MANIFOLDS OF QUASICONFORMAL MAPPINGS AND THE NONLINEAR BELTRAMI EQUATION

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Abstract. In this paper we show that the homeomorphic solutions to each nonlinear Beltrami equation $\partial_{\bar z} f = \mathcal{H}(z, \partial_z f)$ generate a two-dimensional manifold of quasiconformal mappings $\mathcal{F}_{\mathcal{H}} = \{\varphi_a(z)\}_{a\in\mathbb{C}} \subset W^{1,2}_{\mathrm{loc}}(\mathbb{C})$. The process is reversible. To each family of quasiconformal mappings $\mathcal{F}$ we can associate a corresponding nonlinear Beltrami equation. Moreover, we show that there is an interplay between the regularity of $\mathcal{H}$ with respect to both variables (z, w) and the regularity of $\mathcal{F}$ with respect to (z, a). Under regularity assumptions the relation between $\mathcal{F}$ and $\mathcal{H}$ is one-to-one.

1. Introduction

In the context of G-compactness of Beltrami operators (introduced in [15]) it was recently proved that there is a one-to-one correspondence between linear families of quasiconformal mappings (two-dimensional subspaces of quasiconformal mappings) and $\mathbb{R}$-linear Beltrami equations [1], [7], [5, Theorem 16.6.6]. More precisely, given a linear family of quasiconformal maps $\mathcal{F} = \{\alpha f + \beta g : \alpha, \beta \in \mathbb{R},\ \alpha^2 + \beta^2 \neq 0\} \subset W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ there exists a unique couple of measurable functions μ and ν with $\big\| |\mu| + |\nu| \big\|_\infty \leqslant k < 1$ such that $\mathcal{F}$ is exactly the set homeomorphic solutions to the $\mathbb{R}$-linear Beltrami equation

$$\partial_{\bar z} f(z) = \mu(z)\, \partial_z f(z) + \nu(z)\, \overline{\partial_z f(z)}, \qquad \text{for almost every } z \in \mathbb{C}. \tag{1.1}$$

If $\nu \equiv 0$, one gets the classical Beltrami equation and, in this case, the family is not only $\mathbb{R}$-linear, but also $\mathbb{C}$-linear.

The goal of this paper is to study nonlinear families of quasiconformal maps that arise from the nonlinear Beltrami equations

$$\partial_{\bar z} f(z) = \mathcal{H}(z, \partial_z f(z)), \qquad \text{for almost every } z \in \mathbb{C}. \tag{1.2}$$

Starting from the pioneering work of Bojarski and Iwaniec [18], [10], in the last two decades it has been shown that much of the linear theory for the Beltrami equation extends to the nonlinear situation, under basic assumptions

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in the nonlinearity to guarantee the uniform ellipticity. Namely, we consider a field $\mathcal{H} : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ which satisfies

(H1) $\mathcal{H}$ is k-Lipschitz in the second variable, that is, for $z, w_1, w_2 \in \mathbb{C}$,

$$|\mathcal{H}(z, w_1) - \mathcal{H}(z, w_2)| \leqslant k(z)|w_1 - w_2|, \quad 0 \leqslant k(z) = \frac{K(z) - 1}{K(z) + 1} \leqslant k < 1,$$

for almost every $z \in \mathbb{C}$, and the normalization $\mathcal{H}(z, 0) \equiv 0$ holds.

(H2) For every $w \in \mathbb{C}$, the mapping $z \mapsto \mathcal{H}(z, w)$ is measurable on $\mathbb{C}$,

Starting with these assumptions we study the corresponding nonlinear Beltrami equations. The existence and the regularity theory of the nonlinear Beltrami equations resembles that of the linear one, see [18], [10], [6], [5]. However, the uniqueness of a homeomorphic solution, knowing how it maps two points, is subtle and not always true as proved in [3]. Thus we define as follows.

Definition 1.1. We say that a Beltrami equation has *the uniqueness property* if for every $z_0, z_1, \omega_0, \omega_1 \in \mathbb{C}$ there is a unique homeomorphic solution $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ to (1.2) such that $f(z_0) = \omega_0$ and $f(z_1) = \omega_1$.

For instance, all the linear equations (1.1) have the uniqueness property (see [5, Corollary 6.2.4]). In the nonlinear case, the equation (1.2) has the uniqueness property if

$$\limsup_{|z| \to \infty} k(z) < 3 - 2\sqrt{2}, \tag{1.3}$$

but uniqueness property may fail when (1.3) is not true, see [3, Theorem 1.1]. On the other hand, we have the uniqueness property, e.g., if $\mathcal{H}$ is 1-homogeneous in the second variable, see [5, Theorem 8.6.2] or [3, Theorem 1.3]. In the terms of the quasiconformal distortion, the bound (1.3) reads as $K(z) < \sqrt{2}$ near the infinity.

In the rest of the paper we only consider fields $\mathcal{H}$ such that the associated equation has the uniqueness property.

To every such field we can associate a unique family of quasiconformal mappings $\mathcal{F}_{\mathcal{H}} = \{\varphi_a\}_{a \in \mathbb{C}}$ by setting as φ_a, $a \neq 0$, the unique $W^{1,2}_{\mathrm{loc}}(\mathbb{C})$-homeomorphic solution to (1.2) such that

$$\varphi_a(0) = 0, \qquad \varphi_a(1) = a.$$

Often it is convenient to set $\varphi_0(z) \equiv 0$. It follows from the Lipschitz regularity of $\mathcal{H}$, (H1), that if f and g are two solutions to (1.2), then $f - g$ is K-quasiregular. It can also be shown that if the uniqueness property holds for (1.2), the difference $f - g$ is also injective, and thus quasiconformal, see Proposition 3.1. This motivates the following definition.

Definition 1.2. We call $\mathcal{F} = \{\varphi_a\}_{a \in \mathbb{C}} \subset W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ a family of quasiconformal mappings, if the following holds for some $1 \leqslant K < \infty$.

(F1) If $a \neq 0$, then φ_a is a K-quasiconformal mapping with $\varphi_a(0) = 0$ and $\varphi_a(1) = a$. If $a = 0$, $\varphi_0 \equiv 0$.

(F2) The difference $\varphi_a - \varphi_b$ is K-quasiconformal, for $a \neq b$.

If the family arises from a field $\mathcal{H}$, as before, we denote it by $\mathcal{F}_{\mathcal{H}}$.

In Section 2 we prove that if a family of quasiconformal mappings $\mathcal{F}$ is $C^1(a)$, i.e., C^1 in the parameter a, then the family is a C^1-embedded submanifold of the space $L^\infty_{\mathrm{loc}}(\mathbb{C})$, a surface with points consisting of quasiconformal mappings. Here we use the settings and definitions of submanifolds of Fréchet spaces from Lang [20].

The tangent plane, $T_a\mathcal{F}$, at a given point a is given by the homeomorphic solutions of an $\mathbb{R}$-linear Beltrami equation (see Proposition 2.6 and Remark 2.7). Moreover we prove that, if the family is induced by a nonlinear Beltrami equation characterized by the field $\mathcal{H}$, then $\mathcal{F}_\mathcal{H}$ is a submanifold of $W^{1,2}_{\mathrm{loc}}(\mathbb{C})$. In particular, the tangent bundle $T\mathcal{F}_\mathcal{H} := \bigcup_{a\neq 0} T_a\mathcal{F}_\mathcal{H}$ is given by solutions to the $\mathbb{R}$-linear Beltrami equation that is obtained by linearizing the starting equation $\mathcal{H}$.

Theorem 1.3. *Let field $\mathcal{H}(z,w) \in C^1(w)$ and $\mathcal{F}_\mathcal{H} = \{\varphi_a(z)\}_{a\in\mathbb{C}}$. Then for every $a \in \mathbb{C}\setminus\{0\}$ the directional derivatives*

$$\partial_e^a \varphi_a(z) := \lim_{t\to 0^+} \frac{\varphi_{a+te}(z) - \varphi_a(z)}{t}, \qquad e \in \mathbb{C},$$

are quasiconformal mappings of z, all satisfying the same $\mathbb{R}$-linear Beltrami equation

$$\partial_{\bar z} f(z) = \mu_a(z)\, \partial_z f(z) + \nu_a(z)\, \overline{\partial_z f(z)} \qquad a.e. \tag{1.4}$$

It turns out that

$$\mu_a(z) = \partial_w \mathcal{H}\big(z, \partial_z\varphi_a(z)\big) \quad \text{and} \quad \nu_a(z) = \partial_{\bar w} \mathcal{H}\big(z, \partial_z\varphi_a(z)\big), \tag{1.5}$$

and, moreover, $a \mapsto \varphi_a$ is continuously differentiable in $W^{1,2}_{\mathrm{loc}}(\mathbb{C})$.

Next, we investigate to what extent the relation between the field $\mathcal{H}$ and the family $\mathcal{F}_\mathcal{H}$ is unique as in the linear case. Given any family of quasiconformal mappings $\mathcal{F}$ as in Definition 1.2, we can formally associate to it a nonlinear Beltrami equation represented by a field $\mathcal{H}_\mathcal{F}$, simply by starting with the necessary condition

$$\mathcal{H}_\mathcal{F}(z,w) = \partial_{\bar z}\varphi_a(z) \quad \text{if } w = \partial_z\varphi_a(z), \tag{1.6}$$

and, for example, by using Kirzsbraun's extension theorem make the field global. Note that (1.6) gives well defined field by Definition 1.2, (F2). In general such field $\mathcal{H}_\mathcal{F}$ does not need to satisfy the conditions (H1) and (H2). However, starting from a smooth enough $\mathcal{H}$, the family determines the field.

Theorem 1.4. *Suppose that $\mathcal{H}$ is a regular field. Then $\mathcal{F}_\mathcal{H}$ defines $\mathcal{H}$ uniquely, that is, $\mathcal{H}_{\mathcal{F}_\mathcal{H}} = \mathcal{H}$.*

Here, *regular field* means that $\mathcal{H}(z,w) \in C^\alpha_{\mathrm{loc}}(z)$ uniformly in w and $D_w\mathcal{H}(z,w) \in (C^\alpha_{\mathrm{loc}}(z), C^\beta_{\mathrm{loc}}(w))$ locally uniformly in z and w (see Definition 4.12).

Notice that as (1.6) is a necessary requirement, in order to obtain a unique $\mathcal{H}$ the properties of $a \mapsto \partial_z\varphi_a(z) : \mathbb{C} \to \mathbb{C}$ play a fundamental role. Even in the linear case the fact that $a \mapsto \partial_z\varphi_a(z)$ is a bijection is highly nontrivial as it requires the recently proved Wronsky-type theorem, [1], [7]. An important part of the proof of Theorem 1.4 is to show the following theorem.

Theorem 1.5. *Suppose that $\mathcal{H}$ is a regular field and $\mathcal{F}_\mathcal{H} = \{\varphi_a(z)\}_{a\in\mathbb{C}}$. Then the map $a \mapsto \partial_z\varphi_a(z)$ is a homeomorphism on the Riemann sphere, for every fixed $z \in \mathbb{C}$.*

The proof is based on the fact that for families $\mathcal{F}_\mathcal{H}$, arising from regular fields $\mathcal{H}$, $(a, z) \mapsto D_z\varphi_a(z)$ is smooth with respect to both variables and does not degenerate. In the a-variable this is reflected in our Theorem 1.3. Concerning the behaviour in the z-variable, one can argue by perturbation to obtain Schauder-type estimates for nonlinear Beltrami equations. We prove the following theorem.

Theorem 1.6. *Let the field $\mathcal{H}(z, w) \in C^\alpha_{\mathrm{loc}}(z)$ uniformly in w. Then every quasiregular solution f to*

$$\partial_{\bar z} f(z) = \mathcal{H}(z, \partial_z f(z)) \qquad \text{for a.e. } z \in \Omega$$

belongs to $C^{1,\gamma}_{\mathrm{loc}}(\Omega)$, where $\gamma = \alpha$, if $\alpha < \frac{1}{K}$, and $\gamma < \frac{1}{K}$, if $\frac{1}{K} \leqslant \alpha < 1$. Here $K = \frac{1+k}{1-k}$. Moreover, we have a homogeneous norm estimate.

For regular fields, a combination of Theorem 1.3 and Theorem 1.6 shows that the tangent space $T_a\mathcal{F}_\mathcal{H}$ consists of solutions to precise $\mathbb{R}$-linear Beltrami equations whose coefficients (1.5) are Hölder continuous. Thus we can use classical Schauder estimates for linear equations and are entitled to freely change the order of differentiation. This enables us to transfer information from the z-variable to the a-variable and vice versa. We obtain as a consequence the non-degeneracy of $\mathcal{F}_\mathcal{H}$ at points $a \in \mathbb{C} \setminus \{0\}$.

Proposition 1.7. *Let $\mathcal{H}$ be a regular field and $\mathcal{F}_\mathcal{H} = \{\varphi_a(z)\}_{a\in\mathbb{C}}$. Then, for $a \neq 0$, $z \in \mathbb{C}$,*

$$\det[D_a\partial_z\varphi_a(z)] \neq 0,$$

and the determinant does not change sign.

The proof is based on the fact that, after changing the order of differentiation, we are in turn dealing with null Lagrangians $\mathrm{Im}(\partial_z f\, \overline{\partial_z g})$, for solutions f and g to the $\mathbb{R}$-linear Beltrami equation, and these are known not to vanish almost everywhere by the recent works [15], [9], [1], [7].

Now, we have a smooth locally injective mapping from $\mathbb{C} \to \mathbb{C}$, but still this does not imply non-degeneracy at $a = \infty$ for which we need to use the topology of the Riemann sphere. It turns out that, under the Hölder regularity of $\mathcal{H}$ on the z-variable, it is a corollary of the following result.

Theorem 1.8. *Let the field $\mathcal{H}(z, w) \in C^\alpha_{\mathrm{loc}}(z)$ uniformly in w. Then a homeomorphic solution $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ to the nonlinear Beltrami equation*

$$\partial_{\bar z} f(z) = \mathcal{H}(z, \partial_z f(z)) \qquad \text{for a.e. } z \in \mathbb{C}$$

has a positive Jacobian, $J(z, f) > 0$, with an explicit lower bound.

This theorem is well-known in the case when $\mathcal{H}$ is $\mathbb{C}$- or $\mathbb{R}$-linear but the proofs do not extend to the nonlinear case. We present a genuine nonlinear proof based on the analysis of the autonomous system $\partial_{\bar z} F(z) = \mathcal{H}(\partial_z F(z))$ and a perturbation argument.

Thus the family $\mathcal{F}_\mathcal{H}$ is *non-degenerate* in the sense of Definition 5.1 below. A topological argument (Lemma 5.2) shows that such for families $a \mapsto$

$\partial_z \varphi_a(z)$ is a homeomorphism and, furthermore, the family defines a unique $\mathcal{H} = \mathcal{H}_{\mathcal{F}}$. Moreover, regularities of $\mathcal{H}_{\mathcal{F}}$ depend on the regularity properties of $\mathcal{F}$.

Concerning the structure of the paper, in Section 2, we prove basic properties of quasiconformal families $\mathcal{F}$. In Section 3 we study families $\mathcal{F}_{\mathcal{H}}$ and, in particular, show Theorem 1.3 and that $\mathcal{F}_{\mathcal{H}}$ has a manifold structure modelled on $W^{1,2}_{\mathrm{loc}}(\mathbb{C})$. In Section 4, we study smooth fields $\mathcal{H}$ and prove the Schauder estimates (Theorem 1.6), non-vanishing of the Jacobian (Theorem 1.8), and obtain the smoothness of $\mathcal{F}_{\mathcal{H}}$. In Section 5, we show that $\mathcal{F}_{\mathcal{H}}$ is non-degenerate (e.g., Proposition 1.7) and give the topological argument which completes the proofs of Theorems 1.4 and 1.5. We finish the paper by showing how $\mathcal{H}_{\mathcal{F}}$ inherits the regularity of $\mathcal{F}$.

Theorem 1.9. *Let $\mathcal{F} = \{\varphi_a(z)\}_{a\in\mathbb{C}}$ be a non-degenerate family of K-quasiconformal mappings such that for some $s \in (0,1)$,*

$$\|D_z\varphi_a\|_{C^s(\mathbb{D}_R)} \leqslant c(s,R)\,|a|, \qquad R<\infty, \tag{1.7}$$

and $D_aD_z\mathcal{F} \in (C(z), C(a))$ locally uniformly in z and a. Then there is a unique nonlinear Beltrami equation (1.2) *such that every mapping of the family φ_a is a solution to* (1.2)*. The field $\mathcal{H} = \mathcal{H}_{\mathcal{F}}$ satisfies (H1), (H2), $\mathcal{H}(z,w) \in C^s_{\mathrm{loc}}(z)$ locally uniformly in w, and $D_w\mathcal{H}(z,w) \in (C(z), C(w))$ locally uniformly in z and w.*

At the end of the paper we point out some clear obstructions to naive generalizations of our results. However, the line of research seems very promising in two directions. First, it would be clear to investigate the case of less regular fields $\mathcal{H}$. Secondly, the geometric properties of the manifold $\mathcal{F}_{\mathcal{H}}$ will depend of the structure of the field $\mathcal{H}$ in a non-obvious way.

Notation. We will use the usual function spaces of continuous functions, Hölder continuous functions, Sobolev spaces and so on. If the codomain is the complex plane $\mathbb{C}$, we do not state it in the Sobolev or L^p-spaces, e.g., $W^{1,2}_{\mathrm{loc}}(\mathbb{C},\mathbb{C}) = W^{1,2}_{\mathrm{loc}}(\mathbb{C})$. Recall that $L^\infty_{\mathrm{loc}}(\mathbb{C})$ and $W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ are Fréchet spaces, with seminorms $\|\cdot\|_{L^\infty(\mathbb{D}(0,R))}$, $\|\cdot\|_{W^{1,2}(\mathbb{D}(0,R))}$, $R = 1,2,\ldots$, respectively.

Let X, Y be two function spaces. When we say that a function $g(z,w) : \Omega\times\Omega' \to \mathbb{C}$ belongs to $(X(z), Y(w))$, we mean that the map $w \mapsto g(z,w) \in Y$ and also that $z \mapsto g(z,w) \in X$. If we say that $g(z,w) : \Omega\times\Omega' \to \mathbb{C}$ belongs to $(X(z), Y(w))$ locally uniformly in (z,w), $g \in L^\infty_{\mathrm{loc},w}(\mathbb{C}, X(z)) \cap L^\infty_{\mathrm{loc},z}(\mathbb{C}, Y(w))$.

Similarly, if we say $\mathcal{F} = \{\varphi_a(z)\}_{a\in\mathbb{C}}$ belongs to $X(z)$ $(Y(a))$ we mean that $\varphi_a(z)$ belongs to $X(z)$ for every a (it belongs to $Y(a)$ for every z, respectively). If the domain and codomain are clear from the context, we do not state them in $(X(z), Y(w))$-notation.

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2. Manifolds

In this section, $\mathcal{F} = \{\varphi_a(z)\}_{a\in\mathbb{C}}$ will always be a family of K-quasiconformal mappings, satisfying (F1) and (F2). Let $\eta_K : \mathbb{R}_+ \to \mathbb{R}_+$ denote the modulus quasisymmetry of quasiconformal maps [5, Corollary 3.10.4]; we can choose $\eta_K(t) = C(K)\max\{t^K, t^{\frac{1}{K}}\}$. Uniform quasisymmetry of the differences implies that the family is bi-Lipschitz respect to the parameter a.

Proposition 2.1. *Given a family $\mathcal{F} = \{\varphi_a(z)\}_{a\in\mathbb{C}}$ $2 \leqslant p < \frac{2K}{K-1}, R > 0$ it holds that*

(a) $a \mapsto \varphi_a(z)$ is bi-Lipschitz, for every $z \in \mathbb{C}$. In fact,

$$\frac{1}{\eta_K(1/|z|)} \leqslant \frac{|\varphi_a(z) - \varphi_b(z)|}{|a-b|} \leqslant \eta_K(|z|), \quad a \neq b.$$

(b) $a \mapsto \varphi_a : \mathbb{C} \to L^\infty(\mathbb{D}(0,R))$ is bi-Lipschitz

$$\frac{1}{\eta_K(1/R)} \leqslant \frac{\|\varphi_a - \varphi_b\|_{L^\infty(\mathbb{D}(0,R))}}{|a-b|} \leqslant \eta_K(R), \quad a \neq b.$$

(c) $a \mapsto D_z\varphi_a : \mathbb{C} \to L^p(\mathbb{D}(0,R))$ is bi-Lipschitz and

$$\frac{c(p,K)}{\eta_K(1/R)\, R^{1-\frac{2}{p}}} \leqslant \frac{\|D_z\varphi_a - D_z\varphi_b\|_{L^p(\mathbb{D}(0,R))}}{|a-b|} \leqslant \frac{c(p,K)\,\eta_K(R)}{R^{1-\frac{2}{p}}}, \quad a \neq b.$$

Above $\eta_K(t) = \lambda(K)^{2K}\max\{t^K, t^{\frac{1}{K}}\}$, $t \in [0,\infty)$.

Proof. By assumption (F2), $g = \varphi_a - \varphi_b : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal, $a \neq b$. Then, by η_K-quasisymmetry,

$$\begin{aligned}\frac{1}{\eta_K(1/|z|)}|a-b| &= \frac{1}{\eta_K(1/|z|)}|g(1)-g(0)| \leqslant |g(z)-g(0)| = |\varphi_a(z) - \varphi_b(z)| \\ &\leqslant \eta_K(|z|)|g(1)-g(0)| = \eta_K(|z|)|a-b|.\end{aligned}$$

Hence we have shown (a). The statement (b) is immediate.

Next, we prove that the bi-Lipschitz property is inherited to the L^p-norm of the derivatives.

By the η_K-quasisymmetry or straight from Proposition 2.1 (a),

$$\begin{aligned}\frac{\pi}{\eta_K(1/R)^2}|a-b| &\leqslant \pi \inf_{|z|=R} |(\varphi_a - \varphi_b)(z)|^2 \leqslant |(\varphi_a - \varphi_b)(\mathbb{D}(0,R))| \\ &\leqslant \pi \sup_{|z|=R} |(\varphi_a - \varphi_b)(z)|^2 \leqslant \pi\, \eta_K(R)^2|a-b|.\end{aligned}$$

For K-quasiconformal map $\varphi_a - \varphi_b$ the following holds

$$|(\varphi_a - \varphi_b)(\mathbb{D}(0,R))| = \int_{\mathbb{D}(0,R)} J(z, \varphi_a - \varphi_b)\, dA(z),$$

$$\begin{aligned}\frac{1}{K}\int_{\mathbb{D}(0,R)} |D_z(\varphi_a(z) - \varphi_b(z))|^2\, dA(z) &\leqslant \int_{\mathbb{D}(0,R)} J(z, \varphi_a - \varphi_b)\, dA(z) \\ &\leqslant \int_{\mathbb{D}(0,R)} |D_z(\varphi_a(z) - \varphi_b(z))|^2\, dA(z),\end{aligned}$$

and, for any $2 < p < \frac{2K}{K-1}$,

$$\left(\frac{1}{|\mathbb{D}(0,R)|} \int_{\mathbb{D}(0,R)} |D_z(\varphi_a - \varphi_b)|^2 \right)^{\frac{1}{2}} \simeq \left(\frac{1}{|\mathbb{D}(0,R)|} \int_{\mathbb{D}(0,R)} |D_z(\varphi_a - \varphi_b)|^p \right)^{\frac{1}{p}}$$

with constants that depend only on K and p.

Combining the estimates one gets (c). □

In the rest of the section we prove that, if the quasiconformal family $\mathcal{F}$ is C^1-continuous with respect to the family parameter a, then $\mathcal{F}$ is a C^1-embedded submanifold of $L^\infty_{\text{loc}}(\mathbb{C})$. Moreover, we have that the tangent space $T_a\mathcal{F}$ is a two-dimensional family of quasiconformal mappings. Therefore, they span the two-dimensional family of homeomorphic solutions to a linear Beltrami equation

$$\partial_{\bar{z}} f(z) = \mu_a(z)\, \partial_z f(z) + \nu_a(z)\, \overline{\partial_z f(z)} \qquad \text{a.e.}$$

Proposition 2.2. *Assume that $\mathcal{F} = \{\varphi_a(z)\}_{a\in\mathbb{C}}$ is a $C^1(a)$-family of K-quasiconformal mappings. Let $a \in \mathbb{C}$ be fixed. Then there exist $\mu_a, \nu_a \in L^\infty(\mathbb{C})$, $\||\mu_a| + |\nu_a|\|_\infty \leqslant \frac{K-1}{K+1}$, such that for every $e \in \mathbb{C}$ the directional derivative*

$$\partial^a_e \varphi_a = \lim_{t\to 0} \frac{\varphi_{a+te} - \varphi_a}{t}$$

is the unique K-quasiconformal solution to the problem

$$\begin{cases} \partial_{\bar{z}} f(z) = \mu_a(z)\, \partial_z f(z) + \nu_a(z)\, \overline{\partial_z f(z)} & a.e. \\ f(0) = 0, \quad f(1) = e. \end{cases}$$

Proof. By assumption, for every $z \in \mathbb{C}$

$$\lim_{|h|\to 0^+} \left| \frac{\varphi_{a+h}(z) - \varphi_a(z) - D_a\varphi_a(z)\, h}{h} \right| = 0.$$

Observe that without the assumption $\mathcal{F} \in C^1(a)$, $a \mapsto \varphi_a(z)$ is still differentiable almost everywhere as a Lipschitz map (Proposition 2.1 (a)), but the exceptional set might depend on the point z.

In particular, we have directional derivatives and so the pointwise limit

$$\partial^a_e \varphi_a(z) = \lim_{t\to 0^+} \frac{\varphi_{a+te}(z) - \varphi_a(z)}{t}$$

exists and defines a continuous function in a for every z. Notice that each $\eta^e_t = \frac{\varphi_{a+te}-\varphi_a}{t}$ is a K-quasiconformal map $\mathbb{C} \to \mathbb{C}$ that sends 0 to itself and 1 to e. Therefore, the limit mapping $\partial^a_e\varphi_a(z)$ is also a K-quasiconformal mapping sending 0 to itself and 1 to e, by the Montel-type theorem [5, Theorem 3.9.4]. Thus $\partial^a_e\varphi_a(z)$ is continuous in z and the limit defining it can be taken locally uniformly in z, i.e.,

$$\lim_{t\to 0^+} \left\| \frac{\varphi_{a+te} - \varphi_a}{t} - \partial^a_e \varphi_a \right\|_{L^\infty(\mathbb{D}(0,R))} = 0, \tag{2.1}$$

for every $R = 1, 2, \ldots$. Hence $h \mapsto \partial^a_e\varphi_a(z)\, h$ is the directional derivative of

$$a \mapsto \varphi_a : \mathbb{C} \to L^\infty_{\text{loc}}(\mathbb{C}).$$

Note now that the assumed differentiability of $a \mapsto \varphi_a(z)$ allows us to recover $\partial_e^a \varphi_a(z)$ from the full differential,

$$\partial_e^a \varphi_a(z) = D_a \varphi_a(z)\, e = \partial_a \varphi_a(z)\, e + \partial_{\bar a} \varphi_a(z)\, \bar e. \tag{2.2}$$

Thus $\partial_e^a \varphi_a$ depends *linearly* on e. Moreover, if $e_1, e_2 \in \mathbb{C}$ are $\mathbb{R}$-linearly independent then also $\partial_{e_1}^a \varphi_a, \partial_{e_2}^a \varphi_a$ are $\mathbb{R}$-linearly independent. Thus $\{\partial_a^e \varphi_a\}_{e\in\mathbb{C}}$ is a 2-dimensional linear family of K-quasiconformal maps, and so by [5, Theorem 16.6.6] there exists a unique pair of Beltrami coefficients μ_a and ν_a such that $\||\mu_a| + |\nu_a|\|_\infty \leqslant \frac{K-1}{K+1}$ and the equation

$$\partial_{\bar z} f(z) = \mu_a(z)\, \partial_z f(z) + \nu_a(z)\, \overline{\partial_z f(z)} \qquad \text{a.e.}$$

is satisfied by any member of the family $\{\partial_e^a \varphi_a\}_{e\in\mathbb{C}}$. After choosing as generators $e_1 = 1, e_2 = i$, the coefficients μ_a, ν_a may be precisely described as

$$\begin{aligned} \mu_a(z) &= \frac{\partial_{\bar z}(\partial_1^a \varphi_a)\, \overline{\partial_z(\partial_i^a \varphi_a)} - \partial_{\bar z}(\partial_i^a \varphi_a)\, \overline{\partial_z(\partial_1^a \varphi_a)}}{2i\ \mathrm{Im}(\partial_z(\partial_1^a \varphi_a)\, \overline{\partial_z(\partial_i^a \varphi_a)})}, \\ \nu_a(z) &= \frac{\partial_z(\partial_1^a \varphi_a)\, \partial_{\bar z}(\partial_i^a \varphi_a) - \partial_z(\partial_i^a \varphi_a)\, \partial_{\bar z}(\partial_1^a \varphi_a)}{2i\ \mathrm{Im}(\partial_z(\partial_1^a \varphi_a)\, \overline{\partial_z(\partial_i^a \varphi_a)})}. \end{aligned} \tag{2.3}$$

□

Corollary 2.3. *Let $\mathcal{F} = \{\varphi_a(z)\}_{a\in\mathbb{C}}$ be a $C^1(a)$-family of K-quasiconformal mappings. Then*

$$\|D_z D_a \varphi_a\|_{L^2(\mathbb{D}(0,R))} \leqslant c(K,R).$$

Proof. By Proposition 2.2, $\partial_e^a \varphi_a$ is K-quasiconformal map sending 0 to 0 and 1 to e. Moreover, $\{\partial_e^a \varphi_a\}_{e\in\mathbb{C}}$ solve the same $\mathbb{R}$-linear Beltrami equation. Thus $\{\partial_e^a \varphi_a\}_{e\in\mathbb{C}}$ is an $\mathbb{R}$-linear family of K-quasiconformal maps. Hence, by Proposition 2.1 (c),

$$\|D_z(\partial_e^a \varphi_a)\|_{L^2(\mathbb{D}(0,R))} \leqslant c(K,R)\, |e|.$$

Thus

$$\|D_z D_a \varphi_a\|_{L^2(\mathbb{D}(0,R)} \leqslant \max_{|e|=1} \|D_z(\partial_e^a \varphi_a)\|_{L^2(\mathbb{D}(0,R))} \leqslant c(K,R).$$

□

We will repeatedly use the following interpolation estimates.

Theorem 2.4 (Interpolation)**.** *Let $\mathbb{D}_r \subset \mathbb{C}$ be a disk, and $f \in L^p(\mathbb{D}_{2r}) \cap C^\gamma(\mathbb{D}_{2r})$ for some $1 < p < \infty$, $0 < \gamma < 1$. Given any $\theta \in \left(\frac{2}{2+\gamma p}, 1\right]$, set*

$$s = \theta\left(\gamma + \frac{2}{p}\right) - \frac{2}{p}. \tag{2.4}$$

Then

$$\|f\|_{L^\infty(\mathbb{D}_r)} \leqslant \|f\|_{C^s(\mathbb{D}_r)} \leqslant c(\theta, \mathbb{D}_{2r})\, \|f\|_{L^p(\mathbb{D}_{2r})}^{1-\theta}\, \|f\|_{C^\gamma(\mathbb{D}_{2r})}^{\theta}.$$

Proof. Notice that $C^\gamma = B^\gamma_{\infty\infty}$, $L^p \subset B^0_{p,\infty}$, and, by the embedding theorem [8, Theorem 6.5.1], there is a continuous inclusion $B^{\theta\gamma}_{\frac{p}{1-\theta},\infty} \subset B^s_{\infty,\infty}$, where

$B^{\cdot}_{\cdot,\cdot}$ denotes the Besov space. In particular, if $\frac{2}{2+\gamma p} < \theta < 1$, then $0 < s < 1$ and $B^s_{\infty,\infty} = C^s$. By [8, Theorem 6.4.5],

$$(B^0_{p,\infty}, B^\gamma_{\infty,\infty})_{[\theta]} = B^{\theta\gamma}_{\frac{p}{1-\theta},\infty}, \qquad 0 \leqslant \theta \leqslant 1.$$

Thus, for $f \in L^p(\mathbb{C}) \cap C^\gamma(\mathbb{C})$,

$$\|f\|_{L^\infty(\mathbb{C})} \leqslant \|f\|_{C^s(\mathbb{C})} \leqslant c(\theta)\, \|f\|^{1-\theta}_{L^p(\mathbb{C})}\, \|f\|^\theta_{C^\gamma(\mathbb{C})}.$$

The statement follows when we apply the previous estimate to ξf with a suitable cutoff function ξ. ☐

Corollary 2.5. *Let $\mathcal{F} = \{\varphi_a(z)\}_{a\in\mathbb{C}}$ be a $C^1(a)$-family of K-quasiconformal mappings. Then*

$$a \mapsto \varphi_a \in C^1(\mathbb{C}, L^\infty_{\text{loc}}(\mathbb{C})).$$

Proof. By (2.1), we know that $a \mapsto \varphi_a : \mathbb{C} \to L^\infty_{\text{loc}}(\mathbb{C})$ has partial derivatives $\partial^a_e \varphi_a$. We will show that for $\mathcal{T}_a\, h := \partial^a_1 \varphi_a(z)\, h_1 + \partial^a_i \varphi_a(z)\, ih_2$, $h = h_1 + ih_2$,

$$\lim_{|h|\to 0} \frac{\|\varphi_{a+h} - \varphi_a - \mathcal{T}_a\, h\|_{L^\infty(\mathbb{D}(0,R))}}{|h|} = 0, \tag{2.5}$$

that is, $a \mapsto \varphi_a : C \to L^\infty_{\text{loc}}(\mathbb{C})$ is differentiable. Further, we will prove that $a \mapsto \mathcal{T}_a : \mathbb{C} \to L(\mathbb{C}, L^\infty_{\text{loc}}(\mathbb{C}))$ is continuous and thus our claim will be proved.

We start by showing ($e = 1, i$)

$$\lim_{b\to a} \|\partial^a_e \varphi_a - \partial^a_e \varphi_b\|_{L^\infty(\mathbb{D}(0,R))} = 0. \tag{2.6}$$

Note that as quasiconformal maps (by Proposition 2.2) $\partial^a_e \varphi_a$ and $\partial^a_e \varphi_b$ ($e = 1, i$) are locally $\frac{1}{K}$-Hölder continuous and we get by interpolation (Theorem 2.4)

$$\begin{aligned} &\|\partial^a_e \varphi_a - \partial^a_e \varphi_b\|_{L^\infty(\mathbb{D}(0,R))} \\ &\quad \leqslant \|\partial^a_e \varphi_a - \partial^a_e \varphi_b\|^{1-\theta}_{C^{1/K}(\mathbb{D}(0,2R))} \|\partial^a_e \varphi_a - \partial^a_e \varphi_b\|^\theta_{L^2(\mathbb{D}(0,2R))}. \end{aligned} \tag{2.7}$$

Now, there exists $c(K,R)$ such that

$$\|\partial^a_e \varphi_a - \partial^a_e \varphi_b\|_{C^{1/K}(\mathbb{D}(0,2R))} \leq c(K,R)|e| \tag{2.8}$$

Indeed, by [5, Corollary 3.10.3], the seminorm

$$[\partial^a_e \varphi_a]_{C^{1/K}(\mathbb{D}(0,2R))} \leqslant c(K,R) \|D_z(\partial^a_e \varphi_a)\|_{L^2(\mathbb{D}(0,2R))} \leqslant c(K,R)|e|,$$

where the last estimate comes form Corollary 2.3. On the other hand,

$$\|\partial^a_e \varphi_a\|_{L^\infty(\mathbb{D}(0,2R))} \leqslant c(K,R)|e|,$$

by Proposition 2.1 (b) since $\{\partial^a_e \varphi_a\}_{e\in\mathbb{C}}, \{\partial^a_e \varphi_b\}_{e\in\mathbb{C}}$ are $\mathbb{R}$ linear families of quasiconformal mappings.

Next, $\partial^a_e \varphi_b(z) \to \partial^a_e \varphi_a(z)$ pointwise for every $z \in \mathbb{C}$, by assumption $\mathcal{F} \in C^1(a)$. Moreover, $\partial^a_e \varphi_a$ and $\partial^a_e \varphi_b$ have a uniform L^2-bound $c(K,R)|e_2|$, by quasiconformality, e.g., Proposition 2.1 (b). Hence the dominated convergence theorem gives

$$\lim_{b\to a} \|\partial^a_e \varphi_a - \partial^a_e \varphi_b\|_{L^2(\mathbb{D}(0,2R))} = 0. \tag{2.9}$$

Thus plugging (2.8) and (2.9) to (2.7), (2.6) follows.

The continuity of $a \mapsto \mathcal{T}_a : \mathbb{C} \to L(\mathbb{C}, L^\infty_{\text{loc}}(\mathbb{C}))$ is immediate.

We are left to show (2.5). Now, $h = h_1 + ih_2$,

$$\begin{aligned}\frac{\|\varphi_{a+h} - \varphi_a - \mathcal{T}_a\, h\|_{L^\infty(\mathbb{D}(0,R))}}{|h|} &\\ \leqslant \frac{\|\varphi_{a+h_1+ih_2} - \varphi_{a+ih_2} - \partial_1^a\varphi_{a+ih_2}\, h_1\|_{L^\infty(\mathbb{D}(0,R))}}{|h_1|} &\\ + \frac{\|\partial_1^a\varphi_{a+ih_2}\, h_1 - \partial_1^a\varphi_a\, h_1\|_{L^\infty(\mathbb{D}(0,R))}}{|h_1|} &\\ + \frac{\|\varphi_{a+ih_2} - \varphi_a - \partial_i^a\varphi_a\, ih_2\|_{L^\infty(\mathbb{D}(0,R))}}{|h_2|}.\end{aligned}$$

When $|h| \to 0$, the first and third term on the right hand side converge to 0, by (2.1), and the second term goes to 0, by (2.6). □

Proposition 2.6. *Let $\mathcal{F} = \{\varphi_a(z)\}_{a\in\mathbb{C}}$ be a $C^1(a)$-family of quasiconformal mappings. Then $\mathcal{F}$ is a C^1-embedded submanifold of $L^\infty_{\text{loc}}(\mathbb{C})$.*

Proof. By Corollary 2.5, $a \mapsto \varphi_a \in C^1(\mathbb{C}, L^\infty_{\text{loc}}(\mathbb{C}))$.

We follow the setup and definitions of submanifolds of Fréchet dimensional manifolds from [20, Chapter II]. According to this, we need to show that our C^1-map $a \mapsto \varphi_a$ is a topological embedding and an immersion. In fact, the map is a topological embedding, i.e., a homeomorphism onto its image by Proposition 2.1 (a).

As for the immersion it is proved in [20, Proposition 2.3, p. 29] that it is enough that the differential $D_a\varphi_a$ is injective and it splits (see [20, p. 18] for a definition of splitting in this setting).

First, we show that the differential $D_a\varphi_a : T_a\mathbb{C} \to T_{\varphi_a}L^\infty_{\text{loc}}(\mathbb{C})$ is injective. Now, $D_a\varphi_a\, e = \partial_e^a\varphi_a$, by (2.2), and it follows from Proposition 2.2 that $\partial_e^a\varphi_a$ is a quasiconformal mapping sending $0 \mapsto 0$ and $1 \mapsto e$. Hence the kernel of $D_a\varphi_a$ is $\{0\}$, which shows the injectivity.

Now, since $T_a\mathcal{F}$ is finite dimensional (two-dimensional, by Proposition 2.2) and $L^\infty_{\text{loc}}(\mathbb{C})$ is a Fréchet space, Hahn-Banach theorem yields a projection $P : L^\infty_{\text{loc}}(\mathbb{C}) \to T_a\mathcal{F}$ and thus $L^\infty_{\text{loc}}(\mathbb{C})$ is isomorphic to $\big(T_a\mathcal{F}, (\mathrm{Id} - P)(L^\infty_{\text{loc}}(\mathbb{C}))\big)$. Hence $L^\infty_{\text{loc}}(\mathbb{C})$ splits and the proof is concluded. □

Remark 2.7. Combining Proposition 2.2 with Proposition 2.6, we see that the tangent space $T_a\mathcal{F}$ is a two-dimensional family of quasiconformal mappings and they span the two-dimensional family of homeomorphic solutions to a linear Beltrami equation

$$\partial_{\bar z} f(z) = \mu_a(z)\,\partial_z f(z) + \nu_a(z)\,\overline{\partial_z f(z)} \qquad \text{a.e.}$$

3. $\mathcal{H}$-equations

Proposition 3.1. *The Beltrami equation* (1.2) *has the uniqueness property if and only if for every two quasiconformal solutions f and g the difference $f - g$ is either quasiconformal or constant.*

Proof. Suppose that (1.2) has the uniqueness property. Let f and g be homeomorphic solutions to (1.2) and $\omega_0 = (f - g)(z_0)$ for fixed $z_0 \in \mathbb{C}$. Then function $\tilde f = f - \omega_0$ is also a solution to (1.2) and $\tilde f(z_0) = g(z_0)$. By the uniqueness property, if $\tilde f \neq g$, then $\tilde f(z_1) - g(z_1) \neq 0$ for all $z_1 \in \mathbb{C} \setminus \{z_0\}$, that is, $f(z_1) - g(z_1) \neq \omega_0$. Thus ω_0 has a unique pre-image under $f - g$. Since ω_0 is arbitrary (and $f - g$ is a surjection by quasiregularity), it follows that either $f - g$ is a homeomorphism or $\tilde{f} \equiv g$, i.e., $f - g = \omega_0$.

Suppose now that we have quasiconformal solutions f and g to (1.2) such that $f(z_0) = g(z_0)$, $f(z_1) = g(z_1)$, and $f - g$ is quasiconformal or constant. Then $(f - g)(z_0) = (f - g)(z_1) = 0$. Thus $f - g$ is not quasiconformal and it is equal to 0. □

Remark 3.2. The uniqueness property says that no two points are in a special role, unlike in the uniqueness of the normalized homeomorphic solution (mapping 0 to 0 and 1 to 1). The uniqueness property is equivalent with the fact that the uniqueness of the normalized solution to the Beltrami equation (1.2) is preserved under the pre- and the post-composition with similarities. Actually, it follows from the counterexample in [3] that the uniqueness of the normalized solution is a truly weaker feature than the uniqueness property.

In the following results, we study the smoothness of $a \mapsto \varphi_a$ when $\mathcal{F}_{\mathcal{H}} = \{\varphi_a\}_{a\in\mathbb{C}}$ arises from a field $\mathcal{H}$ that is continuously differentiable in the gradient variable. We start with the existence of directional derivatives.

Theorem 3.3 (Theorem 1.3)**.** *Let the field $\mathcal{H}(z, w) \in C^1(w)$ and $\mathcal{F}_{\mathcal{H}} = \{\varphi_a(z)\}_{a\in\mathbb{C}}$. Then the directional derivatives*

$$\partial_e^a \varphi_a(z) := \lim_{t\to 0^+} \frac{\varphi_{a+te}(z) - \varphi_a(z)}{t}, \qquad e \in \mathbb{C},$$

exist, and define quasiconformal mappings of z, all satisfying the same $\mathbb{R}$-linear Beltrami equation

$$\partial_{\bar{z}} f(z) = \mu_a(z)\, \partial_z f(z) + \nu_a(z)\, \overline{\partial_z f(z)} \qquad a.e. \tag{3.1}$$

where

$$\mu_a(z) = \partial_w \mathcal{H}\big(z, \partial_z \varphi_a(z)\big) \qquad \text{and} \qquad \nu_a(z) = \partial_{\bar{w}} \mathcal{H}\big(z, \partial_z \varphi_a(z)\big).$$

Proof. Step 1: Converging subsequence. Let us fix $a \in \mathbb{C}$, and a direction $e \in \mathbb{C}$, and denote

$$\eta_t^e = \frac{\varphi_{a+te} - \varphi_a}{t}, \qquad t \in (0, \infty), \quad e \in \mathbb{C}.$$

Since $\mathcal{H}$ has the uniqueness property, the mappings η_t^e are K-quasiconformal in $\mathbb{C}$, and they map 0 to 0 and 1 to e. Thus $\{\eta_t^e : t \in (0, \infty)\}$ is a normal family, because

$$\{f : \mathbb{C} \to \mathbb{C} : \ f \text{ is } K\text{-quasiconformal}, \ f(0) = 0, \ f(1) = 1\},$$

is compact, by quasisymmetry (see the Montel-type theorem [5, Theorem 3.9.4]). As a consequence, the limit

$$\eta^e = \lim_{t_j \to 0^+} \eta_{t_j}^e$$

exists, at least for a subsequence t_j, and it is a K-quasiconformal homeomorphism mapping 0 to itself and 1 to e. Moreover, the limit is taken locally uniformly in z, so that also the derivatives converge weakly in $L^p_{\mathrm{loc}}(\mathbb{C})$, $p \in \left[2, \frac{2K}{K-1}\right)$. We will show that η^e solves an $\mathbb{R}$-linear Beltrami equation with coefficients given by μ_a and ν_a, and deduce that for any subsequence t_j the limit mapping η^e must be the same.

Step 2: Unique limit. We fix the converging subsequence t_j. As quasiconformal mappings, φ_a and $\eta^e_{t_j}$ are differentiable almost everywhere. Since t_j is countable, we have a set of full measure E such that the derivatives of φ_a and $\eta^e_{t_j}$ exist at any point $z \in E$ and are nonzero (let us remind that quasiconformal mappings have a non-vanishing ∂_z-derivative almost everywhere).

Now, fix one such point $z \in E$. By assumption, $w \mapsto \mathcal{H}(z, w)$ is C^1, so the complex partial derivatives $\partial_w \mathcal{H}(z, \cdot)$, $\partial_{\bar{w}} \mathcal{H}(z, \cdot)$ are defined at the point $w_0 = \partial_z \varphi_a(z)$. Hence we can write

$$
\begin{aligned}
\partial_{\bar z} \eta^e_{t_j}(z) &= \frac{\mathcal{H}\big(z, \partial_z \varphi_{a+t_j e}(z)\big) - \mathcal{H}\big(z, \partial_z \varphi_a(z)\big)}{t_j} \\
&= \frac{\mathcal{H}\big(z, w_0 + t_j\, \partial_z \eta^e_{t_j}(z)\big) - \mathcal{H}\big(z, w_0\big)}{t_j} \\
&= \frac{\partial_w \mathcal{H}\big(z, w_0\big)\, t_j\, \partial_z \eta^e_{t_j}(z) + \partial_{\bar w} \mathcal{H}\big(z, w_0\big)\, \overline{t_j\, \partial_z \eta^e_{t_j}(z)}}{t_j} \\
&\qquad + \frac{\mathcal{H}\big(z, w_0 + t_j\, \partial_z \eta^e_{t_j}(z)\big) - \mathcal{H}\big(z, w_0\big)}{t_j} \\
&\qquad - \frac{\partial_w \mathcal{H}\big(z, w_0\big)\, t_j\, \partial_z \eta^e_{t_j}(z) + \partial_{\bar w} \mathcal{H}\big(z, w_0\big)\, \overline{t_j\, \partial_z \eta^e_{t_j}(z)}}{t_j} \\
&= \mu_a(z)\, \partial_z \eta^e_{t_j}(z) + \nu_a(z)\, \overline{\partial_z \eta^e_{t_j}(z)} + h_{t_j}(z),
\end{aligned} \tag{3.2}
$$

where

$$
\begin{aligned}
\mu_a(z) &= \partial_w \mathcal{H}\big(z, w_0\big) = \partial_w \mathcal{H}\big(z, \partial_z \varphi_a(z)\big), \\
\nu_a(z) &= \partial_{\bar w} \mathcal{H}\big(z, w_0\big) = \partial_{\bar w} \mathcal{H}\big(z, \partial_z \varphi_a(z)\big),
\end{aligned} \tag{3.3}
$$

and

$$
\begin{aligned}
h_{t_j}(z) = &\frac{\mathcal{H}\big(z, w_0 + t_j\, \partial_z \eta^e_{t_j}(z)\big) - \mathcal{H}\big(z, w_0\big)}{t_j} \\
&- \frac{\partial_w \mathcal{H}\big(z, w_0\big)\, t_j\, \partial_z \eta^e_{t_j}(z) + \partial_{\bar w} \mathcal{H}\big(z, w_0\big)\, \overline{t_j\, \partial_z \eta^e_{t_j}(z)}}{t_j}.
\end{aligned}
$$

Let us remind that $w_0 = \partial_z \varphi_a(z)$, hence $t_j\, h_{t_j}(z)$ is precisely the difference between $w \mapsto \mathcal{H}(z, w)$ and its first degree Taylor polynomial at the point w_0.

The coefficients μ_a and ν_a define a genuine Beltrami equation, since they have an ellipticity bound and they are measurable. Indeed, since $w \mapsto \mathcal{H}(z, w)$ is $k(z)$-Lipschitz by (H1), we get

$$
|\mu_a(z)| + |\nu_a(z)| \leqslant |\partial_w \mathcal{H}(z, w_0)| + |\partial_{\bar w} \mathcal{H}(z, w_0)| = |D_w \mathcal{H}(z, w_0)| \leqslant k(z).
$$

Measurability follows as $z \mapsto D_w \mathcal{H}(z, w)$ is measurable as a limit of measurable functions, and $w \mapsto D_w \mathcal{H}(z, w)$ is continuous by assumption. Thus $(z, w) \mapsto D_w \mathcal{H}(z, w)$ is jointly measurable as a Carathéodory function. Since $w_0 = \partial_z \varphi_a(z)$ is a measurable function of z, the measurability of μ_a and ν_a follows.

Now, η^e solves the $\mathbb{R}$-linear Beltrami equation (3.1) defined by μ_a and ν_a, if for every compactly supported test function $\xi \in C_0^\infty(\mathbb{C})$

$$\int_{\mathbb{C}} \xi(z) \left(\partial_{\bar z}\eta^e(z) - \mu_a(z)\,\partial_z\eta^e(z) - \nu_a(z)\,\overline{\partial_z\eta^e(z)}\right) dA(z) = 0. \tag{3.4}$$

We have, by equation (3.2),

$$\begin{aligned}
&\left|\int_{\mathbb{C}} \xi(z)\ \left(\partial_{\bar z}\eta^e(z) - \mu_a(z)\,\partial_z\eta^e(z) - \nu_a(z)\,\overline{\partial_z\eta^e(z)}\right)\ dA(z)\right| \\
&\quad\leqslant \left|\int_{\mathbb{C}} \xi(z)\left(\partial_{\bar z} - \mu_a(z)\partial_z - \nu_a(z)\,\overline{\partial_z}\right)\left(\eta^e(z) - \eta^e_{t_j}(z)\right) dA(z)\right| \\
&\qquad + \left|\int_{\mathbb{C}} \xi(z)\, h_{t_j}(z)\, dA(z)\right|.
\end{aligned}$$

Since $\eta^e_{t_j} \to \eta^e$ locally uniformly, $D_z\eta^e_{t_j} \to D_z\eta^e$ distributionally. Moreover, by Proposition 2.1 (c), it follows that

$$\|D_z\eta^e_{t_j}\|_{L^p(\mathbb{D}(0,R))} \leqslant c(p,K,R) \tag{3.5}$$

and hence the convergence of $D_z\eta^e_{t_j}$ is weak in $L^p(\mathbb{D}(0,R))$. Let $R > 0$ be such that $\operatorname{supp}(\xi) \subset \mathbb{D}(0,R)$. Since both $\xi\,\mu_a$ and $\xi\,\nu_a$ belong to $L^\infty(\mathbb{C})$, they are suitable test functions for the weak $L^p(\mathbb{D}(0,R))$-convergence. Thus the first term in (3.4) converges to 0 and we are left to show that $h_{t_j} \to 0$ distributionally.

We proceed as follows. Set

$$B(R) = \bigcup_{l=1}^{\infty}\left(\bigcap_{k=1}^{\infty}\bigcup_{j=k}^{\infty}\left\{z \in \mathbb{D}(0,R) : t_j\,|\partial_z\eta^e_{t_j}(z)| > 2^{-l}\right\}\right).$$

We choose a sparse enough subsequence of t_j (let us denote it t_j, too) such that $\lim_{k\to\infty}\sum_{j\geqslant k} t_j^p = 0$. Now, for any fixed $l = 1,2,3...$, one has

$$\begin{aligned}
\left|\left\{z \in \mathbb{D}(0,R) : t_j\,|\partial_z\eta^e_{t_j}(z)| > 2^{-l}\right\}\right| &\leqslant 2^{lp}\,t_j^p \int_{\mathbb{D}(0,R)} |\partial_z\eta^e_{t_j}(z)|^p\, dA(z) \\
&\leqslant 2^{lp}\,c(p,K,R)\,t_j^p,
\end{aligned}$$

where (3.5) was used at the last step. Then $|B(R)| = 0$ easily follows.

If we now fix a point $z \in (\mathbb{D}(0,R)\cap E)\setminus B(R)$, then by construction one sees that

$$\lim_{j\to\infty} t_j\,|\partial_z\eta^e_{t_j}(z)| = 0.$$

Thus, using the C^1-smoothness of $w \mapsto \mathcal{H}(z,w)$ one gets at these points z that

$$\lim_{j\to\infty}\frac{|\mathcal{H}(z,w_j) - \mathcal{H}(z,w_0) - D_w\mathcal{H}(z,w_0)\,(w_j - w_0)|}{|w_j - w_0|} = 0, \tag{3.6}$$

where we write $w_0 = \partial_z \varphi_a(z)$ as before, and $w_j = w_0 + t_j\, \partial_z \eta^e_{t_j}(z)$. Summarizing, if $p' = \frac{p}{p-1}$, then

$$\left| \int_{\mathbb{C}} \xi(z)\, h_{t_j}(z)\, dA(z) \right| = \left| \int_{\mathbb{D}(0,R)} \xi(z)\, \frac{h_{t_j}(z)}{\partial_z \eta^e_{t_j}(z)}\, \partial_z \eta^e_{t_j}(z)\, dA(z) \right|$$
$$\leqslant \left(\int_{\mathbb{D}(0,R)} \left| \xi(z)\, \frac{h_{t_j}(z)}{\partial_z \eta^e_{t_j}(z)} \right|^{p'} dA(z) \right)^{\frac{1}{p'}} \left(\int_{\mathbb{D}(0,R)} |\partial_z \eta^e_{t_j}(z)|^p\, dA(z) \right)^{\frac{1}{p}}.$$

The second integral above is bounded independently of j, by (3.5). Concerning the first integral, we note that

$$\frac{|h_{t_j}(z)|}{|\partial_z \eta^e_{t_j}(z)|} = \frac{|\mathcal{H}(z, w_j) - \mathcal{H}(z, w_0) - D_w \mathcal{H}(z, w_0)\,(w_j - w_0)|}{|w_j - w_0|} \leqslant 2k. \tag{3.7}$$

Moreover, we know by (3.6) that

$$\frac{|h_{t_j}(z)|}{|\partial_z \eta^e_{t_j}(z)|} \to 0, \quad j \to \infty, \qquad \text{a.e. on } \mathbb{D}(0,R). \tag{3.8}$$

Since ξ is compactly supported and bounded, the dominated convergence theorem gives that

$$\lim_{j \to \infty} \int_{\mathbb{C}} \xi(z)\, h_{t_j}(z)\, dA(z) = 0$$

as desired. So (3.4) holds, and thus the limit η^e of any converging sequence $\eta^e_{t_j}$ solves the $\mathbb{R}$-linear Beltrami equation (3.1) with coefficients μ_a and ν_a.

If we are now given another converging subsequence $\eta^e_{\tilde{t}_j}$, then by the above argumentation the limit $\tilde{\eta}^e$ solves the same $\mathbb{R}$-linear Beltrami equation (3.1) as η. But $\mathbb{R}$-linear equations have only one K-quasiconformal solution that fixes 0 and maps 1 to a given nonzero point, [5, Theorem 6.2.3]. Hence $\eta^e \equiv \tilde{\eta}^e$.

Step 3: Whole sequence converges. We now prove that $\eta^e_t \to \eta^e$, $t \to 0^+$, which means that the directional derivatives $\partial^a_e \varphi_a = \eta^e$ exist.

Since η^e_t is bounded as a normal family, and all the converging subsequences have the same limit η^e (by Step 2), we obtain that the full sequence converges. Indeed, for fixed z, if there is a sequence $\eta^e_s(z)$ such that $|\eta^e_s(z) - \eta^e(z)| \geqslant \varepsilon$, then the boundedness implies we have a converging subsequence such that $|\eta^e_{s_j}(z) - \eta^e(z)| \geqslant \varepsilon$. This is a contradiction with the fact that all the converging subsequences have the same limit. □

We have just shown that for each z, the mapping $a \mapsto \varphi_a(z)$ has as partial derivatives $\partial^a_e \varphi_a(z)$ a linear family of K-quasiconformal mappings, indexed in the direction e of the differentiation. The following step is proving that these are not just pointwise derivatives, but metric derivatives with the $W^{1,2}_{\mathrm{loc}}$-topology.

Lemma 3.4. *Let the field $\mathcal{H}(z,w) \in C^1(w)$. Then $\partial^a_e \varphi_a$ is the e-directional derivative of*

$$a \mapsto \varphi_a : \mathbb{C} \to L^\infty_{\mathrm{loc}}(\mathbb{C}),$$
$$a \mapsto \varphi_a : \mathbb{C} \to W^{1,2}_{\mathrm{loc}}(\mathbb{C}).$$

Proof. Let $\eta^e_{t_j} = \frac{\varphi_{a+t_je}-\varphi_a}{t_j}$ and $\eta^e = \partial^a_e \varphi_a$. By the proof of Theorem 3.3,

$$\lim_{j\to\infty} \|\eta^e_{t_j} - \eta^e\|_{L^\infty(\mathbb{D}(0,R))} = 0,$$

which shows the first claim. We will show that the limit can be taken in $W^{1,2}_{\mathrm{loc}}(\mathbb{C})$-metrics, too, which proves the lemma. We are left to show the L^2-convergence of the z-derivatives.

Let us denote $L_a = \partial_{\bar z} - \mu_a(z)\,\partial_z - \nu_a(z)\,\overline{\partial_z}$. It has been shown in the proof of Theorem 3.3 that

$$L_a(\eta^e_{t_j}(z)) = h_{t_j}(z)$$
$$L_a(\eta^e) = 0$$

almost everywhere. We know also from the proof of Theorem 3.3 that $h_{t_j} \to 0$ in the sense of distributions. In fact for every $R > 0$

$$\lim_{j\to\infty} \|h_{t_j}\|_{L^2(\mathbb{D}(0,R))} = 0. \tag{3.9}$$

Indeed, for any $2 < p < \frac{2K}{K-1}$ we have

$$\begin{aligned}\int_{\mathbb{D}(0,R)} |h_{t_j}|^2 &= \int_{\mathbb{D}(0,R)} \left|\frac{h_{t_j}}{\partial_z \eta^e_{t_j}}\right|^2 |\partial_z \eta^e_{t_j}|^2 \\ &\leqslant \left(\int_{\mathbb{D}(0,R)} \left|\frac{h_{t_j}}{\partial_z \eta^e_{t_j}}\right|^{\frac{p}{p-2}}\right)^{\frac{p-2}{p}} \left(\int_{\mathbb{D}(0,R)} |\partial_z \eta^e_{t_j}|^p\right)^{\frac{2}{p}}.\end{aligned}$$

By (3.7), (3.8), and the dominated convergence theorem, the first factor converges to 0 as $j \to \infty$. By (3.5), the second factor is uniformly bounded. Thus we get (3.9).

Now, choose a real valued cutoff function $\xi \in C^\infty_0(\mathbb{D}(0, 2R))$, $\xi = 1$ on $\mathbb{D}(0, R)$. Then

$$L_a(\xi\,(\eta^e_{t_j} - \eta^e)) = L_a(\xi)\,(\eta^e_{t_j} - \eta^e) + \xi\, h_{t_j}.$$

Therefore, by the classical L^2-theory of Beltrami operators ($L_a = \partial_{\bar z} - \mu_a(z)\,\mathcal{S}\partial_{\bar z} - \nu_a(z)\,\overline{\mathcal{S}\partial_{\bar z}}$ is invertible in $L^2(\mathbb{C})$), and since $\xi\,(\eta^e_{t_j} - \eta^e) \in W^{1,2}(\mathbb{C})$, we have an a priori L^2-estimate

$$\|\partial_{\bar z}(\xi(\eta^e_{t_j} - \eta^e))\|_{L^2(\mathbb{C})} \leqslant c(K)\, \left(\|L_a(\xi)\,(\eta^e_{t_j} - \eta^e)\|_{L^2(\mathbb{C})} + \|\xi\, h_{t_j}\|_{L^2(\mathbb{C})}\right).$$

The first term on the right hand side converges to 0, due to the locally uniform convergence of $\eta^e_{t_j} \to \eta^e$. The second term also converges to 0 by the dominated convergence theorem and (3.9). Using now that the Beurling transform $\mathcal{S}$ is bounded in $L^2(\mathbb{C})$, we get for the full differentials that

$$\lim_{j\to\infty} \left\| D_z \eta^e_{t_j} - D_z \eta^e \right\|_{L^2(\mathbb{D}(0,R))} = 0. \tag{3.10}$$

□

Remark 3.5. A slight modification of the argument above shows that Sobolev space $W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ can be replaced by $W^{1,p}_{\mathrm{loc}}(\mathbb{C})$ for some $2 < p < \frac{2K}{K-1}$.

It turns out that the derivative is continuous as a function of a respect to various topologies.

Theorem 3.6. *Let the field $\mathcal{H}(z,w) \in C^1(w)$. Then*

(a) $a \mapsto \varphi_a \in C^1(\mathbb{C}, L^\infty_{\mathrm{loc}}(\mathbb{C}))$;

(b) $a \mapsto \varphi_a(z)$ is continuously differentiable on $\mathbb{C}$, for every $z \in \mathbb{C}$;

(c) $a \mapsto \varphi_a \in C^1(\mathbb{C}, W^{1,2}_{\mathrm{loc}}(\mathbb{C}))$.

Proof. (b) follows from (a). For (a) and (c) we have to establish for $\mathcal{T}_a\, h := \partial_1^a \varphi_a(z)\, h_1 + \partial_i^a \varphi_a(z)\, i h_2$, $h = h_1 + i h_2$, that

$$\lim_{|h|\to 0} \frac{\|\varphi_{a+h} - \varphi_a - \mathcal{T}_a\, h\|_X}{|h|} = 0, \tag{3.11}$$

i.e., $a \mapsto \varphi_a : C \to X$ is differentiable and $a \mapsto \mathcal{T}_a : \mathbb{C} \to L(\mathbb{C}, X)$ is continuous, where X is $L^\infty(\mathbb{D}(0,R))$ and $W^{1,2}(\mathbb{D}(0,R))$, respectively.

The argumentation is similar to the proof of Corollary 2.5 and we first establish

$$\lim_{b\to a} \|\partial_e^a \varphi_a - \partial_e^a \varphi_b\|_X = 0. \tag{3.12}$$

First, note that the function $\partial_e^a \varphi_a(z)$ is a normalized quasiconformal map ($0 \mapsto 0$, $1 \mapsto e$) for every fixed $a \in \mathbb{C}$, by Theorem 1.3. Normalized quasiconformal mappings form a normal family, by quasisymmetry (see the Montel-type theorem [5, Theorem 3.9.4]). Set

$$f = \lim_{j\to\infty} \partial_e^a \varphi_{b_j} \tag{3.13}$$

for some subsequence $b_j \to a$. Then f is also a normalized quasiconformal mapping and the convergence is locally uniform in the z-variable (as in page 11). In other words, the convergence at (3.13) is in the L^∞_{loc}-topology. We will show that it holds also in $W^{1,2}_{\mathrm{loc}}$. Set $\mathbb{D}(0,R)$ Notice that as in the proof of Corollary 2.3, Proposition 2.1 (c) yields a constant $c(K,R)$ such that for every $b \in \mathbb{C}, e \in \mathbb{D}$

$$\|D_z \partial_e^a \varphi_b\|_{L^2(\mathbb{D}(0,R))} \leqslant c(K,R). \tag{3.14}$$

In fact, by Proposition 2.1 (c), we know that $\partial_z \varphi_{b_j}(z) \to \partial_z \varphi_a(z)$ for almost every $z \in \mathbb{D}(0,R)$ (at least for a subsequence). We recall now the expressions at (3.3) for the Beltrami coefficients of $\partial_z \varphi_{b_j}$ and $\partial_z \varphi_a$. Since $w \mapsto D_w \mathcal{H}(z,w)$ is continuous and bounded by assumption, dominated convergence gives that for every $1 < p < \infty$

$$\begin{aligned} &\|\mu_{b_j} - \mu_a\|_{L^p(\mathbb{D}(0,R))} \\ &\qquad = \|\partial_w \mathcal{H}(\cdot, \partial_z \varphi_{b_j}) - \partial_w \mathcal{H}(\cdot, \partial_z \varphi_a)\|_{L^p(\mathbb{D}(0,R))} \to 0, \\ &\|\nu_{b_j} - \nu_a\|_{L^p(\mathbb{D}(0,R))} \\ &\qquad = \|\partial_{\bar w} \mathcal{H}(\cdot, \partial_z \varphi_{b_j}) - \partial_{\bar w} \mathcal{H}(\cdot, \partial_z \varphi_a)\|_{L^p(\mathbb{D}(0,R))} \to 0, \end{aligned} \tag{3.15}$$

as $j \to \infty$.

We will now show that $\partial_e^a \varphi_{b_j}$ is a Cauchy sequence in $W^{1,2}_{\mathrm{loc}}(\mathbb{C})$. Take $g_{jk} = \partial_e^a \varphi_{b_j} - \partial_e^a \varphi_{b_k}$. Notice that, by Theorem 1.3, as a quasiconformal maps $\partial_e^a \varphi_{b_j}, \partial_e^a \varphi_{b_k} \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$. Hence $g_{jk} \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$. Moreover,

$$\begin{aligned} &\partial_{\bar z} g_{jk}(z) - \mu_{b_j}(z) \partial_z g_{jk}(z) - \nu_{b_j}(z) \overline{\partial_z g_{jk}}(z) \\ &\qquad = (\mu_{b_j}(z) - \mu_{b_k}(z))\, \partial_z \partial_e^a \varphi_{b_k}(z) + (\nu_{b_j} - \nu_{b_k})\, \overline{\partial_z \partial_e^a \varphi_{b_k}(z)}. \end{aligned}$$

Now, recall the following Caccioppoli-type estimate for a general Sobolev map $g \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$,

$$\begin{aligned}\|D_z g\|_{L^2(\mathbb{D}(0,R))} \\ \leqslant c(k,R)\big(\|g\|_{L^2(\mathbb{D}(0,2R))} + \|\partial_{\bar z} g - \mu_a\,\partial_z g - \nu_a\,\overline{\partial_z g}\|_{L^2(\mathbb{D}(0,2R))}\big),\end{aligned} \tag{3.16}$$

whose proof follows as in [5, Theorem 5.4.3] (in the mentioned theorem there is no ν-term). If we apply (3.16) to $g = g_{jk}$ we get

$$\begin{aligned}\|D_z g_{jk}\|_{L^2(\mathbb{D}(0,R))} &\leqslant c(k,R)\Big(\|g_{jk}\|_{L^2(\mathbb{D}(0,2R))} \\ &+ \|\big(|\mu_{b_j} - \mu_{b_k}| + |\nu_{b_j} - \nu_{b_k}|\big)|\partial_z \partial^a_e \varphi_{b_k}|\|_{L^2(\mathbb{D}(0,2R))}\Big).\end{aligned}$$

Using now (3.14), (3.15) and the fact that $\partial^a_e \varphi_{b_j} \to f$ in $L^\infty_{\mathrm{loc}}(\mathbb{C})$, we get that the sequence $\partial^a_e \varphi_{b_j}$ is Cauchy in $W^{1,2}_{\mathrm{loc}}(\mathbb{C})$. Hence

$$\lim_{j\to\infty} \|D_z f - D_z \partial^a_e \varphi_{b_j}\|_{L^2(\mathbb{D}(0,R))} = 0. \tag{3.17}$$

Thus we have shown that $\partial^a_e \varphi_{b_j}$ converges to f in $W^{1,2}_{\mathrm{loc}}$-topology.

We will show that f solves the same $\mathbb{R}$-linear equation as $\partial^a_e \varphi_a$. Then it follows that $f \equiv \partial^a_e \varphi_a$, by the uniqueness of normalized solutions to $\mathbb{R}$-linear Beltrami equations [5, Theorem 6.2.3]. Thus the limit (3.13) does not depend on the subsequence b_j, and the full sequence converges. This proves (3.12) for $X = L^\infty(\mathbb{D}(0,R))$.

We prove that f solves the same $\mathbb{R}$-linear equation as $\partial^a_e \varphi_a$. We apply a similar argument as above to $g_j = f - \partial^a_e \varphi_{b_j}$ to obtain that

$$\begin{aligned}\partial_{\bar z} f(z) - \mu_a(z)\partial_z f(z) - \nu_a(z)\overline{\partial_z f(z)} &= \partial_{\bar z} g_j(z) - \mu_a(z)\partial_z g_j(z) - \nu_a(z)\overline{\partial_z g_j(z)} \\ &+ (\mu_{b_j}(z) - \mu_a(z))\partial_z \partial^a_e \varphi_{b_j}(z) + (\nu_{b_j}(z) - \nu_a(z))\overline{\partial_z \partial^a_e \varphi_{b_j}(z)}.\end{aligned} \tag{3.18}$$

By (3.15) and (3.17), the right hand side is arbitrarily small in the $L^2(\mathbb{D}(0,R))$-norm and hence

$$\partial_{\bar z} f(z) = \mu_a(z)\partial_z f(z) + \nu_a(z)\overline{\partial_z f(z)} \qquad \text{a.e.}$$

Thus $f = \partial^a_e \varphi_a$ as the normalized solution is unique, [5, Theorem 6.2.3]. This establishes (3.12).

The differentiability (3.11) is immediate. Indeed, $\mathcal{T}_a\, h = \partial^a_1 \varphi_a(z)\, h_1 + \partial^a_i \varphi_a(z)\, i h_2$ and

$$\begin{aligned}\frac{\|\varphi_{a+h} - \varphi_a - \mathcal{T}_a\, h\|_X}{|h|} &\leqslant \frac{\|\varphi_{a+h_1+ih_2} - \varphi_{a+ih_2} - \partial^a_1 \varphi_{a+ih_2}\, h_1\|_X}{|h_1|} \\ &+ \frac{\|\partial^a_1 \varphi_{a+ih_2}\, h_1 - \partial^a_1 \varphi_a\, h_1\|_X}{|h_1|} + \frac{\|\varphi_{a+ih_2} - \varphi_a - \partial^a_i \varphi_a\, i h_2\|_X}{|h_2|}.\end{aligned}$$

When $|h| \to 0$, the first and third term on the right hand side converge to 0, by Lemma 3.4, and the second term goes to 0, by (3.12). We have shown the differentiability and $D_a \varphi_a = \mathcal{T}_a$. Finally, recall that (3.12) yields the continuity of $a \mapsto D_a \varphi_a : \mathbb{C} \to L(\mathbb{C}, X)$ and the proof is concluded. □

It is interesting to compare Theorem 3.6 with Theorem 2.2. The fact that we are starting with the nonlinear equation allows us to get the continuity of the derivative in $W^{1,2}_{\rm loc}(\mathbb{C})$. In the manifolds language this enables us to embed $\mathcal{F}_{\mathcal{H}}$ in $W^{1,2}_{\rm loc}(\mathbb{C})$.

Corollary 3.7. *Let $\mathcal{H} \in C^1(w)$. Then $\mathcal{F}_{\mathcal{H}}$ is a C^1-embedded submanifold of $W^{1,2}_{\rm loc}(\mathbb{C})$.*

Proof. By Theorem 3.6, $a \mapsto \varphi_a \in C^1(\mathbb{C}, W^{1,2}_{\rm loc}(\mathbb{C}))$. Proving that that this yields an embedded manifold is similar to Proposition 2.6. Indeed, the proof that $a \mapsto \varphi_a$ is an immersion is the same and the fact that $a \mapsto \varphi_a : \mathbb{C} \to W^{1,2}_{\rm loc}(\mathbb{C})$ is a topological embedding follows by Proposition 2.1 (c). □

4. Smooth $\mathcal{H}$-equations

In the first two subsections below the uniqueness property of the field $\mathcal{H}$ plays no role. The sections deal with the regularity of solutions to nonlinear Beltrami equations with Hölder continuous coefficients and the lower bound of their Jacobian.

4.1. **Schauder-type estimates.** For the associated families $\mathcal{F}_{\mathcal{H}}$ we need to study the Hölder regularity for the first-order nonlinear Beltrami equations that is natural for the theory of quasiregular mappings. We will give a proof in the autonomous case and give ideas for the general situation. The full proof will be published separately [2]. In what follows we use similar ideas as in the Schauder estimates for the divergence-type equations [16, Chapter 6].

Definition 4.1. Let $\mathcal{H} : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$. We say that $\mathcal{H}(z,w) \in C^{\alpha}_{\rm loc}(z)$ uniformly in w, if given an open bounded set $\Omega' \Subset \Omega$, then

$$|\mathcal{H}(z_1,w) - \mathcal{H}(z_2,w)| \leqslant \mathbf{H}_\alpha\, |z_1 - z_2|^\alpha |w|$$

for $z_1, z_2 \in \Omega'$, where the constant $\mathbf{H}_\alpha = \mathbf{H}_\alpha(\Omega')$ depends only on Ω'.

Theorem 4.2 (Theorem 1.6)**.** *Let $f \in W^{1,2}_{\rm loc}(\Omega)$ be a solution to the nonlinear Beltrami equation*

$$\partial_{\bar z} f(z) = \mathcal{H}(z, \partial_z f(z)) \qquad \text{for a.e. } z \in \Omega, \tag{4.1}$$

where $\mathcal{H}(z,w) \in C^\alpha_{\rm loc}(z)$ uniformly in w.

Then $f \in C^{1,\gamma}_{\rm loc}(\Omega)$ where $\gamma = \alpha$, if $\alpha < \frac{1}{K}$, and $\gamma < \frac{1}{K}$, if $\frac{1}{K} \leqslant \alpha < 1$, and $K = \frac{1+k}{1-k}$. Moreover,

$$\|D_z f\|_{C^\gamma(\mathbb{D}(z_1,R_1/8))} \leqslant c(\mathcal{H}, z_1, R_1)\, \|D_z f\|_{L^2(\mathbb{D}(z_1,R_1))}, \tag{4.2}$$

for $\mathbb{D}(z_1,R_1) \subset \Omega$, where $c(\mathcal{H}, z_1, R_1) = c(k, \alpha, \gamma, \mathbf{H}_\alpha(\mathbb{D}(z_1,R_1)))$.

Towards the proof of Theorem 4.2, we start with an auxiliary result for the nonlinear Beltrami equation with constant coefficients (see [12], [13], [24]).

Proposition 4.3. *Let $F \in W^{1,2}_{\rm loc}(\Omega)$ be a solution to the autonomous nonlinear Beltrami equation*

$$\partial_{\bar z} F(z) = \mathcal{H}(\partial_z F(z)) \qquad \text{for a.e. } z \in \Omega. \tag{4.3}$$

Then the directional derivatives of F are K-quasiregular.

Proof. Let $h > 0$. The difference quotient

$$F_h(z) := \frac{F(z+he) - F(z)}{h}, \qquad |e| = 1,$$

is K-quasiregular, $K = \frac{1+k}{1-k}$. Indeed, by (4.3),

$$\begin{aligned} |\partial_{\bar z} F_h(z)| &= \left| \frac{\mathcal{H}(\partial_z F(z+he)) - \mathcal{H}(\partial_z F(z))}{h} \right| \\ &\leqslant k \, \frac{|\partial_z F(z+he) - \partial_z F(z)|}{|h|} = k \, |\partial_z F_h(z)|. \end{aligned} \tag{4.4}$$

We have a Caccioppoli estimate for F_h, see e.g. [5, Theorem 5.4.2]. Namely, if $\mathbb{D}_R = \mathbb{D}(z_0, R)$, let $\xi \in C_0^\infty(\mathbb{D}_R)$ be a smooth real-valued test function compactly supported on $\mathbb{D}_R$, with $\xi \equiv 1$ on $\mathbb{D}_\rho$ and $|\nabla \xi| \leqslant \frac{c_0}{R-\rho}$. Now, for $0 < \rho \leqslant \frac{R}{2}$,

$$\begin{aligned} \int_{\mathbb{D}_\rho} |D_z F_h|^2 \leqslant \int_{\mathbb{D}_R} \xi^2 |D_z F_h|^2 &\leqslant c(K) \int_{\mathbb{D}_R} |\nabla \xi|^2 |F_h|^2 \\ &\leqslant \frac{c(K)}{R^2} \int_{\mathbb{D}_R} (|D_z F|^2 + 1). \end{aligned} \tag{4.5}$$

Thus we have a uniform bound for the derivative of the difference quotient. Hence the directional derivative $\partial_e^z F \in W^{1,2}_{\text{loc}}(\Omega)$. Further, letting $h \to 0^+$ in (4.4), we see that $\partial_e^z F(z)$ is K-quasiregular. $\square$

Therefore, the directional derivatives inherit the properties of K-quasiregular maps. We state the ones we need as a corollary.

Corollary 4.4. *The map $D_z F$ is locally $\frac{1}{K}$-Hölder continuous. Moreover, the derivative satisfies*

$$\|D_z F - (D_z F)_\rho\|_{L^2(\mathbb{D}(z_0,\rho))} \leqslant c(K) \left(\frac{\rho}{R}\right)^{1+\frac{1}{K}} \|D_z F - (D_z F)_R\|_{L^2(\mathbb{D}(z_0,R))}, \tag{4.6}$$

for every $\mathbb{D}(z_0, \rho) \subset \mathbb{D}(z_0, R) \subset \Omega$, where we denote the integral average $(D_z F)_r = \fint_{\mathbb{D}(z_0,r)} D_z F = \frac{1}{|\mathbb{D}(z_0,r)|} \int_{\mathbb{D}(z_0,r)} D_z F$.

Proof. The $\frac{1}{K}$-Hölder inequality of general K-quasiregular maps f goes back to Morrey [5, Section 3.10]. This follows, e.g., from the isoperimetric inequality and the pointwise equivalence of $\|Df(z)\|^2$ and $J(z, f)$ for quasiregular maps, which together imply that $\psi(r) := r^{-2/K} \int_{\mathbb{D}(z_0,R)} J(z, f)$ is non-decreasing in r, see [5, p. 82]. For our perturbation arguments it is more convenient to write this in the Campanato form. Namely, in combination with Poincaré's inequality and Caccioppoli's inequality we get, for $0 < \rho < R/2$,

$$\begin{aligned} \|f - f_\rho\|_{L^2(\mathbb{D}(z_0,\rho))} \leqslant c\rho \|D_z f\|_{L^2(\mathbb{D}(z_0,\rho))} &\leqslant c\rho \left(\int_{\mathbb{D}(z_0,\rho)} K J(z, f) \right)^{1/2} \\ &\leqslant c(K) \frac{\rho^{1+1/K}}{R^{1/K}} \left(\int_{\mathbb{D}(z_0,R/2)} J(z, f) \right)^{1/2} \\ &\leqslant c(K) \left(\frac{\rho}{R}\right)^{1+\frac{1}{K}} \|f - f_R\|_{L^2(\mathbb{D}(z_0,R))}. \end{aligned} \tag{4.7}$$

The conclusion holds trivially for $R/2 < \rho \leqslant R$, and since $\|D_zF(z)\|^2 = \sum_{j=1}^{2} |D_zF(z)\, e_j|^2 = \sum_{j=1}^{2} |\partial_{e_j}^z F(z)|^2$ for the Hilbert-Schmidt norm, the Corollary follows from (4.7) and the quasiregularity of the directional derivatives. $\square$

Remark 4.5. In view of the Campanato characterization of Hölder continuous functions, for quasiregular mappings the estimate (4.7) is equivalent to Hölder continuity. Indeed, for any function ψ and any $\alpha \in (0,1)$, the integral estimate

$$\|\psi - \psi_\rho\|_{L^2(\mathbb{D}(z_0,\rho))} \leqslant c(K) \left(\frac{\rho}{R}\right)^{1+\alpha} \|\psi - \psi_R\|_{L^2(\mathbb{D}(z_0,R))} \tag{4.8}$$

implies the local α-Hölder continuity of ψ, see [16, Chapter III, Theorem 1.2, p. 70 and Theorem 1.3 p. 72].

We will also need various estimates for the nonlinear Beltrami operators. The complete technical details of the following results will be published separately in [2], but the basic philosophy behind them is well-known, see, e.g., [14] and [5, Chapter 14].

Proposition 4.6. *Let $f \in W^{1,2}_{\mathrm{loc}}(\Omega)$ be a solution to (4.1) and $z_0 \in \Omega$ such that $\mathbb{D}_R = \mathbb{D}(z_0, R) \Subset \Omega$. Then there exists a unique $W^{1,2}(\mathbb{D}_R)$-solution to the following local Riemann-Hilbert problem for the autonomous equation:*

$$\begin{cases} \partial_{\bar z} F(z) = \mathcal{H}(z_0, \partial_z F(z)) & a.e.\ z \in \mathbb{D}_R, \\ \mathrm{Re}(f - F) = 0 & on\ \partial\mathbb{D}_R, \end{cases}$$

Moreover,

(1) For $\rho \leqslant R$ and a constant $c(K)$ depending only on K,

$$\|D_zF\|_{L^2(\mathbb{D}_\rho)} \leqslant c(K)\frac{\rho}{R}\|D_zF\|_{L^2(\mathbb{D}_R)} \leqslant c(K)\frac{\rho}{R}\|D_zf\|_{L^2(\mathbb{D}_R)}.$$

(2) There exists a constant $\mathbf{H}_\alpha = \mathbf{H}_\alpha(\mathbb{D}(z_0,\rho))$ such that if $\rho \leqslant R$

$$\|D_z(F - f)\|_{L^2(\mathbb{D}_\rho)} \leqslant \frac{\mathbf{H}_\alpha\, \rho^\alpha}{1-k}\, \|D_zf\|_{L^2(\mathbb{D}_\rho)}.$$

(3) If $f \in C^\beta_{\mathrm{loc}}(\Omega)$ and $\rho \leqslant \frac{3R}{4}$, then

$$\|D_z(F-f) - (D_z(F-f))_\rho\|_{L^2(\mathbb{D}_\rho)} \leqslant c\, R^{\alpha+\beta}\, \|f\|_{C^\beta(\mathbb{D}_R)}.$$

(4) If $D_zf \in L^\infty_{\mathrm{loc}}(\Omega)$ and $0 < \rho \leqslant R$, then

$$\|D_z(F-f)\|_{L^2(\mathbb{D}_\rho)} \leqslant \frac{c\, \mathbf{H}_\alpha\, \rho^{\alpha+1}}{1-k}\, \|D_zf\|_{L^\infty(\mathbb{D}_\rho)}.$$

Therefore

$$\|D_z(f-F) - (D_z(f-F))_\rho\|_{L^2(\mathbb{D}_\rho)} \leqslant c(K)\, \mathbf{H}_\alpha(\mathbb{D}_R)\, \|D_zf\|_{L^\infty(\mathbb{D}_R)}\, R^{\alpha+1}.$$

The existence of the solution F above is well-known in the study of the Riemann Hilbert problems; the proof is based on the local versions of the classical Cauchy transform and the Beurling transform, see, for instance, [14, Proposition 2]. The listed properties of the solution follow from the mapping properties of these operators in combination with the Cacciopoli inequalities for f, in a manner similar to [14]; for complete details see [2].

Proof of Theorem 4.2. We will give the ideas of the proof. In particular, we will use the Campanato condition (4.8).

Step 1. Splitting. Suppose $\Omega' \Subset \Omega$ is an open and bounded set. Let $z_0 \in \Omega'$ and set $\mathbb{D}_R = \mathbb{D}(z_0, R)$. We split our solution $f = F + (f - F)$, where F solves the Riemann-Hilbert problem of the autonomous equation

$$\begin{cases} \partial_{\bar z} F(z) = \mathcal{H}(z_0, \partial_z F(z)) & \text{a.e. } z \in \mathbb{D}_R, \\ \operatorname{Re}(f - F) = 0 & \text{on } \partial \mathbb{D}_R, \end{cases} \tag{4.9}$$

where $R \leqslant \min\{1, \operatorname{dist}(z_0, \partial\Omega')\}$.

Step 2. Hölder continuity of f. We will show that f is actually locally β-Hölder continuous for every $0 < \beta < 1$.

By the triangle inequality and Proposition 4.6 Facts 1–2,

$$\|D_z f\|_{L^2(\mathbb{D}_\rho)} \leqslant \left(c_1\, \rho^\alpha + c(K)\, \frac{\rho}{R} \right) \|D_z f\|_{L^2(\mathbb{D}_R)}, \qquad 0 < \rho \leqslant R,$$

which implies that for every $\varepsilon > 0$,

$$\|D_z f\|_{L^2(\mathbb{D}_\rho)} \leqslant c(c_1, K, \varepsilon) \left(\frac{\rho}{R}\right)^{1-\varepsilon} \|D_z f\|_{L^2(\mathbb{D}_R)}, \qquad 0 < \rho \leqslant R.$$

Cacciopoli's and Poincaré's inequalities yield that

$$\|f - f_\rho\|_{L^2(\mathbb{D}_\rho)} \leqslant c \left(\frac{\rho}{R}\right)^{1+(1-\varepsilon)} \|f - f_R\|_{L^2(\mathbb{D}_R)}, \qquad 0 < \rho \leqslant R.$$

Step 3: Self-improving Morrey-Campanato estimate. Assume that $1 < \alpha + \beta < 1 + \frac{1}{K}$. We will show that $D_z f \in C^{\alpha+\beta-1}_{\mathrm{loc}}(\Omega)$, and

$$\|D_z f - (D_z f)_\rho\|_{L^2(\mathbb{D}_\rho)} \leqslant c(K, \alpha, \beta) \Bigg[\left(\frac{\rho}{R}\right)^{\alpha+\beta} \|D_z f - (D_z f)_R\|_{L^2(\mathbb{D}_R)} + \mathbf{H}_\alpha(\mathbb{D}_R)\, [f]_{C^\beta(\mathbb{D}_R)} \rho^{\alpha+\beta} \Bigg]$$

whenever $0 < \rho \leqslant R$. In particular, $D_z f$ is locally bounded.

The triangle inequality yields that

$$\begin{aligned} \|D_z f - (D_z f)_\rho\|_{L^2(\mathbb{D}_\rho)} \leqslant{}& \|D_z(f - F) - (D_z(f - F))_\rho\|_{L^2(\mathbb{D}_\rho)} \\ &+ \|D_z F - (D_z F)_\rho\|_{L^2(\mathbb{D}_\rho)}. \end{aligned} \tag{4.10}$$

The first term on the right hand side can be handled with the help of the third fact of Proposition 4.6, because $f \in C^\beta_{\mathrm{loc}}(\Omega)$ for any $0 < \beta < 1$, by Step 2. The second one will be bounded by applying Corollary 4.4 to $D_z F$ and radii ρ and $\frac{3R}{4}$. One gets that

$$\begin{aligned} \|D_z f - (D_z f)_\rho\|_{L^2(\mathbb{D}_\rho)} \leqslant{}& c(K) \left(\frac{\rho}{R}\right)^{1+\frac{1}{K}} \|D_z f - (D_z f)_{3R/4}\|_{L^2(\mathbb{D}_{3R/4})} \\ &+ c(K)\, \mathbf{H}_\alpha(\mathbb{D}_R)\, \|f\|_{C^\beta(\mathbb{D}_R)}\, R^{\alpha+\beta} \left(1 + \left(\frac{\rho}{R}\right)^{1+\frac{1}{K}} \right). \end{aligned}$$

Our choice of the powers ($1 < \alpha + \beta < 1 + \frac{1}{K}$) yields the claim.

Step 4: Final estimate. We go back to (4.10) but with the additional information that $\|D_z f\|_{L^\infty(\mathbb{D}(0,R))} < \infty$. Hence we can use Fact 4 of Proposition 4.6 to improve the power of R in the estimate for the first term. In fact the final restriction in α comes from the term $\|D_z F - (D_z F)_\rho\|_{L^2(\mathbb{D}_\rho)}$ since $D_z F$ is only known to be $\frac{1}{K}$-Hölder continuous. □

Remark 4.7. We do not know if the above result is sharp in terms of the exponent $\frac{1}{K}$. If the field $\mathcal{H}$ is assumed to be C^1 in the gradient variable, one can prove that the solutions are in $C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ and the estimate to the $C^{1,\alpha}_{\mathrm{loc}}(\Omega)$-norm is locally uniform in a. This will appear in [2].

4.2. **Non-vanishing of the Jacobian.** For the nonlinear Beltrami equation with Hölder continuous coefficients one has a lower bound for the Jacobian of a normalized homeomorphic solution.

Theorem 4.8 (Theorem 1.8)**.** *Assume that the field $\mathcal{H}(z,w) \in C^\alpha_{\mathrm{loc}}(z)$ uniformly in w. Then a homeomorphic solution $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ to the nonlinear Beltrami equation*

$$\partial_{\bar z} f(z) = \mathcal{H}(z, \partial_z f(z)) \qquad \text{for a.e. } z \in \mathbb{C} \tag{4.11}$$

has a positive Jacobian, $J(z,f) > 0$.

Further, if f is also normalized, $f(0) = 0$ and $f(1) = 1$, then there is a lower bound for the Jacobian

$$\inf_{z \in \mathbb{D}(0,R_0)} J(z,f) \geqslant c > 0,$$

where $c = c(\mathcal{H}, R_0) = c(k, \alpha, \mathbf{H}_\alpha(\mathbb{D}(0,8R_0)), R_0)$.

In the linear case the statement can be shown by the representation theorem of the quasiregular maps (e.g., [23, Theorems II.5.2 and II.5.47]) or by using the Schauder estimates for the inverse (e.g., the proof of [9, Proposition 5.1]), that is, showing that also the inverse f^{-1} solves a Beltrami equation with Hölder continuous coefficients and hence is locally Hölder continuous, too. Neither of these ideas work in the nonlinear case; we do not have a representation theorem and for the inverse $g = f^{-1}$ we have the nonlinear Beltrami equation

$$\partial_{\bar\omega} g(\omega) = -\frac{1}{J(z,f)} \mathcal{H}\left(g(\omega), J(z,f)\, \overline{\partial_\omega g(\omega)}\right), \qquad \omega = f(z) \qquad \text{a.e.},$$

which would have Hölder continuous coefficients if we *a priori* knew that the Jacobian $J(z,f)$ has a positive lower bound. We give a genuinely nonlinear proof. First, we obtain the corresponding qualitative results for the autonomous and non-autonomous equation. In the first lemma, we deal with the autonomous equation via Stoïlov factorization, Hurwitz theorem, and a compactness argument inspired by [4].

Lemma 4.9. *Let $F : \Omega \to \mathbb{C}$ be a quasiconformal solution to an autonomous nonlinear Beltrami equation $\partial_{\bar z} F(z) = \mathcal{H}(\partial_z F(z))$ a.e.. Then $D_z F(z) \neq 0$ at every point $z \in \Omega$.*

Proof. Let us fix a disk $\mathbb{D}(z_0, 2R) \subset \Omega$ and a point $z_1 \in \mathbb{D}(z_0, R)$ where $J(z_1, F) \neq 0$. The derivatives of F are continuous by Proposition 4.3, and we can assume, for instance, that $\partial_x F(z_1) \neq 0$. We will then show that $\partial_x F(z) \neq 0$ everywhere.

Let us define

$$F_h(z) = \frac{F(z+h) - F(z)}{h}, \qquad h > 0.$$

Clearly F_h is well-defined on $\Omega_h = \{z \in \Omega : d(z, \partial\Omega) > h\}$, and $\mathbb{D}(z_0, 2R) \subset \Omega_h$ for any $0 < h < d(z_0, \partial\Omega) - 2R$. Further, F_h is K-quasiregular on Ω_h, because $\mathcal{H}$ is k-Lipschitz. Moreover, by Proposition 4.3 and (4.5), we know that $\{D_z F_h : 0 < h < 1\}$ is locally uniformly bounded in $W^{1,2}_{\mathrm{loc}}(\Omega)$. Therefore the family $\{F_h : 0 < h < 1\}$ of K-quasiregular mappings is normal, and we can find a subsequence F_{h_j} converging locally uniformly in Ω.

Moreover, since $F(z)$ is a homeomorphism, each $F_h(z)$ is non-vanishing in Ω. We can thus invoke the Hurwitz theorem for quasiregular mappings [21] which tells for any converging subsequence that either the limit $\lim_j F_{h_j}(z) = \partial_x F(z)$ is non-vanishing everywhere, or the limit vanishes identically. As $\partial_x F(z_1) \neq 0$, the lemma follows. □

Next, as in the Schauder estimate we use a perturbation argument to extend the result from the autonomous equation to the non-autonomous equation that has Hölder continuous coefficients.

Lemma 4.10. *Let $f : \Omega \to \mathbb{C}$ be a quasiconformal solution to a nonlinear Beltrami equation*

$$\partial_{\bar z} f(z) = \mathcal{H}(z, \partial_z f(z)) \qquad a.e.$$

and assume that $\mathcal{H}(z, w) \in C^{\alpha}_{\mathrm{loc}}(z)$ uniformly in w. Then $D_z f(z) \neq 0$ at every point $z \in \Omega$.

Proof. Let us assume that $D_z f(z_0) = 0$. Now, $\mathbb{D}(f(z_0), R) \subset \mathbb{C}$ is a bounded, convex, simply connected domain and denote $\Omega = f^{-1}\big(\mathbb{D}(f(z_0), R)\big)$. Since f^{-1} is quasiconformal, we can choose R such that $\partial\Omega$ is a rectifiable Jordan curve (almost every $R > 0$ satisfies the above requirement).

Set F similarly as in Step 1 of the proof of Theorem 1.6, that is, let

$$\begin{cases} \partial_{\bar z} F(z) = \mathcal{H}(z_0, \partial_z F(z)) & \text{a.e. in } \Omega, \\ \mathrm{Re}(f - F) = 0 & \text{on } \partial\Omega, \end{cases}$$

and write $f = f - F + F$. The Riemann-Hilbert problem can be solved as Ω is bounded and simply connected domain whose boundary is a rectifiable Jordan curve (for the corresponding local Cauchy and Beurling transform, see [9, Section 6.1]).

Since $\mathcal{H}$ is Hölder continuous in the z-variable, we know by Theorem 1.6 that $D_z f \in C^{\gamma}_{\mathrm{loc}}(\Omega)$. Thus, if $0 < \rho \leqslant R$,

$$\begin{aligned} \|D_z f\|_{L^2(\mathbb{D}(z_0,\rho))} &= \left(\int_{\mathbb{D}(z_0,\rho)} |D_z f|^2 \right)^{\frac{1}{2}} \\ &= \left(\int_{\mathbb{D}(z_0,\rho)} |D_z f - D_z f(z_0)|^2 \right)^{\frac{1}{2}} \\ &\leqslant \frac{[D_z f]_{C^{\gamma}(\mathbb{D}(z_0,R))} \sqrt{2\pi}}{\sqrt{\gamma+1}}\, \rho^{\gamma+1} = c_1\, \rho^{\gamma+1}. \end{aligned}$$

On the other hand, since F solves an autonomous equation (and is a homeomorphism as the boundary behaviour is given by a quasiconformal map f that maps Ω to a bounded convex domain, [9, Theorem 6.1]), we know by Lemma 4.9 that D_zF does not vanish. Moreover, since D_zF is continuous and does not vanish on z_0 we can find for every $\varepsilon > 0$ a number $\rho_0 = \rho_0(\varepsilon)$ such that if $0 < \rho < \rho_0(\varepsilon)$ then

$$\frac{|D_zF(z)|}{|D_zF(z_0)|} \geqslant 1 - \varepsilon$$

everywhere on $\mathbb{D}(z_0, \rho)$. We obtain

$$\begin{aligned} \|D_z(f-F)\|_{L^2(\mathbb{D}(z_0,\rho))} &\geqslant \|D_zF\|_{L^2(\mathbb{D}(z_0,\rho))} - \|D_zf\|_{L^2(\mathbb{D}(z_0,\rho))} \\ &\geqslant (1-\varepsilon)\,|D_zF(z_0)|\,\sqrt{\pi}\,\rho - c_1\,\rho^{\gamma+1} \\ &= \rho\left((1-\varepsilon)\,|D_zF(z_0)|\,\sqrt{\pi}\ - c_1\,\rho^{\gamma}\right) \end{aligned} \tag{4.12}$$

for every $\rho \in (0, \rho_0)$. However, from the second fact of Proposition 4.6, we have an upper bound

$$\|D_z(f-F)\|_{L^2(\mathbb{D}(z_0,\rho))} \leqslant \frac{\mathbf{H}_\alpha\,\rho^\alpha}{1-k}\,\|D_zf\|_{L^2(\mathbb{D}(z_0,\rho))} \leqslant c\,\rho^{\alpha+\gamma+1}. \tag{4.13}$$

Thus, if we combine (4.12) and (4.13) and let ρ tend to zero, we get a contradiction with the assumption $D_zf(z_0) = 0$. □

Now, the proof of Theorem 4.8 follows from Lemma 4.10 and a compactness argument.

Proof of Theorem 4.8. Let us make a counter assumption: there exist $z_n \in \mathbb{D}(0, R_0)$ and normalized homeomorphic solutions f_n to the nonlinear Beltrami equations with Hölder continuous coefficients of the type (4.11),

$$\partial_{\bar z} f_{n_k}(z) = \mathcal{H}_{n_k}(z, \partial_z f_{n_k}(z)) \qquad \text{a.e.},$$

such that

$$J(z_n, f_n) \leqslant \frac{1}{n}.$$

Now, going to a subsequence n_k, if necessary, $z_{n_k} \to z_\infty \in \overline{\mathbb{D}}(0, R_0)$ and as a normalized family of quasiconformal maps $f_{n_{k_j}} \to f_\infty$ locally uniformly, where f_∞ is quasiconformal and $f_\infty(0) = 0$, $f_\infty(1) = 1$ (the Montel-type theorem [5, Theorem 3.9.4]). Moreover, by the Schauder norm estimate (4.2), for any $R > 0$,

$$\begin{aligned} \|D_zf_{n_{k_j}}\|_{C^\gamma(\mathbb{D}(0,R))} &\leqslant c\,\|D_zf_{n_{k_j}}\|_{L^2(\mathbb{D}(0,8R))} \leqslant c\,\|f_{n_{k_j}}\|_{L^2(\mathbb{D}(0,16R))} \\ &\leqslant c\,\eta_K(16R), \end{aligned} \tag{4.14}$$

where $c = c(\mathcal{H}, R)$ and the second to the last inequality follows by the Caccioppoli inequality and the last one from the η_K-quasisymmetry of quasiconformal maps. Hence derivatives $D_zf_{n_{k_j}}$ have a local uniform C^γ-upper bound and mappings $f_{n_{k_j}}$ converge to f_∞ in $C^{1,\gamma}_{\text{loc}}(\mathbb{C})$, too. Thus $J(z_\infty, f_\infty) = 0$.

The inconsistency follows, since f_∞ solves a nonlinear Beltrami equation with Hölder continuous coefficients of the type (4.11) which contradicts Lemma 4.10. Namely, $\mathcal{H}_{n_{k_j}}$ is locally uniformly equicontinuous on $\mathbb{C} \times \mathbb{C}$, thus passing to a subsequence it converges to $\mathcal{H}_\infty$ which has the same regularity and norm bounds than the family $\mathcal{H}_{n_{k_j}}$. Since the convergence of $D_z f_{n_{k_j}}$ is also locally uniform, it follows that

$$\partial_{\bar z} f_\infty(z) = \mathcal{H}_\infty(z, \partial_z f_\infty(z)) \qquad \text{a.e.}$$

giving us the inconsistency. As γ depends only on k and α, we have proved our statement. □

4.3. **Smooth fields yield smooth families.** In this section we study the smoothness of the associated family $\mathcal{F}_\mathcal{H} = \{\varphi_a(z)\}_{a\in\mathbb{C}}$ to the nonlinear Beltrami equation (1.2), when we are given a smooth field $\mathcal{H}$. The main results are Theorem 4.13 and Corollary 4.15, in which we derive the smoothness $\varphi_a(z)$ with respect to both variables a and z.

We combine first the interpolation (Theorem 2.4) with the Schauder estimates (Theorem 1.6) to obtain continuity of $a \mapsto D_z\varphi_a : \mathbb{C} \to L^\infty_{\mathrm{loc}}(\mathbb{C})$.

Lemma 4.11. *Let the field $\mathcal{H}(z,w) \in C^\alpha_{\mathrm{loc}}(z)$ uniformly in w and $\mathcal{F}_\mathcal{H} = \{\varphi_a\}_{a\in\mathbb{C}}$. Then*

$$\|D_z\varphi_a - D_z\varphi_b\|_{C^s(\mathbb{D}(z_1,R))} \leqslant c(\theta, \mathcal{H}, \mathbb{D}(z_1,R))\, |a-b|^{1-\theta}(|a|+|b|)^\theta, \quad \theta = \theta(s,\gamma),$$

for every $0 < s < \gamma(\alpha, K) \leqslant \alpha$.

In particular, the mapping $a \mapsto D_z\varphi_a(z)$ is continuous and the convergence is locally uniform in z.

Proof. By Proposition 2.1 (c),

$$\|D_z\varphi_{a_n} - D_z\varphi_a\|_{L^p(\mathbb{D}(z_1,2R))} \leqslant c(K, \mathbb{D}(z_1,R))\, |a-b|.$$

Further, we have a $C^{1,\gamma}(\mathbb{D}(z_1,R))$-bound for φ_a and φ_b that is uniform in $|a|$, $|b|$, by combining the Schauder norm estimate (4.2) and Proposition 2.1 (c),

$$\|D_z\varphi_a\|_{C^\gamma(\mathbb{D}(z_1,R))} \leqslant c(\mathcal{H}, \mathbb{D}(z_1,8R))\, \|D_z\varphi_a\|_{L^2(\mathbb{D}(z_1,8R))} \leqslant c(\mathcal{H}, \mathbb{D}(z_1,R))\, |a|.$$

Hence we can use interpolation (Theorem 2.4) to get the claim. □

Next, we state precisely the smoothness assumptions that we need on $\mathcal{H}$.

Definition 4.12. Let $0 < \alpha, \beta < 1$. The field $\mathcal{H}$ is *regular*, if $\mathcal{H}(z,w) \in C^\alpha_{\mathrm{loc}}(z)$ uniformly in w and $D_w\mathcal{H}(z,w) \in (C^\alpha_{\mathrm{loc}}(z), C^\beta_{\mathrm{loc}}(w))$ locally uniformly in z and w. That is, for given disks $\mathbb{D}_r, \mathbb{D}_R \subset \mathbb{C}$ and $z, z_1, z_2 \in \mathbb{D}_r$, $w, w_1, w_2 \in \mathbb{D}_R$

$$|\mathcal{H}(z_1,w) - \mathcal{H}(z_2,w)| \leqslant \mathbf{H}_\alpha(\mathbb{D}_r)\, |z_1 - z_2|^\alpha |w|$$

and

$$\begin{aligned} |D_w\mathcal{H}(z_1,w) - D_w\mathcal{H}(z_2,w)| &\leqslant c(\mathbb{D}_r, \mathbb{D}_R)\, |z_1 - z_2|^\alpha, \\ |D_w\mathcal{H}(z,w_1) - D_w\mathcal{H}(z,w_2)| &\leqslant c(\mathbb{D}_r, \mathbb{D}_R)\, |w_1 - w_2|^\beta. \end{aligned} \tag{4.15}$$

We are now ready to prove the smoothness of $\mathcal{F}_\mathcal{H}$ when $\mathcal{H}$ is a regular field. The regularity of $\mathcal{H}$ is required to guarantee that the Beltrami coefficients (3.3) are Hölder continuous, which enable us to use the classical Schauder estimates.

Theorem 4.13. *Let $\mathcal{H}$ be a regular field and $\mathcal{F}_\mathcal{H} = \{\varphi_a(z)\}_{a\in\mathbb{C}}$. Then* *(a)* $z \mapsto D_z D_a \varphi_a(z) \in C^{\gamma\beta}_{\mathrm{loc}}(\mathbb{C})$, *where* $\gamma = \gamma(\alpha, K) \leqslant \alpha$, *and, moreover,* $\|D_z D_a \varphi_a\|_{C^{\gamma\beta}(\mathbb{D}_r)} \leqslant c(\mathcal{H}, |a|, \mathbb{D}_r)$.

(b) $a \mapsto D_z D_a \varphi_a : \mathbb{C} \to C^s_{\mathrm{loc}}(\mathbb{C})$ *is continuous,* $0 < s < \gamma\beta$.

Remark 4.14. Actually, since $\mathcal{H}(z,w) \in C^1(w)$, $\gamma(\alpha, K) = \alpha$, by Remark 4.7.

Proof. Let us fix a direction $e \in \mathbb{C}$, $|e| = 1$. Given a sequence $a_n \to a$, we denote $f_n = \partial^a_e \varphi_{a_n}$ and $f = \partial^a_e \varphi_a$. We know by Theorems 1.3 and 3.6 that f_n converges to f in the L^∞_{loc}- and in the $W^{1,2}_{\mathrm{loc}}$-topologies and f is a K-quasiconformal solution to an $\mathbb{R}$-linear Beltrami equation

$$\partial_{\bar z} f(z) = \mu_a(z)\, \partial_z f(z) + \nu_a(z)\, \overline{\partial_z f(z)} \qquad \text{a.e.} \tag{4.16}$$

where μ_a and ν_a are given in (3.3). The same is true for f_n, where in (4.16) one has μ_{a_n}, ν_{a_n}.

By the Schauder estimates at Theorem 4.2 and Proposition 2.1 (c) there exists a constant $c = c(\mathcal{H}, \mathbb{D}(z_1, r))$ such that

$$\|D_z \varphi_a\|_{C^\gamma(\mathbb{D}_r))} \leqslant c\, |a| \eta_K(|z_1| + r). \tag{4.17}$$

Further, using (4.15), one gets for $z_1, z_2 \in \mathbb{D}_r$ that

$$\begin{aligned}
|\mu_a(z_1) - \mu_a(z_2)| &= |\partial_w \mathcal{H}(z_1, \partial_z \varphi_a(z_1)) - \partial_w \mathcal{H}(z_2, \partial_z \varphi_a(z_2))| \\
&\leqslant c(\mathbb{D}_r, \mathbb{D}_{\|D_z \varphi_a\|_{L^\infty(\mathbb{D}_r)}}) \left(|z_1 - z_2|^\alpha + |\partial_z \varphi_a(z_1) - \partial_z \varphi_a(z_2)|^\beta \right) \\
&\leqslant c(\mathcal{H}, |a|, \mathbb{D}_r) \left(|z_1 - z_2|^\alpha + [D_z \varphi_a]^\beta_{C^\gamma(\mathbb{D}_r)}]|z_1 - z_2)|^{\gamma\beta} \right) \\
&\leqslant c(\mathcal{H}, |a|, \mathbb{D}_r)\, |z_1 - z_2|^{\gamma\beta}.
\end{aligned} \tag{4.18}$$

Hence $\mu_a \in C^{\gamma\beta}_{\mathrm{loc}}(\mathbb{C})$ (and similarly ν_a, μ_{a_n}, ν_{a_n}) with a norm bound locally uniform in a. Therefore, by the classical Schauder estimates for linear Beltrami equations, [5, Theorem 15.0.6] $D_z f, D_z f_n \in C^{\gamma\beta}_{\mathrm{loc}}(\mathbb{C})$ with a norm estimate $\|D_z f\|_{C^{\gamma\beta}(\mathbb{D}_r)} \leqslant c(\mathcal{H}, |a|, \mathbb{D}_r) \|f\|_{L^2(\mathbb{D}_{2r})}$. Bounding the L^2-norm with Proposition 2.1 (a) proves our statement (a).

To pass from boundedness to continuity we combine again interpolation (Theorem 2.4), continuity in $W^{1,2}_{\mathrm{loc}}(\mathbb{C})$-topology (by Theorem 3.6 (c)) and statement (a) in Theorem 4.13. Namely,

$$\begin{aligned}
\|D_z f_n - D_z f\|_{C^s(\mathbb{D}_r)} &\leqslant c(\theta, \mathbb{D}_r)\, \|D_z f_n - D_z f\|^{1-\theta}_{L^2(\mathbb{D}_{2r})}\, \|D_z f_n - D_z f\|^\theta_{C^{\gamma\beta}(\mathbb{D}_{2r})} \\
&\leqslant c(\theta, \mathbb{D}_r)\, \|D_z f_n - D_z f\|^{1-\theta}_{L^2(\mathbb{D}_{2r})} (\|D_z f_n\|_{C^{\gamma\beta}(\mathbb{D}_{2r})} + \|D_z f\|_{C^{\gamma\beta}(\mathbb{D}_{2r})})^\theta
\end{aligned}$$

and (b) follows. □

Corollary 4.15. *Let* $\mathcal{H}$ *be a regular field and* $\mathcal{F}_{\mathcal{H}} = \{\varphi_a(z)\}_{a \in \mathbb{C}}$. *Then*

(a) *For every* $z \in \mathbb{C}$ *it holds that* $D_a D_z \varphi_a(z) = D_z D_a \varphi_a(z)$.

(b) $D_a D_z \varphi_a(z) \in (C^{\gamma\beta}_{\mathrm{loc}}(z), C(a))$ *locally uniformly in* z *and* a, *where* $\gamma = \gamma(\alpha, K) \leqslant \alpha$.

Proof. In order to prove statement (a), we proceed as follows. Set

$$f_h(z,t) := \frac{(\varphi_{a+he} - \varphi_a)(z + t\tilde{e}) - (\varphi_{a+he} - \varphi_a)(z)}{h\, t}.$$

Since the mapping $a \mapsto \varphi_a$ is differentiable in the L^∞-sense (Theorem 3.6 (a)), the incremental quotients $\frac{\varphi_{a+he}-\varphi_a}{h}$ converge to $\partial_e^a \varphi_a$ locally uniformly in a and z. As a consequence,

$$\lim_{h\to 0^+} f_h(z,t) = \frac{\partial_e^a \varphi_a(z+t\tilde{e}) - \partial_e^a(z)}{t} =: f(z,t)$$

and the convergence is still locally uniform in a, z, and t. Indeed, given a radius $R>0$ and a real number $t>0$, by the proof of Theorem 3.6 we can find for any $\varepsilon>0$ a real number $h_0 = h_0(\varepsilon, R)$ such that

$$|f_h(z,t) - f(z,t)| < \varepsilon \quad \text{whenever } h < h_0, \quad |z| \leqslant \frac{R}{2}, \quad |t| < \frac{R}{2}. \tag{4.19}$$

On the other hand, $z \mapsto \varphi_a(z)$ is also differentiable, locally uniformly in a and z, by the Schauder estimates (Theorem 1.6). Thus the limit

$$\lim_{t\to 0^+} f_h(z,t) = \frac{\partial_{\tilde{e}}^z \varphi_{a+he}(z) - \partial_{\tilde{e}}^z \varphi_a(z)}{h}$$

exists. Similarly, $z \mapsto D_a\varphi_a(z)$ is continuously differentiable (by Theorem 4.13 (a)) and this happens locally uniformly in a and z as well. Thus, again the limit

$$\lim_{t\to 0^+} f(z,t) = \partial_{\tilde{e}}^z(\partial_e^a \varphi_a)(z)$$

exists. Therefore from (4.19) we get that

$$\left| \frac{\partial_{\tilde{e}}^z \varphi_{a+he}(z) - \partial_{\tilde{e}}^z \varphi_a(z)}{h} - \partial_{\tilde{e}}^z(\partial_e^a \varphi_a)(z) \right| \leqslant \varepsilon$$

whenever $h < h_0$ and $|z| \leqslant \frac{R}{2}$. But then we can let $h \to 0^+$ and obtain

$$|\partial_e^a(\partial_{\tilde{e}}^z \varphi_a)(z) - \partial_{\tilde{e}}^z(\partial_e^a \varphi_a)(z)| \leqslant \varepsilon.$$

Since this happens for any $\varepsilon > 0$, we finally obtain

$$\partial_e^a(\partial_{\tilde{e}}^z \varphi_a)(z) = \partial_{\tilde{e}}^z(\partial_e^a \varphi_a)(z)$$

as claimed.

The statement (b) is an easy consequence of (a) together with (a) and (b) of Theorem 4.13. $\square$

5. From $\mathcal{F}$ to $\mathcal{H}$

The main goal of this section is to prove Theorems 1.4 and 1.5 for associated families $\mathcal{F}_{\mathcal{H}}$. It turns out that to obtain a unique and well-defined field $\mathcal{H}_{\mathcal{F}}$ from $\mathcal{F}$, in addition to smoothness properties, we need to $\mathcal{F}$ to satisfy a non-degeneracy condition which we state below.

Definition 5.1. We say that a family $\mathcal{F} = \{\varphi_a(z)\}_{a\in\mathbb{C}}$ of K-quasiconformal maps is *non-degenerate*, if (F1), (F2) hold and the following two conditions are satisfied.

(1) For every disk $\mathbb{D}(0,R)$ there exists a constant $c = c(K,R)$ such that

$$\frac{1}{c} \leqslant \frac{|\partial_z \varphi_a(z)|}{|a|} \leqslant c, \qquad z \in \mathbb{D}(0,R), \quad a \neq 0.$$

(2) For every $z, a \in \mathbb{C}$, $a \neq 0$, one has

$$\det D_a(\partial_z \varphi_a)(z) \neq 0.$$

For non-degenerate and smooth families we can use topology to understand the range of $a \mapsto \partial_z \varphi_a(z)$.

Lemma 5.2. *Let $\mathcal{F} = \{\varphi_a(z)\}_{a \in \mathbb{C}}$ be a non-degenerate family of K-quasiconformal maps. Assume that $\mathcal{F} \in C^1(z)$ and $D_z \mathcal{F} \in C^1(a)$. Then for every $z \in \mathbb{C}$, the mapping*

$$a \mapsto \partial_z \varphi_a(z) =: F_z(a)$$

is a homeomorphism.

Since it will be repeatedly used in this section, we label the inverse of $F_z(a) = \partial_z \varphi_a(z)$.

Definition 5.3. Let $\mathcal{F} = \{\varphi_a(z)\}_{a \in \mathbb{C}}$ be a non-degenerate family of K-quasiconformal maps. Then for each $(z, w) \in \mathbb{C} \times \mathbb{C}$ we denote

$$a(z, w) = F_z^{-1}(w).$$

Now, the proof of Theorem 1.5 immediately follows, if $\mathcal{F}_{\mathcal{H}}$ was non-degenerate for regular $\mathcal{H}$. Fortunately, this is the case.

Proposition 5.4. *Let $\mathcal{H}$ be a regular field. Then the associated family $\mathcal{F}_{\mathcal{H}}$ is non-degenerate.*

We postpone the proofs of Lemma 5.2 and Proposition 5.4 to Sections 5.2 and 5.1.

Proof of Theorem 1.5. We simply notice that for a regular field $\mathcal{H}$, $\mathcal{F}_{\mathcal{H}} = \{\varphi_a(z)\}$ satisfies all requirements in Lemma 5.2 due to Theorem 1.6, Corollary 4.15, and Proposition 5.4. $\square$

In the same way as the fields $\mathcal{H}$ generate a unique family $\mathcal{F}_{\mathcal{H}}$, the smooth families $\mathcal{F}$ determine $\mathcal{H}_{\mathcal{F}}$ uniquely.

Theorem 5.5. *Assume that $\mathcal{F} = \{\varphi_a\}_{a \in \mathbb{C}}$ is a non-degenerate family of K-quasiconformal mappings such that $\mathcal{F} \in C^1(z)$ and $D_a D_z \mathcal{F} \in (C(z), C(a))$. Then there is a unique $\mathcal{H} = \mathcal{H}_{\mathcal{F}}$ satisfying (H1), (H2) such that every member of $\mathcal{F}$ is a K-quasiconformal solution of*

$$\partial_{\bar{z}} \varphi_a(z) = \mathcal{H}(z, \partial_z \varphi_a(z)).$$

We prove the theorem in Section 5.3. Hence, starting from a regular field $\mathcal{H}$, the associated family $\mathcal{F}_{\mathcal{H}}$ defines $\mathcal{H}$ uniquely.

Proof of Theorem 1.4. Since $\mathcal{H}$ is a regular field, we obtain that the associated family $\mathcal{F}_{\mathcal{H}}$ is non-degenerate (Proposition 5.4). Further, $\mathcal{F}_{\mathcal{H}} \in C^{1,\gamma}_{\mathrm{loc}}(z)$ (Theorem 1.6) and $D_a D_z \mathcal{F}_{\mathcal{H}} \in (C^{\gamma\beta}_{\mathrm{loc}}(z), C^s_{\mathrm{loc}}(a))$ (Corollary 4.15 (b)). Hence $\mathcal{F}_{\mathcal{H}}$ defines $\mathcal{H}$ uniquely by Theorem 5.5. $\square$

Remark 5.6. If $\mathcal{F}$ is a smooth non-degenerate family of K-quasiconformal mappings in the sense of Theorem 5.5, which in addition satisfies the ellipticity bound near the infinity (1.3), then the associate $\mathcal{H}_{\mathcal{F}}$ has the uniqueness property by [3, Theorem 1.2] thus the corresponding family $\mathcal{F}_{\mathcal{H}_{\mathcal{F}}}$ must be $\mathcal{F}$.

In the final Section 5.4 we finish the paper by showing how $\mathcal{H}_{\mathcal{F}}$ inherits the regularity of $\mathcal{F}$, i.e., proving Theorem 1.9.

5.1. $\mathcal{F}_{\mathcal{H}}$ **is non-degenerate.** We prove Proposition 5.4 in two steps (Corollary 5.7 and Proposition 5.8). The following is a corollary of Theorems 1.6 and 4.8.

Corollary 5.7. *Let $\mathcal{H}(z,w) \in C^{\alpha}_{\mathrm{loc}}(z)$ uniformly in w and $\mathcal{F}_{\mathcal{H}} = \{\varphi_a(z)\}$. There exists a constant $c = c(K,R) > 0$ such that*

$$\frac{1}{c} \leqslant \frac{|\partial_z \varphi_a(z)|}{|a|} \leqslant c, \qquad z \in \mathbb{D}(0,R). \tag{5.1}$$

In particular, $a \mapsto \partial_z \varphi_a(z)$ admits a continuous extension at the point $a = \infty$.

Proof. The mappings $F = \frac{1}{a}\varphi_a$ are normalized quasiconformal solutions ($0 \mapsto 0$ and $1 \mapsto 1$) to the following nonlinear Beltrami equations

$$\partial_{\bar z} F(z) = \tilde{\mathcal{H}}(z, \partial_z F(z)) \qquad \text{a.e.,}$$

where $\tilde{\mathcal{H}}(z,w) = \frac{1}{a}\mathcal{H}(z,aw)$. Clearly $\tilde{\mathcal{H}}$ satisfies (H1) and (H2). Moreover, if $z_1, z_2 \in \Omega'$, the Hölder bounds for $\mathcal{H}$ give us that

$$\left|\tilde{\mathcal{H}}(z,w) - \tilde{\mathcal{H}}(z_2,w)\right| \leqslant \frac{\mathbf{H}_\alpha(\Omega')}{|a|}\,|z_1 - z_2|^\alpha\,|a\,w| = \mathbf{H}_\alpha(\Omega')\,|z_1 - z_2|^\alpha |w|.$$

In particular the Hölder constant of $\tilde{\mathcal{H}}$ does not depend on a. Thus, first by Theorem 4.2 and Proposition 2.1 (c) (with $b = 0$) we get an upper bound

$$\|D_z F\|_{C^\gamma(\mathbb{D}(0,R))} \leqslant c\,\|D_z F\|_{L^2(\mathbb{D}(0,8R))} \leqslant c,$$

where $c = c(\mathcal{H}, R)$. Since α, γ, and the field are fixed and $K = \frac{1+k}{1-k}$, the upper bound at (5.1) follows. For the lower bound, we use Theorem 4.8 and get

$$(1-k)\left|\frac{\partial_z \varphi_a}{a}\right| \geqslant J\left(z, \frac{\varphi_a}{a}\right) \geqslant c(\mathcal{H}, R) > 0,$$

for every $z \in \mathbb{D}(0,R)$, which proves (5.1).

Finally By Lemma 4.11, $a \mapsto \partial_z \varphi_a(z)$ is continuous in $\mathbb{C}$, uniformly on compact subsets of z. The estimates (5.1) yield that $\lim_{|a|\to\infty} |\partial_z \varphi_a(z)| = \infty$ for every $z \in \mathbb{C}$. $\square$

We have proved the first non-degeneracy condition (Definition 5.1). We are left to show the non-vanishing of the Jacobian.

Note that the mapping $a \mapsto \partial_z \varphi_a(z)$, even being continuously differentiable, might still have a vanishing Jacobian for some a and z as, for instance, $\varphi_a(z) = a\,z|z|^2$. However, the fact that the partial derivatives are solutions to the linear Beltrami equation with Hölder continuous coefficients prevents this kind of behaviour. We find very interesting this surprising application of the nonvanishing of new null Lagrangians, a recent theme of research (see, e.g., [9], [5], [1], [7], [19]).

Proposition 5.8 (Proposition 1.7)**.** *Let $\mathcal{H}$ be a regular field and $\mathcal{F}_{\mathcal{H}} = \{\varphi_a(z)\}$. Then, for $a \neq 0$, $z \in \mathbb{C}$,*

$$|\det[D_a \partial_z \varphi_a(z)]| \geqslant c(\mathcal{H}, |a|, \mathbb{D}_r) > 0,$$

and the determinant does not change sign.

Proof. Let

$$\begin{aligned} f(z) &:= \partial_1^a \varphi_a(z) = \partial_a \varphi_a(z) + \partial_{\bar a} \varphi_a(z) \\ g(z) &:= \partial_i^a \varphi_a(z) = -i\big(\partial_{\bar a} \varphi_a(z) - \partial_a \varphi_a(z)\big). \end{aligned}$$

Then

$$\operatorname{Im}\big(\partial_z f \, \overline{\partial_z g}\big) = |\partial_a \partial_z \varphi_a|^2 - |\partial_{\bar a} \partial_z \varphi_a|^2 = \det[D_a \partial_z \varphi_a(z)],$$

since we can exchange the order of differentiation by Corollary 4.15 (a).

Now, f and g solve the $\mathbb{R}$-linear Beltrami equation (1.4) by Theorem 1.3, and from (4.18) we know that $\|\mu_a\|_{\mathbb{C}^{\gamma\beta}(\mathbb{D}_r)} + \|\nu_a\|_{\mathbb{C}^{\gamma\beta}(\mathbb{D}_r)} \leqslant c(\mathcal{H}, |a|, \mathbb{D}_r)$. As a consequence, by [9, Proposition 5.1], we obtain

$$\operatorname{Im}(\partial_z f(z) \, \overline{\partial_z g(z)}) \neq 0 \qquad \text{everywhere}$$

and it does not change sign, [9, Lemma 7.1.] or [5, Theorem 6.3.2].

Actually, the argument from [9, Lemma 7.1.] from can easily be made quantitative. Let $(t,s) \in S^1$, then $|\operatorname{Im}(\partial_z f(z) \, \overline{\partial_z g(z)})|^2$ is the discriminant of the quadratic form $|t \, \partial_z f(z) + s \, \partial_z g(z)|^2$. We have for the first eigenvalue λ_1 that

$$\iota := \inf_{(t,s)\in S^1} \frac{|t \, \partial_z f(z) + s \, \partial_z g(z)|^2}{t^2 + s^2} = \lambda_1$$

and thus

$$|\operatorname{Im}(\partial_z f(z) \, \overline{\partial_z g(z)})|^2 = 4 \left((|\partial_z f(z)|^2 + |\partial_z g(z)|^2)\iota - \iota^2 \right).$$

As in cite [9, Proposition 5.1] is that map $(t\, f + s\, g)^{-1}$ solves also an $\mathbb{R}$-linear Beltrami equation for every t, s with Hölder continuous coefficients. Moreover it is easy to theck that and, in our case, the Hölder norm of the coefficients is bounded by $c(\mathcal{H}, |a|, \mathbb{D}_r)$.

Now, from our lower bound for the Jacobian (Theorem 1.8) or the classical Schauder estimates, one bounds ι from below by the Hölder norm of the coefficients and the L^2-norm of the solution and hence

$$|\operatorname{Im}(\partial_z f(z) \, \overline{\partial_z g(z)})| \geqslant c(\mathcal{H}, |a|, \mathbb{D}_r).$$

□

Remark 5.9. If we strengthen our assumptions by a condition of the differential quotients like

$$\begin{aligned} &\left| \frac{\mathcal{H}(z_1, w_1 + h_1) - \mathcal{H}(z_1, w_1)}{h_1} - \frac{\mathcal{H}(z_2, w_2 + h_2) - \mathcal{H}(z_2, w_2)}{h_2} \right| \\ &\qquad \leqslant c(\mathbb{D}_r) \left(|z_1 - z_2|^\alpha + |w_1 - w_2|^\beta \right) \end{aligned} \tag{5.2}$$

for every $z_1, z_2 \in \mathbb{D}_r \subset \mathbb{C}$ and $w_1, w_2 \in \mathbb{C}$, $h_1, h_2 \in \mathbb{C} \setminus \{0\}$, then it actually holds that

$$\frac{1}{c_a(K, \mathbb{D}_r)} \leqslant \left| \frac{\partial_z \varphi_a(z) - \partial_z \varphi_b(z)}{a - b} \right| \leqslant c_a(K, \mathbb{D}_r), \qquad z \in \mathbb{D}_r, \; a \neq b.$$

and thus $a \mapsto \partial_z \varphi_a(z)$ is a local homeomorphism in $\hat{\mathbb{C}}$ and thus a global homeomorphism.

5.2. **Global homeomorphism.** The key observation is the following topological result, whose proof we included for completness.

Theorem 5.10. *Let $h : \mathbb{C} \to \mathbb{C}$ be continuous and locally injective map. Assume that*

$$\lim_{|z|\to\infty} |h(z)| = \infty. \tag{5.3}$$

Then $h : \hat{\mathbb{C}} \mapsto \hat{\mathbb{C}}$ is a global homeomorphism.

Proof. We will prove that $h : \mathbb{C} \to \mathbb{C}$ is a covering map. Then, by [22, Theorem 54.4], it is a global homeomorphism.

Step 1. Locally homeomorphic. By assumption, $h : \mathbb{C} \to \mathbb{C}$ is a local injection, thus it is locally homeomorphic, by the invariance of the domain (Brouwer's fixed point theorem).

Step 2. Surjective. We will prove that $h : \mathbb{C} \to \mathbb{C}$ is a surjection. By Step 1, $h : \mathbb{C} \to \mathbb{C}$ is locally homeomorphic. In particular, h is an open mapping. So $h(\mathbb{C})$ is an open subset of $\mathbb{C}$. On the other hand, clearly h has a continuous extension on the Riemann sphere $\hat{\mathbb{C}}$, and therefore it sends compact subsets to compact subsets. In particular, $h(\hat{\mathbb{C}})$ is a compact subset of $\hat{\mathbb{C}}$. However,

$$h(\hat{\mathbb{C}}) = h(\mathbb{C}) \cup \{h(\infty)\} = h(\mathbb{C}) \cup \{\infty\}.$$

As a compact set on the sphere, $h(\hat{\mathbb{C}})$ is closed on the sphere, and so it consists of the point at infinity together with a closed subset of the plane. It follows that $h(\mathbb{C})$ is both open and closed on the plane. As a consequence, $h : \mathbb{C} \to \mathbb{C}$ is onto.

To prove that $h : \mathbb{C} \to \mathbb{C}$ is a covering map, we are left to show the existence of the so-called evenly covered neighbourhood E_{ω_0}. We do this in three steps.

Step 3. Closed. Map h is closed. Indeed, as $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is continuous, it sends compact sets to compact sets. So if F is a closed set on $\mathbb{C}$, then $h(F \cup \{\infty\})$ is a compact set on $\hat{\mathbb{C}}$. However, $h(F \cup \{\infty\}) = h(F) \cup h(\{\infty\}) = h(F) \cup \{\infty\}$, and moreover this union is disjoint. Thus the only option is that $h(F)$ is closed in $\mathbb{C}$.

Step 4. Pre-image of every point is finite. Indeed $h^{-1}(\omega_0)$ is finite as it is a discrete (since h is local homeomorphism, by Step 1) and bounded (by (5.3)).

Step 5. Evenly covered neighbourhood. Now, we will define the evenly covered neighbourhood E_{ω_0} for given $\omega_0 \in \mathbb{C}$. We have shown in Step 4 that the pre-image $h^{-1}(\omega_0)$ is finite, say, $h^{-1}(\omega_0) = \{z_1, \ldots, z_n\}$. Since h is a local homeomorphism (Step 1), there are pairwise disjoint open neighbourhoods U_j of the z_j such that for each j, $h|_{U_j} : U_j \to h(U_j)$ is a homeomorphism to an open subset of $\mathbb{C}$ containing ω_0.

Let $E_{\omega_0} = \Big(\bigcap_{j=1}^n h(U_j)\Big) \setminus \Big(h\big[h^{-1}\big(\overline{\bigcap_{j=1}^n h(U_j)}\big) \setminus \bigcup_{j=1}^n U_j)\big]\Big)$. Then E_{ω_0} is the required evenly covered open neighbourhood of ω_0. Indeed, E_{ω_0} is open, since h is a continuous open and closed map. And $\omega_0 \in E_{\omega_0}$, because $\bigcup_{j=1}^n U_j$ contains all pre-images of ω_0. Finally, by construction, $h^{-1}(E_{\omega_0})$ is the disjoint union of the $h^{-1}(E_{\omega_0}) \cap U_j$, each of which is homeomorphically mapped onto E_{ω_0} by h.

We have shown that $h : \mathbb{C} \to \mathbb{C}$ is a covering map. Alternatively, by Steps 1, 3, and 4, h is a proper local homeomorphism. Thus it is a global homeomorphism, by [17]. $\square$

The exponential map $h(z) = e^z$ shows that the continuity at ∞ is crucial.

Proof of Lemma 5.2. From $\mathcal{F} \in C^1(z)$ and $D_z\mathcal{F} \in C^1(a)$, we know that $a \mapsto \partial_z\varphi_a(z)$ is well-defined and continuously differentiable, for every fixed $z \in \mathbb{C}$. On the other hand, by the first condition of the non-degeneracy (Definition 5.1)

$$\frac{1}{c(K,R)} \leqslant \frac{|\partial_z\varphi_a(z)|}{|a|} \leqslant c(K,R).$$

The second condition of the non-degeneracy guarantees the local injectivity on $\mathbb{C}$, by the Inverse Function Theorem. We are thus in the situation of Theorem 5.10, and so the claim follows. $\square$

5.3. **Construction of $\mathcal{H}_\mathcal{F}$.** Under the assumptions of Theorem 5.5, the existence, uniqueness, and regularity properties of $\mathcal{H}_\mathcal{F}$ depend on those of the function a introduced in Definition 5.3. Namely given $a(z,w)$, we set $\mathcal{H} = \mathcal{H}_\mathcal{F}$ as follows,

$$\mathcal{H}(z,w) = \partial_{\bar z}\varphi_a(z)|_{a=a(z,w)}. \tag{5.4}$$

Equivalently, $(z,w) \mapsto \mathcal{H}(z,w)$ is defined as the composition of

$$(z,w) \mapsto a(z,w) \qquad \text{with} \qquad a \mapsto \partial_{\bar z}\varphi_a(z), \tag{5.5}$$

thus it holds that

$$\mathcal{H}(z, \partial_z\varphi_a(z)) = \partial_{\bar z}\varphi_a(z).$$

We first show the continuity of $a(z,w)$ in both variables.

Lemma 5.11. *Let $\mathcal{F}$ be a non-degenerate family of quasiconformal maps such that $\mathcal{F} \in C^1(z)$ and $D_aD_z\mathcal{F} \in (C(z), C(a))$. Then*

(a) $w \mapsto a(z,w)$ is continuous and

(b) $z \mapsto a(z,w)$ is continuous.

Proof. We know, by Lemma 5.2, that $a \mapsto F_z(a) = \partial_z\varphi_a(z)$ is a homeomorphism. Since $a(z,w) = F_z^{-1}(w)$, (a) follows.

Next, we will prove (b). Fix $z_0, w \in \mathbb{C}$. Take a sequence $z_n \to z_0$, $z_n \in \mathbb{D}(z_0, r)$. By Lemma 5.2, there is a_0 and a_n such that $F_{z_0}(a_0) = \partial_z\varphi_{a_0}(z_0) = w$ and $F_{z_n}(a_n) = \partial_z\varphi_{a_n}(z_n) = w$. Now,

$$|a_0 - a_n| = |F_{z_n}^{-1} \circ F_{z_n} \circ F_{z_0}^{-1}(w) - F_{z_n}^{-1}(w)| =: |F_{z_n}^{-1}(h_1) - F_{z_n}^{-1}(h_2)|. \tag{5.6}$$

Further,

$$|h_1 - h_2| = |F_{z_n}(a_0) - F_{z_0}(a_0)| = |\partial_z\varphi_{a_0}(z_n) - \partial_z\varphi_{a_0}(z_0)| \to 0, \tag{5.7}$$

when $n \to \infty$, because, for fixed $a_0 \in \mathbb{C}$, $\varphi_{a_0} \in C^1(z)$ by assumption. So we are reduced to study the modulus of continuity of $F_{z_n}^{-1}$. To this end, we will show that the Jacobian of F_{z_n} with respect to a has a positive lower bound

c_1 that does not depend on z_n. Then $F_{z_n}^{-1}$ is $\frac{1}{c_1}$-Lipschitz. Thus we have by Lipschitz estimate in (5.6) and by (5.7)

$$\lim_{n\to\infty} |a_0 - a_n| \leqslant \lim_{n\to\infty} \left(\frac{1}{c_1} |h_1 - h_2| \right) = 0$$

and the continuity of $z \mapsto a(z,w)$ follows

We are left to show the lower bound for the Jacobian of F_{z_n}. First, by 1. condition of non-degeneracy (Definition 5.1),

$$|a_n| \leqslant c(K,|z_0|,r)\,|F_{z_n}(a_n)| = c(K,|z_0|,r)\,|w|. \tag{5.8}$$

Hence we can study the situation locally $a_n \in \overline{\mathbb{D}\,(0, c(K,|z_0|,r)\,|w|)}$.

The Jacobian $J(a, F_{z_0}) \neq 0$ due to the 2. condition of non-degeneracy. As $F_{z_0} \in C^1(\mathbb{C})$, by assumption $D_a D_z \mathcal{F} \in C(a)$, the Jacobian is continuous and we can assume without loss of generality $J(a, F_{z_0}) > 0$.

Now, by the continuity of the map $a \mapsto J(a, F_{z_0})$ and a compactness argument,

$$J(a, F_{z_0}) \geqslant c_2 = c_2(|w|, c(K,|z_0|,r)), \qquad a \in \overline{\mathbb{D}\,(0, c(K,|z_0|,r)\,|w|)}.$$

Further, $z \mapsto J(a, F_z)$ is continuous, because $D_a D_z \mathcal{F} \in C(z)$ by assumption. Hence, when r is small enough and $|z_n - z_0| < r$, then $J(a, F_{z_n}) \geqslant \frac{c_2}{2} = c_1 > 0$. We have shown our claim. □

Proof of Theorem 5.5. By assumption $\mathcal{F} \in C^1(z)$, $\partial_{\bar z}\varphi_a(z)$ exists at every point z and a. Hence the bijectivity of $w \mapsto a(z,w)$ in Definition 5.3 (which comes from Lemma 5.2) guarantees that the function $w \mapsto \mathcal{H}(z,w)$ given by (5.4), i.e., $\mathcal{H}(z,w) = \partial_{\bar z}\varphi_{a(z,w)}(z)$, is well-defined for every $(z,w) \in \mathbb{C} \times \mathbb{C}$. Furthermore, by (F2),

$$|\partial_{\bar z}\varphi_a(z) - \partial_{\bar z}\varphi_b(z)| \leqslant \frac{K-1}{K+1} |\partial_z\varphi_a(z) - \partial_z\varphi_b(z)|$$

and, since $z \mapsto D_z\varphi_a(z)$ is continuous, by assumption $\mathcal{F} \in C^1(z)$, the inequality holds at every point z. Thus, for fixed z, we get straight from the definition (5.4)

$$\begin{aligned}
|\mathcal{H}(z,w_1) - \mathcal{H}(z,w_2)| &= |\partial_{\bar z}\varphi_{a(z,w_1)}(z) - \partial_{\bar z}\varphi_{a(z,w_2)}(z)| \\
&\leqslant \frac{K-1}{K+1} |\partial_z\varphi_{a(z,w_1)}(z) - \partial_z\varphi_{a(z,w_2)}(z)| \\
&= \frac{K-1}{K+1} |w_1 - w_2|
\end{aligned}$$

and $w \mapsto \mathcal{H}(z,w)$ is $k(z)$-Lipschitz for every z, with $\|k\|_\infty \leqslant \frac{K-1}{K+1}$. Also, $\mathcal{H}(z,0) \equiv 0$ by (5.4), since $\varphi_0 \equiv 0$. Hence (H1) holds.

To see (H2), we recall that $\mathcal{H}$ has been defined in (5.5) as the composition of two continuous functions, since $z \mapsto a(z,w)$ is continuous, by Lemma 5.11, and $a \mapsto \partial_{\bar z}\varphi_a(z)$ is continuous, by assumption $D_a D_z \mathcal{F} \in C(a)$. As a continuous function $z \mapsto \mathcal{H}(z,w)$ is, in particular, measurable. □

5.4. **Regularity of $\mathcal{H}_{\mathcal{F}}$.** We study the smoothness of $\mathcal{H}_{\mathcal{F}}$. Looking at how we define $\mathcal{H}_{\mathcal{F}}$ at (5.5), it is clear that we first need to understand the smoothness of $(z,w) \mapsto a(z,w)$.

Lemma 5.12. *Let $\mathcal{F} = \{\varphi_a(z)\}_{a\in\mathbb{C}}$ be a non-degenerate family of K-quasiconformal mappings such that for some $s \in (0,1)$*

$$\|D_z\varphi_a\|_{C^s(\mathbb{D}_r)} \leqslant c(\mathbb{D}_r)\,|a|,$$

and $D_aD_z\mathcal{F} \in (C(z), C(a))$ locally uniformly in z and a. Then

(a) $z \mapsto a(z,w) \in C^s_{\mathrm{loc}}(\mathbb{C})$ locally uniformly in w and

(b) $D_w a(z,w) \in (C(z), C(w))$ locally uniformly in z and w.

Proof. We start by proving (a). Let us denote

$$F(z,a,w) = \partial_z\varphi_a(z) - w.$$

Fix a disk $\mathbb{D}_r \subset \mathbb{C}$ and two points $z_0, z_1 \in \mathbb{D}_r$.

Notice that by (1.7)

$$\begin{aligned}\frac{|F(z_1,a(z_1,w),w) - F(z_0,a(z_1,w),w)|}{|z_1-z_0|^s} &\leqslant \|D_z\varphi_a\|_{C^s(\mathbb{D}_r)} \\ &\leqslant c(\mathbb{D}_r)\,|a(z_1,w)|.\end{aligned} \tag{5.9}$$

Since $D_a((\partial_z\varphi_a)(z)) \neq 0$ and it is continuous with respect to z and a, by assumption, and $z \mapsto a(z,w)$ is continuous (Lemma 5.11), we get

$$\inf_{z\in\mathbb{D}_r, |\xi|=1} |D_a((\partial_z\varphi_a)(z))|_{a(z,w)}\,\xi| \geqslant c(w,\mathbb{D}_r) > 0.$$

Further, as $a(z,w)$ is continuous on w (Lemma 5.11), the infimum stays bounded for $z_0, z_1 \in \mathbb{D}_r$ and w in a compact set.

Thus, by the mean-value theorem,

$$\begin{aligned}&\frac{|F(z_0,a(z_0,w),w) - F(z_0,a(z_1,w),w)|}{|a(z_0,w)-a(z_1,w)|} \\ &\qquad \geqslant \inf_{z\in\mathbb{D}_r,|\xi|=1} |D_a((\partial_z\varphi_a)(z))|_{a(z,w)}\,\xi| \geqslant c(w,\mathbb{D}_r)\end{aligned} \tag{5.10}$$

locally uniformly in w.

Then, for each $w \in \mathbb{C}$

$$F(z_1,a(z_1,w),w) = F(z_0,a(z_0,w),w) = 0.$$

Hence

$$\begin{aligned}&\frac{|F(z_1,a(z_1,w),w) - F(z_0,a(z_1,w),w)|}{|z_1-z_0|^s} \\ &\qquad = \frac{|F(z_0,a(z_0,w),w) - F(z_0,a(z_1,w),w)|}{|a(z_0,w)-a(z_1,w)|}\,\frac{|a(z_0,w)-a(z_1,w)|}{|z_0-z_1|^s}\end{aligned}$$

and we have an upper bound by (5.9), non-degeneracy (as in (5.8) it implies that $|a(z,w)| \leq C(r)|w|$), and (5.10)

$$\frac{|a(z_0,w)-a(z_1,w)|}{|z_0-z_1|^s} \leqslant \frac{c(\mathbb{D}_r)\,|a(z_1,w)|}{c(w,r)} \leqslant C(w,\mathbb{D}_r)$$

locally uniformly in w.

Next we show (b). By assumption $a \mapsto \partial_z\varphi_a(z) \in C^1(\mathbb{C})$ locally uniformly in z. Moreover, from the second condition of non-degeneracy (Definition 5.1), we know that $\det D_a(\partial_z\varphi_a)(z) \neq 0$ at every point a, for every $z \in \mathbb{C}$. Thus, the Inverse Function Theorem asserts that $a \mapsto w = \partial_z\varphi_a(z)$ has a C^1 local inverse in a neighbourhood of every point. However, the (global) inverse is precisely $w \mapsto a = a(z,w)$. Therefore $D_w a(z,w) \in C(w)$ locally uniformly in z.

Moreover, the chain rule holds and thus from

$$\partial_z\varphi_a(z)|_{a=a(z,w)} = w$$

we get that

$$D_w a(z,w) = \Big(D_a(\partial_z\varphi_a)(z)|_{a=a(z,w)} \Big)^{-1}. \tag{5.11}$$

Now,

$$\begin{aligned}
&\left|D_a(\partial_z\varphi_a)(z_1)|_{a=a(z_1,w)} - D_a(\partial_z\varphi_a)(z_2)|_{a=a(z_2,w)}\right| \\
&\quad \leqslant \left|D_a(\partial_z\varphi_a)(z_1)|_{a=a(z_1,w)} - D_a(\partial_z\varphi_a)(z_2)|_{a=a(z_1,w)}\right| \\
&\qquad + \left|D_a(\partial_z\varphi_a)(z_2)|_{a=a(z_1,w)} - D_a(\partial_z\varphi_a)(z_2)|_{a=a(z_2,w)}\right|.
\end{aligned}$$

By assumption $D_a(\partial_z\varphi_a) \in C(z)$ locally uniformly in a and hence locally uniformly in w, since by the first condition of non-degeneracy (Definition 5.1)

$$|a(z,w)| \leqslant c(\mathbb{D}_r)|w|.$$

Further, as $D_a(\partial_z\varphi_a) \in C(a)$ (by assumption) and $a \in C^s_{\mathrm{loc}}(z)$ locally uniformly in w (by (a)), we get $D_a(\partial_z\varphi_a)(z)|_{a=a(z,w)} \in C(z)$ locally uniformly in w.

As $D_a(\partial_z\varphi_a)(z)|_{a=a(z,w)}$ is nonzero by the non-degeneracy, it thus follows from (5.11) that $D_w a(z,w) \in C(z)$ locally uniformly in w. □

Proof of Theorem 1.9. By Theorem 5.5 we have a unique $\mathcal{H}$ that satisfies (H1) and (H2) and every map in $\mathcal{F}$ is a solution to the nonlinear Beltrami equation defined by $\mathcal{H}$.

We are left to show the regularity of $\mathcal{H}$. Let $z, z_1, z_2 \in \mathbb{D}_r$ and $w, w_1, w_2 \in \mathbb{D}_R$. From the definition of $\mathcal{H}$, (5.4),

$$\begin{aligned}
|\mathcal{H}(z_1,w) - \mathcal{H}(z_2,w)| &= |\partial_{\bar z}\varphi_{a(z_1,w)}(z_1) - \partial_{\bar z}\varphi_{a(z_2,w)}(z_2)| \\
&\leqslant |\partial_{\bar z}\varphi_{a(z_1,w)}(z_1) - \partial_{\bar z}\varphi_{a(z_2,w)}(z_1)| + |\partial_{\bar z}\varphi_{a(z_2,w)}(z_1) - \partial_{\bar z}\varphi_{a(z_2,w)}(z_2)| \\
&\leqslant c(\mathbb{D}_r,\mathbb{D}_R)|a(z_1,w) - a(z_2,w)| + c(\mathbb{D}_r)|z_1 - z_2|^s|a(z_2,w)| \\
&\leqslant c(\mathbb{D}_r,\mathbb{D}_R)|z_1 - z_2|^s,
\end{aligned}$$

i.e., $\mathcal{H}(z,w) \in C^s_{\mathrm{loc}}(z)$ locally uniformly in w. The second to the last inequality follows as $D_z\varphi_a \in C^1(a)$ (and thus in particularly locally Lipschitz) and by (1.7). In the last inequality we use Lemma 5.12 (a) and the non-degeneracy as in (5.8).

Similarly for $D_w\mathcal{H}(z,w)$

$$\begin{aligned}
&|D_w\mathcal{H}(z,w_1) - D_w\mathcal{H}(z,w_2)| \\
&\quad = |(D_a\partial_{\bar z}\varphi_a)|_{a(z,w_1)}\, D_w a(z,w_1) - (D_a\partial_{\bar z}\varphi_a)|_{a(z,w_2)}\, D_w a(z,w_2)| \\
&\quad \leqslant |(D_a\partial_{\bar z}\varphi_a)|_{a(z,w_1)} - (D_a\partial_{\bar z}\varphi_a)|_{a(z,w_2)}|\,|D_w a(z,w_1)| \\
&\qquad + |(D_a\partial_{\bar z}\varphi_a)|_{a(z,w_2)}|\,|D_w a(z,w_1) - D_w a(z,w_2)|
\end{aligned}$$

and

$$\begin{aligned}
&|D_w\mathcal{H}(z_1,w) - D_w\mathcal{H}(z_2,w)| \\
&\quad = |(D_a\partial_{\bar z}\varphi_a)|_{a(z_1,w)}\, D_w a(z_1,w) - (D_a\partial_{\bar z}\varphi_a)|_{a(z_2,w)}\, D_w a(z_2,w)| \\
&\quad \leqslant |(D_a\partial_{\bar z}\varphi_a)|_{a(z_1,w)} - (D_a\partial_{\bar z}\varphi_a)|_{a(z_2,w)}|\, |D_w a(z_1,w)| \\
&\qquad + |(D_a\partial_{\bar z}\varphi_a)|_{a(z_2,w)}|\, |D_w a(z_1,w) - D_w a(z_2,w)|.
\end{aligned}$$

Thus $D_w\mathcal{H}(z,w) \in (C(z), C(w))$ locally uniformly in z and w, since we have $D_aD_z\varphi_a(z) \in (C(z), C(a))$ (by assumption), $D_w a(z,w) \in (C(z), C(w))$ (Lemma 5.12 (b)), and $a(z,w) \in C(z)$ (Lemma 5.11 or Lemma 5.12 (a)). □

Remark 5.13. As we have discussed several times in the paper, in absence of regularity, one can start with a general family of quasiconformal mappings and define an equation $\mathcal{H}_\mathcal{F}$. Unfortunately with no further assumptions (H1) and (H2) do not need to hold. As for the ellipticity, by quasiconformality it holds that, for fixed a, b, there is a set of measure zero $N_{a,b}$ such that

$$|\partial_{\bar z}\varphi_a(z) - \partial_{\bar z}\varphi_b(z)| \leqslant k|\partial_z\varphi_a(z) - \partial_z\varphi_b(z)| \qquad \text{for every } z \in \mathbb{C} \setminus N_{a,b}.$$

Thus, for fixed w_1, w_2, the corresponding $\mathcal{H}_\mathcal{F}$ would be only elliptic up to measure zero depending on w_1, w_2. The measurability is also not clear unless one assumes extra conditions. It would suffice, for example, to prove that $a(z,w)$ is measurable and satisfies the Lusin condition N^{-1} in order that the composition $\mathcal{H}(z,w) = \partial_{\bar z}\varphi_a(z)|_{a=a(z,w)}$ is measurable [5, page 73] or to assume joint measurability properties of the mapping $(z,a) \mapsto D_z\varphi_a(z)$ but we will not pursue this issue here.

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