NONLINEAR BELTRAMI OPERATORS, SCHAUDER ESTIMATES AND BOUNDS FOR THE JACOBIAN

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Abstract. We provide Schauder estimates for nonlinear Beltrami equations and lower bounds of the Jacobians for homeomorphic solutions. The results were announced in [1] but here we give detailed proofs.

1. Introduction

This note is devoted to establish properties of solutions to the nonlinear Beltrami equation

$$\partial_{\bar{z}} f(z) = \mathcal{H}(z, \partial_z f(z)) \qquad \text{a.e.} \tag{1.1}$$

under additional regularity of $\mathcal{H}$. Recall that the strong ellipticity of the equation is encoded in the fact that the function $\mathcal{H}(z,\xi)$ is k-Lipchitz on its second variable where $k < 1$.

In the recent monograph [4] on quasiconformal mappings and elliptic equations it was established that the nonlinear Beltrami equation governs effectively all nonlinear planar elliptic systems. The nonlinear equation was introduced by Bojarski and Iwaniec in [6, 8, 14] and its basic L^p-properties were obtained in [5]. On the other hand, to study oscillating properties of sequences of gradients of Sobolev mappings in [10, 12], it was vital to associate to them a corresponding nonlinear Beltrami equation.

The nonlinear Beltrami equation shares the existence properties of homeomorphic solutions with the linear one [4] but, for example, the uniqueness fails in general as proved in [2]. In [1] it was proved that the set of homeomorphic solutions forms an embedded submanifold of $W^{1,2}_{\mathrm{loc}}(\mathbb{C},\mathbb{C})$ and that under regularity assumptions the manifold of homeomorphic solutions defines uniquely the field $\mathcal{H}$. The arguments in [1] rely on regularity properties of the solutions, which we prove in the current paper.

Let us state our regularity assumptions on the field $\mathcal{H}(z,\xi)$. Throughout this paper we will assume Hölder continuity of $\mathcal{H}$ in the first variable and

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k-Lipschitz dependence on the second one. More precisely, given an open bounded set $\Omega \subset \mathbb{C}$, we assume that

$$\begin{aligned} &|\mathcal{H}(z_1,\xi_1) - \mathcal{H}(z_2,\xi_2)| \leqslant \mathbf{H}_\alpha(\Omega)\, |z_1 - z_2|^\alpha \big(|\xi_1| + |\xi_2|\big) + k\, |\xi_1 - \xi_2|, \\ &\mathcal{H}(z_1, 0) \equiv 0, \end{aligned} \tag{1.2}$$

for all $z_1, z_2 \in \Omega$, $\xi_1, \xi_2 \in \mathbb{C}$, where $k = \frac{K-1}{K+1} < 1$ and $\alpha \in (0,1)$ are fixed.

Some of our results also require C^1-regularity of $\mathcal{H}(z,\xi)$ in the variable ξ. Furthermore, in our main result Theorem 1.1 we assume that this C^1-dependence is uniform in the variable z, requiring that

$$\mathcal{H}_\xi(z,\xi) \;\text{ and }\; \mathcal{H}_{\overline{\xi}}(z,\xi) \quad \text{are continuous in } (z,\xi). \tag{1.3}$$

It should be mentioned that condition (1.3) will only be needed in the set $\Omega \times \mathbb{D}(0,r)$ for some $r > 0$. This condition is also seemingly natural as it arises in our primary application to the study of manifolds of quasiconformal mappings, see [1].

In case $\mathcal{H}(z,\xi)$ is linear in the second variable, (1.2) implies that the derivatives of the solutions to Beltrami equation are Hölder continuous and that the Jacobian of a homeomorphic solution does not vanish (see [4, 21]). We are aiming for similar regularity in the general nonlinear case. We start with the second question, the main goal of our paper.

Theorem 1.1. *Suppose the field $\mathcal{H}(z,\xi)$ satisfies* (1.2) *and* (1.3). *Then a homeomorphic solution $f \in W^{1,2}_{\mathrm{loc}}(\Omega,\mathbb{C})$ to the nonlinear Beltrami equation* (1.1) *has a positive Jacobian, $J(z,f) > 0$.*

Further, if $\Omega = \mathbb{C}$ and $f : \mathbb{C} \to \mathbb{C}$ is a normalised solution, i.e. $f(0) = 0$ and $f(1) = 1$, then there is a lower bound for the Jacobian

$$\inf_{z \in \mathbb{D}(0,R_0)} J(z,f) \geqslant c(\mathcal{H}, R_0) > 0, \qquad 0 < R_0 < \infty.$$

Besides of intrinsic interest, the non-vanishing of the Jacobian is, e.g., a key property needed in the study of manifolds of quasiconformal maps in [1]. In the linear case the statement can be shown by the representation theorem of the quasiregular maps (e.g., [21, Theorems II.5.2 and II.5.47]) or by using the Schauder estimates for the inverse (e.g., proof of [7, Proposition 5.1]), i.e., showing that also f^{-1} solves a Beltrami equation with Hölder continuous coefficients and hence the inverse is locally Hölder continuous, too. In the nonlinear case it is much harder to establish a suitable equation for the inverse. If we denote $g = f^{-1}$ then g satisfies the nonlinear Beltrami equation

$$\partial_{\overline{\omega}} g(\omega) = -\frac{1}{J(z,f)} \mathcal{H}\left(g(\omega), J(z,f)\, \overline{\partial_\omega g(\omega)}\right), \qquad \omega = f(z) \qquad \text{a.e.},$$

which would have Hölder continuous coefficients if we *a priori* knew that the Jacobian $J(z,f)$ has a positive lower bound.

In Section 3 we show that it is also possible to recover a nonlinear equation (that satisfies (1.2)) for g when the Jacobian of f is small, giving us the required regularity to be able to conclude that the Jacobian must be positive everywhere. We also give a separate proof in the case of a completely autonomous equation

$$\partial_{\overline{z}} f(z) = \mathcal{H}(\partial_z f(z)) \qquad \text{a.e.}$$

with the added benefit of having to assume only (1.2) on the structural field $\mathcal{H}$.

Next we turn to the regularity of the gradient. Nowadays the term Schauder estimates refers to various types of Hölder regularity results in the theory of PDEs. Juliusz Schauder pioneered these topics in [22, 23]. His papers deal mostly with linear, quasilinear and nonlinear elliptic equations of second order. The importance of his ideas (freezing the equation, i.e., viewing equations with Hölder regular coefficients locally as perturbations of equations with constant coefficients) is reflected in a enormous number of applications and generalisations. These ideas were successfully used to deal with quasilinear equations in [18] and the nonlinear divergence equations with C^1-dependence on the gradient variable [13, Chapter 6], see also [17] for recent developments. Notice that quasilinear elliptic equations relate to the nonlinear Beltrami equation through the two dimensional Hodge operator [4], though the relation to the regularity of $\mathcal{H}$ is not clear.

Schauder estimates for general nonlinear fields $\mathcal{H}(z,\xi)$, which are only Lipschitz in the gradient variable ξ and Hölder continuous in z form an important step in proving Theorem 1.1. The required estimates do not seem to appear in literature in this generality, and therefore we give a quasiregular proof for the Schauder estimates in this setting. A different quasiregular approach of Schauder estimates for linear and quasilinear Beltrami equations is considered in [4, Chapter 15].

Theorem 1.2. *Assuming* (1.2)*, suppose* $f \in W^{1,2}_{\mathrm{loc}}(\Omega,\mathbb{C})$ *is a solution to the nonlinear Beltrami equation*

$$\partial_{\bar z} f(z) = \mathcal{H}(z, \partial_z f(z)) \qquad \textit{a.e.} \quad \textit{in } \Omega.$$

Then $f \in C^{1,\gamma}_{\mathrm{loc}}(\Omega,\mathbb{C})$*, where* $\gamma=\alpha$*, if* $\alpha < \frac{1}{K}$*; otherwise one can take any* $\gamma < \frac{1}{K}$*. Moreover, we have a norm bound, when* $\mathbb{D}(\omega,2R) \Subset \Omega$*,*

$$\|D_z f\|_{C^\gamma(\mathbb{D}(\omega,R))} \leqslant c(K,\alpha,\gamma,\omega,R,\mathbf{H}_\alpha(\Omega))\, \|D_z f\|_{L^2(\mathbb{D}(\omega,2R))}. \tag{1.4}$$

Let us emphasise that there is a restriction $\gamma < \frac{1}{K}$ on the exponent. We do not know whether this bound is sharp. Note, however, that the difficulty occurs already at the level of the autonomous equation

$$\partial_{\bar z} f(z) = \mathcal{H}(\partial_z f(z)) \qquad \text{a.e.,}$$

Corollary 2.3. Thus the optimal Hölder regularity of $D_z f$ remains an interesting open problem.

If in addition to (1.2), the field $\mathcal{H}$ is assumed to be C^1 in the gradient variable, then for every $0<\alpha<1$ one can prove that the solutions are in $C^{1,\alpha}_{\mathrm{loc}}(\Omega,\mathbb{C})$. The estimate of the $C^{1,\alpha}_{\mathrm{loc}}(\Omega,\mathbb{C})$-norm is locally uniform in L^2-norm, but the dependence is not linear (as it is in (1.4)).

Theorem 1.3. *Let* $f \in W^{1,2}_{\mathrm{loc}}(\Omega,\mathbb{C})$ *be a solution to the nonlinear Beltrami equation* (1.1)*, where we assume in addition that* $\xi \mapsto \mathcal{H}(z,\xi) \in C^1(\mathbb{C},\mathbb{C})$*. Then* $f \in C^{1,\alpha}_{\mathrm{loc}}(\Omega,\mathbb{C})$*.*

We will study first the autonomous case (Section 2.1) and then in the spirit of Schauder estimates tackle the general case by perturbation. The proof of

Theorem 1.2 will be given in Section 2.3 and Theorem 1.3 is considered in Section 2.4.

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2. Schauder-type estimates

2.1. **Autonomous equation and integral estimates.** We start with an auxiliary result for the nonlinear Beltrami equation with constant coefficients (see [24, 10, 11]). In this case $\mathcal{H}$ depends only on the gradient variable, and the requirement (1.2) reduces to $\mathcal{H}(0) = 0$ with $|\mathcal{H}(\xi_1) - \mathcal{H}(\xi_2)| \leqslant k|\xi_1 - \xi_2|$.

Proposition 2.1. *Let $F \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{C})$ be a solution to the autonomous nonlinear Beltrami equation*

$$\partial_{\bar z} F(z) = \mathcal{H}(\partial_z F(z)) \qquad \text{for a.e. } z \in \Omega. \tag{2.1}$$

Then the directional derivatives of F are K-quasiregular, $K = \frac{1+k}{1-k}$.

Proof. Let $h > 0$. The difference quotients

$$F_h(z) := \frac{F(z + he) - F(z)}{h}, \qquad |e| = 1$$

are K-quasiregular. Indeed, by (2.1),

$$\begin{aligned} |\partial_{\bar z} F_h(z)| &= \left| \frac{\mathcal{H}(\partial_z F(z + he)) - \mathcal{H}(\partial_z F(z))}{h} \right| \\ &\leqslant k \, \frac{|\partial_z F(z + he) - \partial_z F(z)|}{|h|} = k\, |\partial_z F_h(z)|. \end{aligned} \tag{2.2}$$

Now, we have a Caccioppoli estimate for F_h, see e.g. [4, Theorem 5.4.2]. For $\rho < R$ and any constant c

$$\int_{\mathbb{D}_\rho} |D_z F_h|^2 \leqslant \frac{c(K)}{(R - \rho)^2} \int_{\mathbb{D}_R} |F_h - c|^2, \tag{2.3}$$

where we denote $\mathbb{D}_r = \mathbb{D}(z_0, r)$. Thus $c(K) \fint_{\mathbb{D}_R} (|D_z F|^2 + 1)$ is a uniform bound for the derivative of the difference quotient for the range $0 < \rho \leqslant \frac{R}{2}$. Hence the directional derivative $\partial_e F \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{C})$. Further, letting $h \to 0$ in (2.2), we see that $\partial_e F(z)$ is K-quasiregular. □

Therefore, the directional derivatives inherit the properties of K-quasiregular maps. We will need few integral estimates that we prove next.

Proposition 2.2. *Let $g \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{C})$ be K-quasiregular. Then*

$$\|g\|_{L^2(\mathbb{D}(z_0,\rho))} \leqslant c(K)\, \frac{\rho}{R}\, \|g\|_{L^2(\mathbb{D}(z_0,R))} \tag{2.4}$$

for $\mathbb{D}(z_0, \rho) \subset \mathbb{D}(z_0, R) \subset \Omega$. Moreover, g is locally $\frac{1}{K}$-Hölder continuous; formulated in a Morrey-Campanato form we have

$$\|g - g_\rho\|_{L^2(\mathbb{D}(z_0,\rho))} \leqslant c(K) \left(\frac{\rho}{R}\right)^{1+\frac{1}{K}} \|g - g_R\|_{L^2(\mathbb{D}(z_0,R))} \tag{2.5}$$

for any $\rho \leqslant R$, where $g_r = \fint_{\mathbb{D}(z_0,r)} g$.

Proof. We start by proving (2.4). Denote $\mathbb{D}_r = \mathbb{D}(z_0, r)$. Since g is K-quasiregular, we have by Caccioppoli's inequality and weak reverse Hölder inequalities, [4, Theorem 5.4.2], [15, Proposition 1], for $\frac{2K}{K+1} < p < \frac{2K}{K-1}$,

$$\|D_z g\|_{L^p(\mathbb{D}_{R/2})} \leqslant c_0(p,K,R)\, \|g\|_{L^p(\mathbb{D}_{2R/3})} \leqslant c_1(p,K,R)\, \|g\|_{L^2(\mathbb{D}_R)}. \tag{2.6}$$

Now, for $\rho \leqslant \frac{R}{2}$,

$$\begin{aligned} \|g\|_{L^2(\mathbb{D}_\rho)} &\leqslant \sqrt{\pi}\, \rho \sup_{\mathbb{D}_\rho} |g| \leqslant c(R)\, \rho\, \|g\|_{W^{1,p}(\mathbb{D}_{R/2})} \\ &\leqslant c(p,K,R)\, \rho\, \|g\|_{L^2(\mathbb{D}_R)}, \end{aligned}$$

where the second to the last inequality follows from the Sobolev embedding, by Morrey's inequality (choose $p > 2$), and the last one from Caccioppoli's inequality (2.6). By rescaling, one sees that $c(p,K,R) = c(p,K)R^{-1}$. Hence

$$\|g\|_{L^2(\mathbb{D}(z_0,\rho))} \leqslant c(p,K)\, \frac{\rho}{R}\, \|g\|_{L^2(\mathbb{D}(z_0,R))},$$

for $\rho \leqslant R$ (above we show the estimate for $\rho \leqslant \frac{R}{2}$ and it is trivial for $\frac{R}{2} < \rho \leqslant R$ with possibly a bigger constant). We have thus proved the integral estimate (2.4).

The $\frac{1}{K}$-Hölder inequality of K-quasiregular maps goes back to Morrey [20, 4, Section 3.10]. For later purposes we recall how this follows using the isoperimetric inequality for Sobolev spaces, in combination with Caccioppoli's and Poincaré's inequalities, and the pointwise equivalence of $|Dg(z)|^2$ and $J(z,g)$ for quasiregular maps.

We have, by the isoperimetric inequality in the Sobolev space and the Hölder inequality, that the mapping $\psi(r) := r^{-\frac{2}{K}} \int_{\mathbb{D}_r} J(z,g)\, dA(z)$ is non-decreasing. Indeed,

$$\begin{aligned} \int_{\mathbb{D}_r} J(z,g)\, dA(z) &\leqslant \frac{1}{4\pi} \left(\int_{\partial\mathbb{D}_r} |D_z g(z)|\, dz \right)^2 \\ &\leqslant \frac{K|\partial\mathbb{D}_r|}{4\pi} \int_{\partial\mathbb{D}_r} \frac{|D_z g(z)|^2}{K}\, dz \leqslant \frac{Kr}{2} \int_{\partial\mathbb{D}_r} J(z,g)\, dz, \end{aligned}$$

for $\mathbb{D}_r \subset \Omega$, where the last inequality follows by quasiregularity (i.e., $|Dg(z)|^2 \leqslant KJ(z,g)$ almost everywhere). In other words $\psi'(r) \geqslant 0$.

The non-decreacing of ψ implies that

$$\int_{\mathbb{D}_\rho} J(z,g)\, dA(z) \leqslant \left(\frac{\rho}{R}\right)^{2/K} \int_{\mathbb{D}_R} J(z,g)\, dA(z), \tag{2.7}$$

for $\mathbb{D}_\rho \subset \mathbb{D}_R \subset \Omega$. Now, by Poincaré's inequality, K-quasiregularity, (2.7), and Caccioppoli's inequality (2.3), we get for $\rho \leqslant \frac{R}{2}$

$$\begin{aligned} \|g - g_\rho\|_{L^2(\mathbb{D}_\rho)} &\leqslant c\, \rho \|D_z g\|_{L^2(\mathbb{D}_\rho)} \leqslant c\, \rho \left(\int_{\mathbb{D}_\rho} K\, J(z,g) \right)^{\frac{1}{2}} \\ &\leqslant c(K)\, \frac{\rho^{1+1/K}}{R^{1/K}} \left(\int_{\mathbb{D}_{R/2}} J(z,g) \right)^{\frac{1}{2}} \leqslant c(K)\, \frac{\rho^{1+1/K}}{R^{1/K}} \|D_z g\|_{L^2(\mathbb{D}_{R/2})} \\ &\leqslant c(K) \left(\frac{\rho}{R}\right)^{1+1/K} \|g - g_R\|_{L^2(\mathbb{D}_R)}. \end{aligned} \tag{2.8}$$

For $\frac{R}{2} < \rho \leqslant R$, (2.8) holds trivially. Hence we have shown the integral estimate (2.5). $\square$

The formulation of Proposition 2.2 will be particularly useful when applied to the derivatives D_zF of a solution to the autonomous equation (2.1).

Corollary 2.3. *If F is as in Proposition 2.1, the derivative D_zF is locally $\frac{1}{K}$-Hölder continuous. Moreover,*

(1) for every $\mathbb{D}(z_0,\rho) \subset \mathbb{D}(z_0,R) \subset \Omega$,

$$\|D_zF\|_{L^2(\mathbb{D}(z_0,\rho))} \leqslant c(K)\,\frac{\rho}{R}\,\|D_zF\|_{L^2(\mathbb{D}(z_0,R))}.$$

(2) For every $\mathbb{D}(z_0,\rho) \subset \mathbb{D}(z_0,R) \subset \Omega$,

$$\|D_zF - (D_zF)_\rho\|_{L^2(\mathbb{D}(z_0,\rho))} \leqslant c(K)\left(\frac{\rho}{R}\right)^{1+\frac{1}{K}}\|D_zF - (D_zF)_R\|_{L^2(\mathbb{D}(z_0,R))}$$

where $(D_zF)_r = \fint_{\mathbb{D}(z_0,r)} D_zF$.

Proof. Since for the Hilbert-Schmidt norm $\|D_zF(z)\|^2 = \sum_{j=1}^2 |D_zF(z)\,e_j|^2 = \sum_{j=1}^2 |\partial_{e_j}F(z)|^2$ and $\|D_zF - (D_zF)_r\|^2 = \sum_{j=1}^2 |\partial_{e_j}F - (\partial_{e_j}F)_r|^2$, the corollary follows from Proposition 2.2 and the quasiregularity of the directional derivatives. $\square$

2.2. **Riemann-Hilbert problem.** The solution of the Riemann Hilbert problem is well-known; the proof is based on the local versions of the classical Cauchy transform and the Beurling transform, see, for instance, [12, Proposition 2]. We sketch a proof for the reader's convenience in the situation we need in this paper.

Proposition 2.4. *Let f be a solution to the nonlinear Beltrami equation (1.1), and suppose $\mathbb{D}(z_0,R) \Subset \Omega$. Then there exists a unique solution $F \in W^{1,2}(\mathbb{D}(z_0,R),\mathbb{C})$ to the following local Riemann-Hilbert problem for the autonomous equation*

$$\begin{cases} \partial_{\overline{z}}F(z) = \mathcal{H}(z_0,\partial_zF(z)) & a.e.\ z \in \mathbb{D}(z_0,R), \\ \mathrm{Re}(f-F) = 0 & on\ \partial\mathbb{D}(z_0,R). \end{cases} \tag{2.9}$$

Furthermore, $\|\partial_{\overline{z}}F - \partial_{\overline{z}}f\|_{L^2(\mathbb{D}_R)} = \|\partial_zF - \partial_zf\|_{L^2(\mathbb{D}_R)}$ and we have a norm bound

$$\|D_zF\|_{L^2(\mathbb{D}_R)} \leqslant 2K\|D_zf\|_{L^2(\mathbb{D}_R)}. \tag{2.10}$$

Proof. The local Cauchy transform in $\mathbb{D}_R := \mathbb{D}(z_0,R)$ is obtained from the Cauchy transform on the unit disk by conformal change of variables (see, e.g., [7, Section 6.1]). Namely, the local Cauchy transform of $\psi \in L^2(\mathbb{D}_R,\mathbb{C})$ is given by

$$(\mathcal{C}_{\mathbb{D}_R}\psi)(z) = \frac{1}{\pi}\int_\Omega \left(\frac{\psi(\zeta)}{z-\zeta} - \frac{(z-z_0)\,\overline{\psi(\zeta)}}{R^2 - (z-z_0)\,\overline{(\zeta - z_0)}}\right) dA(\zeta)$$

and the local Beurling transform by $\mathcal{S}_{\mathbb{D}_R}\psi = \partial_z\,\mathcal{C}_{\mathbb{D}_R}\psi$, that is,

$$(\mathcal{S}_{\mathbb{D}_R}\psi)(z) = -\frac{1}{\pi}\int_\Omega \left(\frac{\psi(\zeta)}{(z-\zeta)^2} + \frac{R^2\,\overline{\psi(\zeta)}}{(R^2 - (z-z_0)\,\overline{(\zeta-z_0)})^2}\right) dA(\zeta).$$

By definition, $\partial_z \mathcal{C}_{\mathbb{D}_R}\psi = \mathcal{S}_{\mathbb{D}_R}\psi$, $\partial_{\bar z} \mathcal{C}_{\mathbb{D}_R}\psi = \psi$, and $\mathcal{C}_{\mathbb{D}_R}\psi \in W^{1,2}(\mathbb{D}_R, \mathbb{C}) \cap C(\overline{\mathbb{D}_R}, \mathbb{C})$.

As the integrand in the definition of $\mathcal{C}_{\mathbb{D}_R}$ is purely imaginary on the boundary, $\mathrm{Re}(\mathcal{C}_{\mathbb{D}_R}\psi) = 0$ on $\partial\mathbb{D}_R$, i.e., $\mathrm{Re}(\mathcal{C}_{\mathbb{D}_R}\psi)$ is in the closure of $C_0^\infty(\mathbb{D}_R, \mathbb{C})$ in $W^{1,2}(\mathbb{D}_R, \mathbb{C})$. Now, we can use Green's theorem, [4, Theorem 2.9.1], to see that the local Beurling transform $\mathcal{S}_{\mathbb{D}_R} : L^2(\mathbb{D}_R, \mathbb{C}) \to L^2(\mathbb{D}_R, \mathbb{C})$ is an isometry, that is,

$$\|\mathcal{S}_{\mathbb{D}_R}\psi\|_{L^2(\mathbb{D}_R)} = \|\psi\|_{L^2(\mathbb{D}_R)}.$$

Indeed, let $\mathcal{C}_{\mathbb{D}_R}\psi = u + iv$,

$$\begin{aligned}\int_{\mathbb{D}_R} |\mathcal{S}_{\mathbb{D}_R}\psi|^2 - |\psi|^2 &= \int_{\mathbb{D}_R} |\partial_z \mathcal{C}_{\mathbb{D}_R}\psi|^2 - |\partial_{\bar z} \mathcal{C}_{\mathbb{D}_R}\psi|^2 = \int_{\mathbb{D}_R} J(z, \mathcal{C}_{\mathbb{D}_R}\psi) \\ &= -\frac{i}{2}\int_{\mathbb{D}_R} \partial_z u\, \partial_{\bar z} v - \partial_{\bar z} u\, \partial_z v = \frac{1}{4}\int_{\partial\mathbb{D}_R} u(\partial_z v + \partial_{\bar z} v) = 0,\end{aligned}$$

as $u = 0$ on $\partial\mathbb{D}_R$.

The isometry of $\mathcal{S}_{\mathbb{D}_R}$ implies that the Beltrami operator

$$(\mathcal{B}\psi)(z) = \mathcal{H}(z_0, (\mathcal{S}_{\mathbb{D}_R}\psi)(z) + \partial_z f(z)) - \mathcal{H}(z_0, \partial_z f(z))$$

is a contraction on $L^2(\mathbb{D}_R, \mathbb{C})$;

$$\begin{aligned}&\|\mathcal{B}\psi_1 - \mathcal{B}\psi_2\|_{L^2(\mathbb{D}_R)} \\ &\quad = \|\mathcal{H}(z_0, (\mathcal{S}_{\mathbb{D}_R}\psi_1)(z) + \partial_z f(z)) - \mathcal{H}(z_0, (\mathcal{S}_{\mathbb{D}_R}\psi_2)(z) + \partial_z f(z))\|_{L^2(\mathbb{D}_R)} \\ &\quad \leqslant k\|\psi_1 - \psi_2\|_{L^2(\mathbb{D}_R)}.\end{aligned}$$

Thus there is a unique fixed point $\Psi \in L^2(\mathbb{D}_R, \mathbb{C})$ of $\mathcal{B}$.

We define $F = \mathcal{C}_{\mathbb{D}_R}\Psi + f$, since then $\partial_{\bar z}F = \Psi + \partial_{\bar z} f$, $\partial_z F = \mathcal{S}_{\mathbb{D}_R}\Psi + \partial_z f$, and $\mathrm{Re}\, F = \mathrm{Re}(\mathcal{C}_{\mathbb{D}_R}\Psi) + \mathrm{Re}\, f = \mathrm{Re}\, f$ (i.e., F solves (2.9)).

The L^2-estimate is obtained in the similar fashion. For the fixed point Ψ

$$\begin{aligned}\|\Psi\|_{L^2(\mathbb{D}_R)} &= \|\mathcal{B}\Psi\|_{L^2(\mathbb{D}_R)} \\ &= \|\mathcal{H}(z_0, (\mathcal{S}_{\mathbb{D}_R}\Psi)(z) + \partial_z f(z)) - \mathcal{H}(z_0, \partial_z f(z))\|_{L^2(\mathbb{D}_R)} \\ &\leqslant k\|\mathcal{S}_{\mathbb{D}_R}\Psi\|_{L^2(\mathbb{D}_R)} + 2k\|\partial_z f\|_{L^2(\mathbb{D}_R)}.\end{aligned}$$

Now, using that $\mathcal{S}_{\mathbb{D}_R}$ is also an isometry,

$$\begin{aligned}\|D_z F\|_{L^2(\mathbb{D}_R)} &\leqslant \|\Psi\|_{L^2(\mathbb{D}_R)} + \|\mathcal{S}_{\mathbb{D}_R}\Psi\|_{L^2(\mathbb{D}_R)} + 2\|D_z f\|_{L^2(\mathbb{D}_R)} \\ &\leqslant \left(\frac{4k}{1-k} + 2\right)\|D_z f\|_{L^2(\mathbb{D}_R)} = 2\,\frac{1+k}{1-k}\|D_z f\|_{L^2(\mathbb{D}_R)}.\end{aligned}$$

□

2.3. **Schauder estimates by freezing the coefficients.** We will use the Morrey-Campanato integral characterisation of Hölder continuous functions [13, Chapter III, Theorem 1.2, p. 70, and Theorem 1.3, p. 72]. Namely, the integral estimate

$$\|g - g_\rho\|_{L^2(\mathbb{D}(z_0,\rho))} \leqslant M\,\rho^{1+\gamma} \tag{2.11}$$

for $z_0 \in \Omega$ and every $\rho \leqslant \min\{R_0, \mathrm{dist}(z_0, \partial\Omega)\}$ (for some R_0) gives the local γ-Hölder continuity of g in Ω. Moreover, for $\tilde\Omega \Subset \Omega$, (2.11) implies the Hölder seminorm bound

$$[g]_{C^\gamma(\tilde\Omega)} \leqslant c(\gamma, \tilde\Omega)\, M \tag{2.12}$$

and the L^∞-bound

$$\|g\|_{L^\infty(\tilde\Omega)} \leqslant c(\gamma,\tilde\Omega)\left(M\,\mathrm{diam}(\Omega)^\gamma + \|g\|_{L^2(\Omega)}\right), \tag{2.13}$$

see the proofs of Proposition 1.2 and Theorem 1.2 in pages 68–72 of [13, Chapter III].

Next, we apply the ideas of freezing the coefficients to get few basic estimates for solutions to (1.1). We start with the following

Lemma 2.5. *Suppose $\mathcal{H}$ satisfies the conditions* (1.2) *and let $f \in W^{1,2}_{\mathrm{loc}}(\Omega,\mathbb{C})$ be a solution to*

$$\partial_{\bar z} f(z) = \mathcal{H}(z,\partial_z f(z)) \qquad a.e. \quad in\ \Omega.$$

If $\mathbb{D}(z_0,R) \Subset \Omega$, then for each $0<\rho\leqslant R$ we have

$$\begin{aligned}\|D_z f - (D_z f)_\rho\|_{L^2(\mathbb{D}_\rho)} \leqslant c(K)\left(\frac{\rho}{R}\right)^{1+\frac1K} &\|D_z f - (D_z f)_R\|_{L^2(\mathbb{D}_R)}\\ &+ c(K)\,\mathbf{H}_\alpha(\Omega)\,R^\alpha\,\|\partial_z f\|_{L^2(\mathbb{D}_R)},\end{aligned}$$

where $\mathbb{D}_r = \mathbb{D}(z_0,r)$.

Proof. The required estimate to prove is then the same as in Corollary 2.3, claim (2), up to the correction term $c(K)\,\mathbf{H}_\alpha(\Omega) R^\alpha \|\partial_z f\|_{L^2(\mathbb{D}_R)}$. This will arise from a comparison of f and the solution F to an autonomous equation, the local Riemann-Hilbert problem

$$\begin{cases}\partial_{\bar z} F(z) = \mathcal{H}(z_0,\partial_z F(z)) & \text{a.e. } z\in\mathbb{D}_R,\\ \mathrm{Re}(f-F)=0 & \text{on } \partial\mathbb{D}_R.\end{cases}$$

The existence of F follows by Proposition 2.4. Furthermore, by (1.2),

$$\begin{aligned}&\|\partial_{\bar z}(f-F)\|_{L^2(\mathbb{D}_R)}\\ &\qquad\leqslant \|\mathcal{H}(z,\partial_z f) - \mathcal{H}(z_0,\partial_z f)\|_{L^2(\mathbb{D}_R)} + \|\mathcal{H}(z_0,\partial_z f) - \mathcal{H}(z_0,\partial_z F)\|_{L^2(\mathbb{D}_R)}\\ &\qquad\leqslant \mathbf{H}_\alpha(\Omega)\,R^\alpha\,\|\partial_z f\|_{L^2(\mathbb{D}_R)} + k\,\|\partial_z(f-F)\|_{L^2(\mathbb{D}_R)}.\end{aligned}$$

Since the Beurling transform $\mathcal{S}_{\mathbb{D}_R}$ of the disk $\mathbb{D}_R$ is an isometry $L^2(\mathbb{D}_R)\to L^2(\mathbb{D}_R)$, we end up with

$$\|D_z f - D_z F\|_{L^2(\mathbb{D}_R)} \leqslant \frac{2}{1-k}\,\mathbf{H}_\alpha(\Omega)\,R^\alpha\,\|\partial_z f\|_{L^2(\mathbb{D}_R)}. \tag{2.14}$$

On the other hand, Corollary 2.3 (2) gives

$$\begin{aligned}\|D_z f - (D_z f)_\rho\|_{L^2(\mathbb{D}_\rho)} &\leqslant \|D_z F - (D_z F)_\rho\|_{L^2(\mathbb{D}_\rho)} + 2\|D_z f - D_z F\|_{L^2(\mathbb{D}_\rho)}\\ &\leqslant c(K)\left(\frac{\rho}{R}\right)^{1+\frac1K}\|D_z F - (D_z F)_R\|_{L^2(\mathbb{D}_R)} + 2\|D_z f - D_z F\|_{L^2(\mathbb{D}_R)}\\ &\leqslant c(K)\left(\frac{\rho}{R}\right)^{1+\frac1K}\|D_z f - (D_z f)_R\|_{L^2(\mathbb{D}_R)} + 2\,c(K)\|D_z f - D_z F\|_{L^2(\mathbb{D}_R)},\end{aligned}$$

$\rho\leqslant R$. Combining this with (2.14) gives the claim. □

If we use claim (1) of Corollary 2.3, instead of claim (2), the same argument as above leads to

Lemma 2.6. *Suppose $\mathcal{H}$ satisfies the conditions* (1.2)*. If $f \in W^{1,2}_{\mathrm{loc}}(\Omega,\mathbb{C})$ and $\mathbb{D}(z_0,R)$ are as in Lemma 2.5, then for each $0<\rho\leqslant R$,*

$$\|D_z f\|_{L^2(\mathbb{D}_\rho)} \leqslant c(K)\,\frac{\rho}{R}\,\|D_z f\|_{L^2(\mathbb{D}_R)} + c(K)\,\mathbf{H}_\alpha(\Omega)\,R^\alpha\,\|\partial_z f\|_{L^2(\mathbb{D}_R)}.$$

Since the $W^{1,2}_{\mathrm{loc}}$-solutions to (1.1) are a priori K-quasiregular, we have the Caccioppoli estimates (2.3) at our use. These are convenient to present in the following form.

Lemma 2.7. *Suppose $\mathcal{H}$ and $f \in W^{1,2}_{\mathrm{loc}}(\Omega,\mathbb{C})$ are as in Lemma 2.5. Let $\mathbb{D}(z_0,R)\subset\Omega''\Subset\Omega'\Subset\Omega$. If $f\in C^\beta(\Omega',\mathbb{C})$ for some $0<\beta\leqslant 1$, then*

$$\|D_z f\|_{L^2(\mathbb{D}(z_0,R))} \leqslant c(K,\Omega',\Omega'')\,[f]_{C^\beta(\Omega')}\,R^\beta.$$

Lastly, let us recall

Lemma 2.8 (Lemma 2.1, p. 86, in [13, Chapter III])**.** *Let Ψ be non-negative, non-decreasing function such that*

$$\Psi(\rho)\leqslant a\left[\left(\frac{\rho}{R}\right)^\lambda+\sigma\right]\Psi(R)+bR^\gamma$$

for every $0<\rho\leqslant R\leqslant R_0$, where a is non-negative constant and $0<\gamma<\lambda$. Then there exists $\sigma_0=\sigma_0(a,\lambda,\gamma)$ such that, if $\sigma<\sigma_0$,

$$\Psi(\rho)\leqslant c(a,\lambda,\gamma)\left[\left(\frac{\rho}{R}\right)^\gamma\Psi(R)+b\rho^\gamma\right]$$

for all $0<\rho\leqslant R\leqslant R_0$.

With these tools and estimates at our disposal we are ready for the Schauder estimates.

Proof of Theorem 1.2. Denote $\mathbb{D}_r=\mathbb{D}(z_0,r)$.

Step 1. Hölder continuity of f. We will show that f is actually locally β-Hölder continuous for every $0<\beta<1$.

Namely, according to Lemma 2.6 we have

$$\|D_z f\|_{L^2(\mathbb{D}_\rho)}\leqslant c_0(K)\left(\frac{\rho}{R}+\mathbf{H}_\alpha(\Omega)R^\alpha\right)\|D_z f\|_{L^2(\mathbb{D}_R)},$$

whenever $0<\rho\leqslant R$ and $\mathbb{D}_R=\mathbb{D}(z_0,R)\subset\Omega$. Applying Lemma 2.8 to $\Psi(\rho)=\|D_z f\|_{L^2(\mathbb{D}_\rho)}$, with $b=0,\lambda=1$ and $\sigma=\mathbf{H}_\alpha(\Omega)R^\alpha$, we see that

$$\|D_z f\|_{L^2(\mathbb{D}_\rho)}\leqslant c_1(K)\left(\frac{\rho}{R}\right)^{1-\varepsilon}\|D_z f\|_{L^2(\mathbb{D}_R)},$$

where $0<\rho\leqslant R\leqslant\min\{R_0,\operatorname{dist}(z_0,\partial\Omega)\}$. Here R_0 is small enough; how small R_0 needs to be taken depends on $c_0(K),\mathbf{H}_\alpha(\Omega)$ and $\varepsilon>0$ but not on z_0. Thus the same upper bound R_0 works throughout the bounded domain Ω.

Combining with the Poincaré inequality gives

$$\|f-f_\rho\|_{L^2(\mathbb{D}_\rho)}\leqslant\rho\,\|D_z f\|_{L^2(\mathbb{D}_\rho)}\leqslant c_1(K)\,\rho^{2-\varepsilon}\,R^{\varepsilon-1}\,\|D_z f\|_{L^2(\mathbb{D}_R)},$$

for $0<\rho\leqslant R\leqslant\min\{R_0,\operatorname{dist}(z_0,\partial\Omega)\}$.

Let $\mathbb{D}(\omega,4R)\subset\Omega$. Now, for $\mathbb{D}(z_0,\rho)\subset\mathbb{D}(\omega,2R)$,

$$\|f-f_\rho\|_{L^2(\mathbb{D}_\rho)}\leqslant c_1(K)\,\rho^{2-\varepsilon}\,\min\{R_0,R\}^{-\beta}\|D_z f\|_{L^2(\mathbb{D}(\omega,3R))}.$$

In view of (2.11) we see that $f \in C^{\beta}_{\rm loc}(\mathbb{D}(\omega, 2R), \mathbb{C})$ for every $0 < \beta < 1$. The estimate (2.12) gives a bound for the local Hölder norm,

$$[f]_{C^{\beta}(\mathbb{D}(\omega,R))} \leqslant c_2(K, \beta, R, \mathbf{H}_\alpha(\Omega)) \, \|D_z f\|_{L^2(\mathbb{D}(\omega,3R))}. \tag{2.15}$$

Step 2: Self-improving Morrey-Campanato estimate. Claim: Assume that $1 < \alpha + \beta < 1 + \frac{1}{K}$. Then $D_z f \in C^{\alpha+\beta-1}_{\rm loc}(\Omega, \mathbb{C})$.

Let $\Omega'' \Subset \Omega' \Subset \Omega$. We first show the claim for $\beta < 1$, and start with estimates from Lemma 2.5,

$$\begin{aligned}\|D_z f - (D_z f)_\rho\|_{L^2(\mathbb{D}_\rho)} \leqslant c_0(K) \left(\frac{\rho}{R}\right)^{1+\frac{1}{K}} \|D_z f - (D_z f)_R\|_{L^2(\mathbb{D}_R)} \\ + c_0(K) \, \mathbf{H}_\alpha(\Omega) \, R^\alpha \, \|\partial_z f\|_{L^2(\mathbb{D}_R)},\end{aligned}$$

when $\mathbb{D}(z_0, R) \subset \Omega''$. Here, by the Caccioppoli estimate of Lemma 2.7

$$\|\partial_z f\|_{L^2(\mathbb{D}_R)} \leqslant c_1(K, \Omega', \Omega'') \, [f]_{C^\beta(\Omega')} \, R^\beta, \tag{2.16}$$

which by Step 1 is finite for every $\beta < 1$.

We will now apply Lemma 2.8 to the non-decreasing function $\Psi(\rho) = \|D_z f - (D_z f)_\rho\|_{L^2(\mathbb{D}_\rho)} = \inf_{a \in \mathbb{C}} \|D_z f - a\|_{L^2(\mathbb{D}_\rho)}$ and the parameters $\lambda = 1 + \frac{1}{K}$, $\sigma = 0$ and $b = \mathbf{H}_\alpha(\Omega) \, [f]_{C^\beta(\Omega')}$. We obtain that

$$\begin{aligned}\|D_z f - (D_z f)_\rho\|_{L^2(\mathbb{D}_\rho)} \leqslant c_2 \left(\frac{\rho}{R}\right)^{\alpha+\beta} \|D_z f - (D_z f)_R\|_{L^2(\mathbb{D}_R)} \\ + c_2 \, \rho^{\alpha+\beta} \, \mathbf{H}_\alpha(\Omega) \, [f]_{C^\beta(\Omega')}\end{aligned} \tag{2.17}$$

whenever $\rho \leqslant R$.

In terms of the Morrey-Campanato estimate (2.11) in the set Ω'', we see that $D_z f \in C^{\alpha+\beta-1}_{\rm loc}(\Omega'', \mathbb{C})$, which is enough for our claim if $\alpha \geqslant 1/K$. The norm estimate (1.4) follows from combining (2.12) with (2.15) and (2.17).

In case $\alpha < 1/K$ we need to continue to show that $f \in C^{1,\alpha}_{\rm loc}(\Omega, \mathbb{C})$. But what we have above proves that $D_z f$ is locally bounded. Thus the bound in (2.16) remains finite for $\beta = 1$, and we can repeat the proof of (2.17) with $\beta = 1$. Accordingly, (2.12) gives $f \in C^{1,\alpha}_{\rm loc}(\Omega, \mathbb{C})$, with norm bound

$$\begin{aligned}[D_z f]_{C^\alpha(\mathbb{D}(\omega,R))} \leqslant c(K, \alpha, \omega, R) \Big[\|D_z f\|_{L^2(\mathbb{D}(\omega,2R))} \\ + \mathbf{H}_\alpha(\Omega) \, \|D_z f\|_{L^\infty(\mathbb{D}(\omega,2R))} \Big].\end{aligned}$$

To estimate the L^∞-norm in $\mathbb{D}(\omega, 2R)$, we note that for $\mathbb{D}(z_0, \rho) \subset \mathbb{D}(\omega, \frac{5R}{2})$ (2.17) holds with $\Omega' = \mathbb{D}(\omega, 3R)$ and thus once more by Morrey-Campanato norm estimate (2.11) (with (2.13))

$$\begin{aligned}\|D_z f\|_{L^\infty(\mathbb{D}(\omega,2R))} \leqslant c(K, \alpha, \omega, R) \Big[\|D_z f\|_{L^2(\mathbb{D}(\omega,3R))} \\ + \mathbf{H}_\alpha(\Omega) \, [f]_{C^{\beta'}(\mathbb{D}(\omega,3R))} \Big],\end{aligned}$$

where $\beta' < 1$. It remains to combine with (2.15) to obtain

$$\|D_z f\|_{C^\gamma(\mathbb{D}(\omega,R))} \leqslant c(K, \alpha, \gamma, \omega, R, \mathbf{H}_\alpha(\Omega)) \, \|D_z f\|_{L^2(\mathbb{D}(\omega,9R))},$$

and we have the norm bound (1.4) by rescaling. □

2.4. **Schauder estimates with C^1 gradient dependence.**

Proof of Theorem 1.3. As we see in Step 2 of the proof of Theorem 1.2, the restriction on Hölder continuity comes from the autonomous case. Hence it is enough to show that, when the dependence on the gradient is C^1, we may improve the norm estimates in Corollary 2.3.

Proposition 2.9. *Let $F \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{C})$ be a solution to the autonomous nonlinear Beltrami equation (2.1), where in addition $\xi \mapsto \mathcal{H}(\xi) \in C^1(\mathbb{C}, \mathbb{C})$. Then, for every $\varepsilon > 0$ and $\mathbb{D}(z_0, \rho) \subset \mathbb{D}(z_0, R) \subset \Omega'' \Subset \Omega' \Subset \Omega$,*

$$\|D_z F - (D_z F)_\rho\|_{L^2(\mathbb{D}(z_0,\rho))} \leqslant c \left(\frac{\rho}{R}\right)^{2-\varepsilon} \|D_z F - (D_z F)_R\|_{L^2(\mathbb{D}(z_0,R))}$$

where $(D_z F)_r = \fint_{\mathbb{D}(z_0,r)} D_z F$ and the constant c depends on the parameters K, Ω', Ω'', $\|Df\|_{L^2(\Omega')}$ and the modulus of continuity of $\mathcal{H}_\xi$ and $\mathcal{H}_{\overline{\xi}}$.

Proof. We know by Proposition 2.1 that $\partial_z F(z) \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{C})$. If we differentiate the autonomous equation $\partial_{\overline{z}} F(z) = \mathcal{H}(\partial_z F(z))$ with respect to z, we get for $g = \partial_z F$ that

$$g_{\overline{z}} = \mathcal{H}_\xi(g)\, \partial_z g + \mathcal{H}_{\overline{\xi}}(g)\, \overline{\partial_{\overline{z}} g}, \qquad \text{a.e.} \quad \text{in } \Omega.$$

By isolating $g_{\overline{z}}$ we obtain the $\mathbb{R}$-linear Beltrami equation

$$g_{\overline{z}} = \mu(g)\, g_z + \nu(g)\, \overline{g_z} \tag{2.18}$$

with the coefficients

$$\mu(g) = \frac{\mathcal{H}_\xi(g)}{1 - |\mathcal{H}_{\overline{\xi}}(g)|^2}, \quad \nu(g) = \frac{\overline{\mathcal{H}_\xi(g)}\, \mathcal{H}_{\overline{\xi}}(g)}{1 - |\mathcal{H}_{\overline{\xi}}(g)|^2}, \tag{2.19}$$

satisfying

$$|\mu(g)| + |\nu(g)| \leqslant k < 1,$$

by k-Lipschitz property of $\mathcal{H}$, $|D_\xi \mathcal{H}(g)| = |\mathcal{H}_\xi(g)| + |\mathcal{H}_{\overline{\xi}}(g)| \leqslant k < 1$.

There are now two natural ways to proceed. First, we have a quick way to deduce the $W^{2,p}_{\mathrm{loc}}$-regularity of the solution F for all $1 < p < \infty$ using (2.18) and the fact that the coefficients μ, ν are continuous. In fact, following the path from [5, 9, 16], for any linear Beltrami equation with coefficients in VMO all $W^{1,2}_{\mathrm{loc}}$-solutions are actually $W^{1,p}_{\mathrm{loc}}$-regular for every $1 < p < \infty$. However, these arguments rely on applying Fredholm theory to the Beltrami equation and as such do not yield the explicit bounds we need in a straightforward manner.

Another approach is to use the Morrey-Campanato method to improve the norm estimates in Corollary 2.3. Here we split $g = G + (g - G)$, where G solves the Riemann-Hilbert problem of a linear equation with constant coefficients,

$$\begin{cases} G_{\overline{z}} = \mu((\partial_z F)_R)\, G_z + \nu((\partial_z F)_R)\, \overline{G_z} & \text{a.e. } z \in \mathbb{D}_R = \mathbb{D}(z_1, R), \\ \mathrm{Re}(g - G) = 0 & \text{on } \partial\mathbb{D}_R. \end{cases} \tag{2.20}$$

Above $z_1 \in \Omega''$, $R \leqslant \operatorname{dist}(z_1, \partial\Omega'')$, and $(\partial_z F)_R = \fint_{\mathbb{D}_R} \partial_z F$. Similarly, as we already saw in Proposition 2.4, the existence of G is based on the local versions of the classical Cauchy transform and the Beurling transform. Moreover,

$$\|D_z G\|_{L^2(\mathbb{D}_R)} \leqslant c(K) \|D_z g\|_{L^2(\mathbb{D}_R)}. \tag{2.21}$$

For a solution G to (2.20), we have $D_z G \in W^{1,2}_{\mathrm{loc}}(\mathbb{D}_R, \mathbb{C})$ and the directional derivatives $\partial_e G(z)$, $|e| = 1$, are K-quasiregular in $\mathbb{D}_R$, by using difference quotients G_h as in the proof of Proposition 2.1.

We will show that, for every $\varepsilon > 0$,

$$\|D_z^2 F\|_{L^2(\mathbb{D}_\rho)} \leqslant c(K,\varepsilon) \left(\frac{\rho}{R}\right)^{1-\varepsilon} \|D_z^2 F\|_{L^2(\mathbb{D}_R)}, \tag{2.22}$$

whenever $\rho \leqslant R \leqslant \min\{R_0, \operatorname{dist}(z_1, \partial\Omega'')\}$, where R_0 will be chosen later.

As $\partial_{\overline{z}} F = \mathcal{H}(\partial_z F)$ and $D_\xi \mathcal{H}(\xi)$ is uniformly bounded by k, it is enough to show the claim for $g = \partial_z F$, that is,

$$\|D_z g\|_{L^2(\mathbb{D}_\rho)} \leqslant c(K,\varepsilon) \left(\frac{\rho}{R}\right)^{1-\varepsilon} \|D_z g\|_{L^2(\mathbb{D}_R)}, \tag{2.23}$$

whenever $\rho \leqslant R \leqslant \min\{R_0, \operatorname{dist}(z_1, \partial\Omega'')\}$.

Since $\|D_z G(z)\|^2 = \sum_{j=1}^2 |D_z G(z)\, e_j|^2 = \sum_{j=1}^2 |\partial_{e_j} G(z)|^2$ for the Hilbert-Schmidt norm, the quasiregularity of $\partial_e G$ with integral estimate (2.4) of Proposition 2.2 implies

$$\|D_z G\|_{L^2(\mathbb{D}_\rho)} \leqslant c(K)\, \frac{\rho}{R}\, \|D_z G\|_{L^2(\mathbb{D}_R)}.$$

Hence, by triangle inequality,

$$\begin{aligned} \|D_z g\|_{L^2(\mathbb{D}_\rho)} &\leqslant \|D_z G\|_{L^2(\mathbb{D}_\rho)} + \|D_z (g-G)\|_{L^2(\mathbb{D}_\rho)} \\ &\leqslant c(K)\, \frac{\rho}{R}\, \|D_z G\|_{L^2(\mathbb{D}_R)} + \|D_z (g-G)\|_{L^2(\mathbb{D}_R)} \\ &\leqslant c(K)\, \frac{\rho}{R}\, \|D_z g\|_{L^2(\mathbb{D}_R)} + \|D_z (g-G)\|_{L^2(\mathbb{D}_R)}, \end{aligned} \tag{2.24}$$

where the last estimate follows by (2.21). Thus we need to estimate $D_z(g-G)$. Below we use the uniform bound $|\mu| + |\nu| \leqslant k$ to get that

$$\begin{aligned} &\|(g-G)_{\overline{z}}\|_{L^2(\mathbb{D}_R)} \\ &\quad = \|\mu(g)\, g_z + \nu(g)\, \overline{g_z} - \mu((\partial_z F)_R)\, G_z - \nu((\partial_z F)_R)\, \overline{G_z}\|_{L^2(\mathbb{D}_R)} \\ &\quad \leqslant \|\mu((\partial_z F)_R)\, (g-G)_z + \nu((\partial_z F)_R)\, \overline{(g-G)_z}\|_{L^2(\mathbb{D}_R)} \\ &\qquad + \|(\mu(g) - \mu((\partial_z F)_R))\, g_z + (\nu(g) - \nu((\partial_z F)_R))\, \overline{g_z}\|_{L^2(\mathbb{D}_R)} \\ &\quad \leqslant k\, \|(g-G)_z\|_{L^2(\mathbb{D}_R)} \\ &\qquad + \sup_{z\in\mathbb{D}_R} \big[|(\mu(g) - \mu((\partial_z F)_R)| + |\nu(g) - \nu((\partial_z F)_R))|\big]\, \|g_z\|_{L^2(\mathbb{D}_R)}. \end{aligned}$$

Hence, combining with (2.24) and using that the local Beurling transform of the disk is an isometry to absorb the term $k\, \|(g-G)_z\|_{L^2(\mathbb{D}_R)}$ into the left hand side, we have

$$\|D_z g\|_{L^2(\mathbb{D}_\rho)} \leqslant c(K) \left(\frac{\rho}{R} + \sigma(R)\right) \|D_z g\|_{L^2(\mathbb{D}_R)},$$

where

$$\sigma(R) := \sup_{z\in\mathbb{D}_R} \left[|(\mu(\partial_z F) - \mu((\partial_z F)_R)| + |\nu(\partial_z F) - \nu((\partial_z F)_R))| \right].$$

Now, (2.23) follows by Lemma 2.8 if we can make $\sigma(R)$ as small as we wish by reducing R. This is actually possible since $\partial_z F$ is $\frac{1}{K}$-Hölder continuous by Corollary 2.3 and μ and ν are continuous by the fact that $\mathcal{H}$ is C^1. Here R_0 has to be so small that $\sigma(R_0) \leqslant \sigma_0(K,\varepsilon)$, where the constant σ_0 is from Lemma 2.8. Moreover, we can choose R_0 uniformly in the compact set Ω''.

We collect now the dependence of R_0 on the parameters. From the proof we see that it depends on the modulus of continuity of $\mathcal{H}_\xi$ and $\mathcal{H}_{\overline{\xi}}$ on the set $\partial_z F(\Omega'')$ as well as the numbers $[\partial_z F]_{C^{1/K}(\Omega'')}$, K and ε. It is also possible, via Corollary 2.3 and the Morrey-Campanato norm estimates (2.11)–(2.13), to bound the size of the set $\partial_z F(\Omega'')$ and $[\partial_z F]_{C^{1/K}(\Omega'')}$ in terms of $c(K,\Omega',\Omega'')\,\|DF\|_{L^2(\Omega')}$.

Using Poincaré's inequality on the left hand side of (2.22) and Caccioppoli's inequality on the right we deduce, for $\rho \leqslant R \leqslant \min\{R_0, \mathrm{dist}(z_1, \partial\Omega'')\}$,

$$\begin{aligned} &\|D_z F - (D_z F)_\rho\|_{L^2(\mathbb{D}_\rho)} \\ &\qquad \leqslant c(K,\varepsilon)\left(\tfrac{\rho}{R}\right)^{1+(1-\varepsilon)} \|D_z F - (D_z F)_R\|_{L^2(\mathbb{D}_R)}. \end{aligned} \tag{2.25}$$

As we have seen before, because of the Caccioppoli estimate, we have (2.25) first for $\rho \leqslant \frac{R}{2}$. The full range $\rho \leqslant R$ holds with possible bigger constant.

The claim follows by covering $\mathbb{D}(z_0,R)$ with disks of radius R_0. □

The (nonlinear) L^2-norm dependence of R_0 is reflected in the final C^α-norm estimate of $D_z f$, that is, we do not have linear dependence on the L^2-norm as in Theorem 1.2 (i.e., (1.4)). □

3. Positivity of the Jacobian

In this section we prove Theorem 1.1. After that we will give a simple and separate argument for the autonomous case, where for the structural field we do not have to assume C^1-dependence on the gradient variable.

Proof of Theorem 1.1. Let us first prove the positivity of the Jacobian when f is a solution to (1.1) on a bounded domain $\Omega \subset \mathbb{C}$. Since f is a quasiconformal map, we can conclude several things. First of all, the null set of the Jacobian, defined by

$$G = \{z \in \Omega : J(z,f) = 0\}$$

is of zero Lebesgue measure. Secondly, we know that the Jacobian $J_f = J(z,f) = |f_z|^2 - [f_{\overline{z}}|^2$ is comparable to the expression $|f_z|^2$. We also have additional regularity of f, since by Theorem 1.2 we know that f is in $C^{1,\gamma}(\Omega,\mathbb{C})$ for some positive number $\gamma > 0$. In particular, the set G is closed.

We now assume that the set G is nonempty and find a contradiction. For each $r > 0$ consider the open r-neighbourhood

$$G_r = \{z \in \Omega : 0 < \mathrm{dist}(z,G) < r\}.$$

Let g denote the inverse of f. Then g is quasiconformal in the whole domain $f(\Omega)$ and, in particular, it lies in $W^{1,2}_{\mathrm{loc}}(f(\Omega),\mathbb{C})$. The whole point of the proof is to show the following lemma.

Lemma 3.1. *For $r > 0$ small enough, the inverse map g solves a nonlinear Beltrami equation of the form*

$$g_{\overline{\omega}}(\omega) = \mathcal{H}^*(g(\omega), g_\omega(\omega)) \quad \text{in the set } f(G_r), \tag{3.1}$$

where $\mathcal{H}^*(g, \xi)$ satisfies the conditions (1.2).

Proof of the lemma. We first find a nonlinear equation for the inverse map g. In the set G_r, with r to be determined later, we have that

$$f_{\overline{z}} = \frac{-g_{\overline{\omega}}}{J_g} \quad \text{and} \quad f_z = \frac{\overline{g_\omega}}{J_g}, \qquad \omega = f(z). \tag{3.2}$$

Thus g satisfies the equation

$$-g_{\overline{\omega}} = \left(|g_\omega|^2 - |g_{\overline{\omega}}|^2\right) \mathcal{H}\left(g, \frac{\overline{g_\omega}}{|g_\omega|^2 - |g_{\overline{\omega}}|^2}\right).$$

Our strategy is to show that in the set $f(G_r)$, the expression $g_{\overline{\omega}}$ can be solved from this equation in terms of g_ω and g. Notice the estimate (as $f \in C^{1,\gamma}(\Omega, \mathbb{C})$)

$$|f_z(z)| = |f_z(z) - 0| \leqslant c_1 \operatorname{dist}(z, G)^\gamma \leqslant c_1 r^\gamma \quad \text{for } z \in G_r.$$

This implies that

$$|J(\omega, g)| \geqslant \frac{c_2}{r^{2\gamma}} \quad \text{and} \quad |g_\omega| \geqslant \frac{c_3}{r^\gamma}, \qquad \omega \in f(G_r).$$

We collect the bounds as $|J_g| \geqslant M$ and $|g_\omega| \geqslant M$ for some large number M, which can be as big as we want by choosing r small enough.

Consider now the equation

$$-\zeta = \left(t^2 - s^2\right) \mathcal{H}\left(g, \frac{\overline{\xi}}{t^2 - s^2}\right), \tag{3.3}$$

where we use the shorthand notations $t = |\xi|$ and $s = |\zeta|$. This equation involves three variables:

$$\begin{cases} \xi, \text{ a complex variable with } |\xi| \geqslant M, \\ \zeta, \text{ which satisfies } |\zeta| \leqslant k|\xi| \text{ due to quasiconformality of } g, \\ g, \text{ a variable that belongs to the bounded set } \Omega. \end{cases} \tag{3.4}$$

We must then show that for fixed ξ and g equation (3.3) admits a unique solution ζ.

Step 1. Existence. Note that if we know what $s = |\zeta|$ is in terms of ξ and g, then we can also solve ζ since the right hand side of (3.3) only depends on s, g and ξ. We hence take absolute values on both sides of equation (3.3) and consider the continuous function

$$\varphi(s) = s - \left(t^2 - s^2\right) \left|\mathcal{H}\left(g, \frac{\overline{\xi}}{t^2 - s^2}\right)\right|.$$

It is enough to show that $\varphi(s) = 0$ for some $s \in [0, kt]$. Since $\mathcal{H}$ is k-Lipschitz in the second variable with $0 \mapsto 0$, we have that

$$\varphi(s) \geqslant s - kt, \quad \text{implying that} \quad \varphi(kt) \geqslant 0.$$

At the point $s = 0$,

$$\varphi(0) = -t^2 \left|\mathcal{H}\left(g, \frac{\overline{\xi}}{t^2}\right)\right| \leqslant 0.$$

Hence φ has a zero between 0 and kt and thus existence is proved.

Step 2. Uniqueness. We define

$$\psi(\zeta) = \left(|\xi|^2 - |\zeta|^2\right) \mathcal{H}\left(g, \frac{\overline{\xi}}{|\xi|^2 - |\zeta|^2}\right).$$

It will be enough to show that ψ is a strict contraction, since then the mapping $\zeta \mapsto \zeta + \psi(\zeta)$ must be injective. Since by (1.3), ψ is C^1 with respect to ζ, it is enough to show that $|\psi_\zeta(\zeta)| \leqslant \varepsilon$ for sufficiently small ε, because then also $|\psi_{\overline{\zeta}}(\zeta)| \leqslant \varepsilon$ due to the fact that $\psi(\zeta) = \psi(\overline{\zeta})$. We use another shorthand notation

$$\tau = \frac{\overline{\xi}}{|\xi|^2 - |\zeta|^2}.$$

A calculation now gives that

$$\psi_\zeta(\zeta) = -\overline{\zeta}\big(\mathcal{H}(g,\tau) - \tau\mathcal{H}_\tau(g,\tau) - \overline{\tau}\mathcal{H}_{\overline{\tau}}(g,\tau)\big).$$

Notice that $|\tau|$ is dominated by a constant times $\frac{1}{t}$ and thus is as small as we wish (by our choice of r). We now use the formula

$$\mathcal{H}(g,\tau) = \tau \int_0^1 \mathcal{H}_\tau(g, R\tau)dR + \overline{\tau}\int_0^1 \mathcal{H}_{\overline{\tau}}(g,R\tau)dR$$

which is valid since $\mathcal{H}(g,\tau)$ is C^1 with respect to τ. This gives that

(3.5)
$$\begin{aligned} &-\overline{\zeta}\left(\mathcal{H}(g,\tau) - \tau\mathcal{H}_\tau(g,\tau) - \overline{\tau}\mathcal{H}_{\overline{\tau}}(g,\tau)\right) \\ &\quad = -\overline{\zeta}\tau \int_0^1 \left[\mathcal{H}_\tau(g,R\tau) - \mathcal{H}_\tau(g,\tau)\right]dR - \overline{\zeta}\overline{\tau}\int_0^1 \left[\mathcal{H}_{\overline{\tau}}(g,R\tau) - \mathcal{H}_{\overline{\tau}}(g,\tau)\right]dR. \end{aligned}$$

Note that $|\zeta\tau| \leqslant kt|\tau|$, which is bounded above by a constant. We now get that

$$|\psi_\zeta(\zeta)| \leqslant c\left(\sup_{0\leqslant R\leqslant 1}|\mathcal{H}_\tau(g,R\tau) - \mathcal{H}_\tau(g,\tau)| + \sup_{0\leqslant R\leqslant 1}|\mathcal{H}_{\overline{\tau}}(g,R\tau) - \mathcal{H}_{\overline{\tau}}(g,\tau)|\right).$$

Since $\mathcal{H}(g,\tau)$ is in C^1 with respect to τ, the right hand side is as small as we wish. Let us collect this as the estimate

$$|\psi_\zeta(\zeta)| \leqslant \varepsilon. \tag{3.6}$$

This estimate also holds uniformly in g, since by decreasing Ω if necessary we may assume that the variable g lies in a fixed compact set and we can then apply (1.3). In fact, we will later also use the estimate

$$|\xi||\mathcal{H}(g,\tau) - \tau\mathcal{H}_\tau(g,\tau) - \overline{\tau}\mathcal{H}_{\overline{\tau}}(g,\tau)| \leqslant \varepsilon' \tag{3.7}$$

which follows from (3.5) and holds uniformly in g with $\varepsilon' > 0$ as small as we wish. In any case, we have proved the uniqueness part as well.

Now, for any g and ξ satisfying (3.4), equation (3.3) admits a unique solution ζ, denoted by

$$\zeta = \mathcal{H}^*(g,\xi). \tag{3.8}$$

Next we prove the necessary regularity conditions for $\mathcal{H}^*$.

Step 3. Hölder regularity. We need to prove the estimate

$$|\mathcal{H}^*(g_1,\xi)-\mathcal{H}^*(g_2,\xi)|\leqslant c\,|g_1-g_2|^\alpha|\xi|. \tag{3.9}$$

To do this we define

$$\zeta_1=\mathcal{H}^*(g_1,\xi),\ s_1=|\zeta_1| \quad\text{and}\quad \zeta_2=\mathcal{H}^*(g_2,\xi),\ s_2=|\zeta_2|$$

to obtain from (3.3) the equations

$$-\zeta_1=\left(t^2-s_1^2\right)\mathcal{H}\left(g_1,\frac{\overline{\xi}}{t^2-s_1^2}\right)\quad\text{and}\quad -\zeta_2=\left(t^2-s_2^2\right)\mathcal{H}\left(g_2,\frac{\overline{\xi}}{t^2-s_2^2}\right).$$

Taking the difference of the two equations, we obtain that

$$\begin{aligned}
|\zeta_1-\zeta_2| &= \left|\left(t^2-s_1^2\right)\mathcal{H}\left(g_1,\frac{\overline{\xi}}{t^2-s_1^2}\right)-\left(t^2-s_2^2\right)\mathcal{H}\left(g_2,\frac{\overline{\xi}}{t^2-s_2^2}\right)\right|\\
&\leqslant \left|\left(t^2-s_1^2\right)\mathcal{H}\left(g_1,\frac{\overline{\xi}}{t^2-s_1^2}\right)-\left(t^2-s_2^2\right)\mathcal{H}\left(g_1,\frac{\overline{\xi}}{t^2-s_2^2}\right)\right|\\
&\quad+\left|\left(t^2-s_2^2\right)\mathcal{H}\left(g_1,\frac{\overline{\xi}}{t^2-s_2^2}\right)-\left(t^2-s_2^2\right)\mathcal{H}\left(g_2,\frac{\overline{\xi}}{t^2-s_2^2}\right)\right|\\
&\leqslant 2\varepsilon\,|\zeta_1-\zeta_2|+\left(t^2-s_2^2\right)\frac{c\,|g_1-g_2|^\alpha|\xi|}{t^2-s_2^2}.
\end{aligned}$$

The last estimate comes from (3.6) and the Hölder continuity of $\mathcal{H}$ (1.2). In total, we get that

$$|\mathcal{H}^*(g_1,\xi)-\mathcal{H}^*(g_2,\xi)|=|\zeta_1-\zeta_2|\leqslant\frac{c}{1-2\varepsilon}|g_1-g_2|^\alpha|\xi|,$$

which is the estimate we wanted to prove.

Step 4. Ellipticity. First note that $\mathcal{H}^*$ is C^1 in the second variable; one sees this from the implicit function theorem, when keeping the other variable g fixed. We then need to prove

$$|\mathcal{H}^*(g,\xi_1)-\mathcal{H}^*(g,\xi_2)|\leqslant k'|\xi_1-\xi_2|$$

for some constant $k'<1$. We obtain this by finding estimates for the ξ-derivatives of the function $\mathcal{H}^*$. These estimates are derived implicitly from the equation

$$-\mathcal{H}^*=\left(\xi\overline{\xi}-\mathcal{H}^*\overline{\mathcal{H}^*}\right)\mathcal{H}\left(g,\frac{\overline{\xi}}{\xi\overline{\xi}-\mathcal{H}^*\overline{\mathcal{H}^*}}\right). \tag{3.10}$$

The equation (3.10) is just (3.3) with the definition of $\mathcal{H}^*$ (3.8) plugged in. For the following computation we use the shorthand notations

$$\begin{cases}\tau=\dfrac{\overline{\xi}}{\xi\overline{\xi}-\mathcal{H}^*\overline{\mathcal{H}^*}}\\[2ex] Q=\mathcal{H}-\tau\mathcal{H}_\tau-\overline{\tau}\mathcal{H}_{\overline{\tau}}\end{cases}$$

Taking the ξ-derivative of both sides of (3.10) gives

$$-\mathcal{H}^*_\xi=\left(\overline{\xi}-\mathcal{H}^*_\xi\overline{\mathcal{H}^*}-\mathcal{H}^*\overline{\mathcal{H}^*_{\overline{\xi}}}\right)Q+\mathcal{H}_{\overline{\tau}}. \tag{3.11}$$

We also take the $\overline{\xi}$-derivative of (3.10) to obtain

$$-\mathcal{H}^*_{\overline{\xi}} = \left(\xi - \mathcal{H}^*\overline{\mathcal{H}^*_{\xi}} - \mathcal{H}^*_{\overline{\xi}}\overline{\mathcal{H}^*}\right) Q + \mathcal{H}_\tau. \tag{3.12}$$

Notice that by the definition of Q and the estimate (3.7) we may assume that $|\xi Q| < \varepsilon$ for arbitrarily small $\varepsilon > 0$. Since $|\mathcal{H}^*| \leqslant k|\xi|$, we also have that $|\mathcal{H}^* Q| \leqslant \varepsilon$. Along with the bound $|\mathcal{H}_\tau| + |\mathcal{H}_{\overline{\tau}}| \leqslant k < 1$ we may now estimate from (3.11) that

$$|\mathcal{H}^*_\xi| \leqslant \varepsilon + \varepsilon|\mathcal{H}^*_\xi| + \varepsilon|\mathcal{H}^*_{\overline{\xi}}| + |\mathcal{H}_{\overline{\tau}}|$$

and from (3.12) that

$$|\mathcal{H}^*_{\overline{\xi}}| \leqslant \varepsilon + \varepsilon|\mathcal{H}^*_\xi| + \varepsilon|\mathcal{H}^*_{\overline{\xi}}| + |\mathcal{H}_\tau|.$$

Adding these together, we get the estimate

$$|\mathcal{H}^*_\xi| + |\mathcal{H}^*_{\overline{\xi}}| \leqslant \frac{2\varepsilon + |\mathcal{H}_\tau| + |\mathcal{H}_{\overline{\tau}}|}{1 - 2\varepsilon} \leqslant \frac{2\varepsilon + k}{1 - 2\varepsilon}.$$

For sufficiently small ε, the utmost right hand side is less than one. Thus the ellipticity of $\mathcal{H}^*$ has been proven.

Step 5. Extension. Above we have achieved finding the function $\mathcal{H}^*(g, \xi)$, with required ellipticity and regularity properties, but defined only in the subset $\Omega \times \{\xi : |\xi| \geqslant M\}$. Extending to the entire ξ-plane is, however, no problem: We may set

$$\mathcal{H}^*(g, \rho\xi) = \rho\mathcal{H}^*(g, \xi), \quad \text{when } 0 \leqslant \rho|\xi| \leqslant M.$$

This gives $\mathcal{H}^*$ the properties (1.2) in $\Omega \times \mathbb{C}$, and completes the proof of Lemma 3.1. □

Conclusion. By Lemma 3.1, we know that $g \in W^{1,2}_{\mathrm{loc}}(f(\Omega), \mathbb{C})$ satisfies the equation (3.1) in $f(G_r)$, hence almost everywhere in the open set $U := f(G_r) \cup f(G)$ since $f(G)$ is of zero measure (as f is quasiconformal). Because g is quasiconformal it is Hölder continuous to some exponent $\beta > 0$, which also gives the Hölder estimate

$$|\mathcal{H}^*(g(\omega_1), \xi) - \mathcal{H}^*(g(\omega_2), \xi)| \leqslant c_1 \, |g(\omega_1) - g(\omega_2)|^\alpha \leqslant c_2 \, |\omega_1 - \omega_2|^{\alpha\beta}.$$

We can now use Theorem 1.2 to see that g is in $C^{1,\gamma'}_{\mathrm{loc}}(U)$ for some $\gamma' > 0$. Thus J_g is locally bounded with $J_f(z) J_g(f(z)) \equiv 1$, a contradiction. This shows that $J_f > 0$ in Ω, finishing the first part of Theorem 1.1.

To complete the proof of Theorem 1.1 we next use a compactness argument to show that for a normalised homeomorphic solution f to (1.1) there is a lower bound for the Jacobian in each disk $\mathbb{D}(0, R_0)$, that is,

$$\inf_{z \in \mathbb{D}(0,R_0)} J(z, f) \geqslant c(\mathcal{H}, R_0) > 0.$$

We also collect the dependence of the constant $c(\mathcal{H}, R_0)$ on $\mathcal{H}$ and R_0. It will be shown that $c(\mathcal{H}, R_0)$ only depends on the numbers $R_0, k, \alpha, \mathbf{H}_\alpha(\mathbb{D}(0, 8R_0))$ and the modulus of continuity of $\mathcal{H}_\xi(z, \xi)$ and $\mathcal{H}_{\overline{\xi}}(z, \xi)$ in the set $\mathbb{D}(0, R_0) \times \mathbb{D}(0, 1)$.

Let us make a counter-assumption: there exist $z_n \in \mathbb{D}(0, R_0)$ and normalised homeomorphic solutions f_n to the nonlinear Beltrami equations of the type (1.1) with the regularity (1.2) and (1.3), i.e.,

$$\partial_{\bar{z}} f_n(z) = \mathcal{H}_n(z, \partial_z f_n(z)) \qquad \text{a.e.},$$

such that

$$J(z_n, f_n) \leqslant \frac{1}{n}.$$

In particular, the Hölder constant $\mathbf{H}_\alpha(\mathbb{D}(0, 8R_0))$, Hölder exponent α and ellipticity k are assumed to be the same for each $\mathcal{H}_n$.

Now, we may pass to the subsequence, if necessary, to assume that $z_n \to z_\infty \in \overline{\mathbb{D}(0, R_0)}$ and as a normalised family of quasiconformal maps $f_n \to f_\infty$ locally uniformly, where f_∞ is quasiconformal and $f_\infty(0) = 0$, $f_\infty(1) = 1$ (see the Montel-type theorem [4, Theorem 3.9.4]). Moreover, by the Schauder norm estimate (1.4), for any $R > 0$,

$$\begin{aligned} \|D_z f_n\|_{C^\gamma(\mathbb{D}(0,R))} &\leqslant c\, \|D_z f_n\|_{L^2(\mathbb{D}(0,2R))} \leqslant c\, \|f_n\|_{L^2(\mathbb{D}(0,4R))} \\ &\leqslant c\, \eta_K(4R), \end{aligned} \tag{3.13}$$

where $c = c(\mathcal{H}, R)$ and the second to the last inequality follows by Caccioppoli's inequality and the last one from the η_K-quasisymmetry of quasiconformal maps. Hence derivatives $D_z f_n$ have a local uniform C^γ-upper bound and mappings f_n converge to f_∞ in $C^{1,\gamma}_{\text{loc}}(\mathbb{C}, \mathbb{C})$, too. Thus $J(z_\infty, f_\infty) = 0$.

We will show that the inconsistency follows from the fact that f_∞ also solves a nonlinear Beltrami equation

$$\partial_{\bar{z}} f_\infty(z) = \mathcal{H}_\infty(z, \partial_z f_\infty(z)) \qquad \text{a.e.}, \tag{3.14}$$

where $\mathcal{H}_\infty$ will satisfy the assumptions (1.2) and (1.3).

We first find $\mathcal{H}_\infty$ as a limit of the fields $\mathcal{H}_n$. Namely, $\mathcal{H}_n$ is locally uniformly equicontinuous on $\mathbb{C} \times \mathbb{C}$. Indeed, given open, bounded sets Ω', Ω'' and $(z_i, \xi_i) \in \Omega' \times \Omega''$, by assumption

$$|\mathcal{H}_n(z_1, \xi_1) - \mathcal{H}_n(z_2, \xi_2)| \leqslant \mathbf{H}_\alpha(\Omega')|z_1 - z_2|^\alpha(|\xi_1| + |\xi_2|) + k\, |\xi_1 - \xi_2|.$$

This gives the equicontinuity. Thus passing to a subsequence it converges to a function $\mathcal{H}_\infty$ locally uniformly, where $\mathcal{H}_\infty$ has the same regularity and norm bounds (1.2) as the family $\mathcal{H}_n$, and satisfies (1.3) in the set $\mathbb{D}(0, R_0) \times \mathbb{D}(0, 1)$.

As $\mathcal{H}_\infty$ has the required regularity properties, we must only show that f_∞ satisfies equation (3.14). But this is immediate from the fact that the convergence of $D_z f_n$ is also locally uniform (they converge in the Hölder class as seen above). By the earlier part of the proof of Theorem 1.1, we now know that $J_{f_\infty} > 0$ in the set $\mathbb{D}(0, R_0)$, a contradiction to the fact that $J(z_\infty, f_\infty) = 0$. Hence there must be a lower bound for the Jacobian, and we have proven Theorem 1.1. □

For the autonomous case we present a different argument based on Stoïlov factorisation, Hurwitz theorem and a compactness argument inspired by [3]. The key point is that we do not need to assume C^1-regularity of $\mathcal{H}$ in the gradient variable.

Theorem 3.2. *Assume $\mathcal{H} : \mathbb{C} \to \mathbb{C}$ is k-Lipschitz, where $k < 1$, with $\mathcal{H}(0) = 0$ and let $F \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{C})$ be a homeomorphic solution to*

$$\partial_{\bar{z}} F(z) = \mathcal{H}(\partial_z F(z)) \qquad a.e.$$

Then $J(z, F) \neq 0$ at every point $z \in \Omega$.

Proof. Let us fix a disk $\mathbb{D}(z_0, 2R) \subset \Omega$ and a point $z_1 \in \mathbb{D}(z_0, R)$ where $J(z_1, F) \neq 0$. The derivatives of F are continuous by Proposition 2.1 and we can assume, for instance, that $\partial_x F(z_1) \neq 0$ and we will show that $\partial_x F(z) \neq 0$ everywhere. This is enough, since $|DF|^2 \leqslant K J_F$.

Let us define

$$F_h(z) = \frac{F(z+h) - F(z)}{F(z_1+h) - F(z_1)}, \qquad h > 0. \tag{3.15}$$

Clearly F_h is well-defined on $\Omega_h = \{z \in \Omega : d(z, \partial\Omega) > h\}$, and $\mathbb{D}(z_0, 2R) \subset \Omega_h$ for any $h < d(z_0, \partial\Omega) - 2R$. Further, F_h is K-quasiregular on Ω_h, as we saw in (2.2). Moreover, by Proposition 2.1, we know that $D_z F \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{C})$ and $F \in C^{1, \frac{1}{K}}_{\mathrm{loc}}(\Omega)$.

We can factor, by Stoïlow factorisation,

$$F_h = H_h \circ \Phi_h$$

where $\Phi_h : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal, and we choose the normalisation $\Phi_h(z_0) = 0$, $\Phi_h(z_1) = 1$, and $H_h : \Phi_h(\Omega_h) \to \mathbb{C}$ is holomorphic. Moreover, $H_h(1) = 1$, by the definition of F_h and the above normalisation of Φ_h. Since Φ_h are normalised K-quasiconformal maps, there exists a limit K-quasiconformal map

$$\Phi = \lim_{h \to 0^+} \Phi_h,$$

with locally uniform convergence, at least for a subsequence, see the Montel-type theorem [4, Theorem 3.9.4]. Similarly, for the same subsequence $\Phi_h^{-1} \to \Phi^{-1}$ locally uniformly in $\Phi(\mathbb{D}(z_0, R))$.

Note further that since F is continuously differentiable and $\partial_x F(z_1) \neq 0$, the functions F_h in (3.15) converge locally uniformly in $\mathbb{D}(z_0, R)$, hence also $H_h = F_h \circ \Phi_h^{-1}$ converges locally uniformly in $\Phi(\mathbb{D}(z_0, R))$. Let us now fix a compact set $E \subset \Phi(\mathbb{D}(z_0, R))$ with 1 as an interior point. Since $\Phi_h(\mathbb{D}(z_0, R))$ converges in the Hausdorff metric to $\Phi(\mathbb{D}(z_0, R))$, for every h small enough $\Phi(\mathbb{D}(z_0, R)) \Subset \Phi_h(\mathbb{D}(z_0, 2R))$. Thus $E \subset \Phi(\mathbb{D}(z_0, R)) \Subset \Phi_h(\Omega_h)$, and so H_h, $h < h_0$, is well-defined family of functions analytic on a neighbourhood of E, with limit

$$H = \lim_{h \to 0^+} H_h$$

at least for a subsequence. Of course, the limit mapping H is holomorphic on a neighbourhood of E and $H(1) = 1$. Then it follows that

$$\lim_{h \to 0^+} H_h \circ \Phi_h = H \circ \Phi$$

uniformly on compact subsets of $\mathbb{D}(z_0, R)$. In particular,

$$\frac{\partial_x F(z)}{\partial_x F(z_1)} = H(\Phi(z)) \qquad \text{for every } z \in \mathbb{D}(z_0, R).$$

But the analytic functions H_h do not have zeros in $\Phi_h(\Omega_h)$, since F is a homeomorphism. By the Hurwitz theorem H as well is non-vanishing on E, that is, $\frac{\partial_x F(z)}{\partial_x F(z_1)}$ does not have zeros in $\mathbb{D}(z_0, R)$. We have shown our claim. $\square$

Remark 3.3. Alternatively in the proof of Theorem 3.2 one can invoke the Hurwitz theorem for quasiregular mappings [19] which tells for any converging subsequence that either the limit $\lim_j F_{h_j}(z)$ is non-vanishing everywhere, or the limit vanishes identically.

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