb{Z}/p & \end{array}$$

which implies again that  $B\phi^*(y) = \psi(y)$ ,  $B\phi^*(y') = \psi(y')$ , and  $B\phi^*(u) = \psi(u)$ , and

$$(\text{res} \circ B\phi)(y) = (\text{res}^* \circ \psi)(y) = y,$$

$$(\text{res} \circ B\phi)^*(y') = (\text{res}^* \circ \psi)(y) = y', \text{ and}$$

$$(\text{res} \circ B\phi)(u) = (\text{res}^* \circ \psi)(u) = u.$$

Again, applying Proposition 4.5 or Theorem 4.3, we obtain that  $\phi|_{P(p,n)}$  is an automorphism of  $P(p, n)$ , that is,  $P(p, n)$  is a complement for  $N = \text{Ker } \phi \leq G$ .  $\square$

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