A Note on Transport Equation in Quasiconformally Invariant Spaces

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Abstract In this note, we study the well-posedness of the Cauchy problem for the transport equation in the BMO space and certain Triebel-Lizorkin spaces.

1 Introduction

In fluid mechanics, the Euler equation

$$\begin{cases} \dfrac{dv}{dt} + v \cdot \nabla v = 0, \\ \operatorname{div}(v) = 0 \end{cases}$$

describes the motion of an incompressible, inviscid fluid with velocity $v : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ whose initial state $v(0,\cdot) = v_0$ is given. When $n = 2$, one can reformulate the system in scalar terms. Namely, one uses the *vorticity* $\omega : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$, which is the scalar curl of $v = (v_1, v_2)$,

$$\omega = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}.$$

The Biot-Savart law then makes it possible to recover $v(t,\cdot)$ from the vorticity $\omega(t,\cdot)$ by means of the convolution with the complex valued kernel $\frac{i}{2\pi\,\bar{z}}$. Moreover, one obtains for ω the following equation,

$$(1) \qquad \frac{d\omega}{dt} + v \cdot \nabla \omega = 0, \qquad \text{where } v(t,\cdot) = \frac{i}{2\pi\bar{z}} * \omega(t,\cdot),$$

together with the initial condition $\omega(0,\cdot) = \frac{i}{2\pi\bar{z}} * v(0,\cdot)$. This can be seen as a *scalar* transport equation for ω, still nonlinear because the velocity field v depends on the unknown ω. Under the assumption $\omega_0 \in L^1 \cap L^\infty$, Yudovich proved global existence and uniqueness of solutions $\omega \in L^\infty(0,T;L^\infty)$ (cf. [Yu63],[Yu95],[MB02, Chapter 8]). In the recent years, there has been many attempts to understand the case of unbounded vorticities. Particular attention is devoted to spaces that stay close to BMO, the space of functions of *bounded mean oscillation*. This space arises naturally since it contains the image of L^∞ under

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any Calderón-Zygmund singular integral operator. Notice that, according to Bourgain-Li [BL15], (1) is strongly ill-posed in the borderline space $W^{1,2}(\mathbb{R}^2)$, while the equation (1) is not completely understood in BMO. Recently Bernicot-Keraani [BK14] extended the well-posedness of (1) to a sub-class of BMO, which in particular contains unbounded vorticities; see also [BH14].

Scalar nonlinear transport equations do not only arise from the Euler equation. Other examples include the surface quasigeostrophic equation, and the aggregation equation. The general model is

$$\begin{cases} \dfrac{du}{dt} + b_u \cdot \nabla u = 0 \\ u(0,\cdot) = u_0 \end{cases} \tag{2}$$

with unknown $u : [0,T] \times \mathbb{R}^n \to \mathbb{R}$. The nonlinearity comes from the velocity field $b = b_u$, which may depend on the unknown u.

To study the nonlinear Cauchy problem (2), one of the methods is to first deal with the corresponding linear problem, i.e., b is independent of u. For example, in the Euler equation, one can first find a suitable condition on b to solve the linear case and then use the explicit formula of b in terms of the solution u to play the compactness argument; see [BK14] for instance.

Our central problem here is to find suitable conditions on the vector field b to solve the Cauchy problems for the linear transport equation with initial value in BMO. In the case of bounded u_0, these problems were successfully treated with the DiPerna-Lions scheme (cf. [DPL89]) and the notion of renormalized solution, as well as the more recent extensions by Ambrosio (cf. [Am04]) in the bounded variation setting. In both approaches, the starting point is the classical Cauchy-Lipschitz theory, which allows to write the solution $u = u(t,x)$ of

$$\begin{cases} \dfrac{d}{dt} u - b \cdot \nabla u = 0 \\ u(0,\cdot) = u_0 \end{cases} \tag{3}$$

as the composition

$$u(t,x) = u_0 \circ \phi_t(x) \tag{4}$$

where $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ is the flow generated by the velocity field b,

$$\begin{cases} \dfrac{d}{dt}\phi_t(x) = b(t,\phi_t(x)), \\ \phi_0(x) = x, \end{cases} \tag{5}$$

at least for smooth enough b. Towards finding explicit solutions $u \in L^\infty(0,T;\mathrm{BMO})$ of the problem (3) for a given $u_0 \in \mathrm{BMO}$, there are two things to be analyzed. First, describing the class $\mathcal{Q}$ of homeomorphisms ϕ_t under which (4) defines a bounded operator in BMO. Second, describing the class of velocity fields b such that (5) has a solution ϕ_t that falls into $\mathcal{Q}$. Both questions were analyzed by Reimann [Re74, Re76] in the 70's. In the first case (cf. [Re74]), quasiconformality was found to be the fundamental notion. In the second (cf. [Re76]), uniform bounds for the anticonformal part of Db were proven to be enough.

The novelty of this work is to apply some known results from quasiconformal theory to the transport equation with initial data in quasiconformally invariant spaces.

For a vector field $b:[0,T]\times\mathbb{R}^n\to\mathbb{R}^n$ being such that $b\in L^1(0,T;W^{1,1}_{\mathrm{loc}})$, let Db be the gradient matrix of b and

$$S_Ab := \frac{1}{2}(Db+Db^t)-\frac{\operatorname{div} b}{n}\, I_{n\times n}$$

the anticonformal part of Db. Let's mention that if $S_Ab(t,\cdot)$ is in $L^\infty(\mathbb{R}^n)$ then $b(t,\cdot)$ is in the Zygmund class [Re76]. Our main result is the following.

Theorem 1. *Let $b:[0,T]\times\mathbb{R}^n\to\mathbb{R}^n$ be such that $b\in L^1(0,T;W^{1,1}_{\mathrm{loc}})$ and*

$$\frac{b(t,x)}{1+|x|\log^+|x|}\in L^1(0,T;L^\infty). \tag{6}$$

If $S_Ab\in L^1(0,T;L^\infty)$, then for each $u_0\in \mathrm{BMO}$, the problem (3) admits a unique weak solution $u\in L^\infty(0,T;\mathrm{BMO})$. Moreover, for each $t\in(0,T]$, it holds

$$\|u\|_{L^\infty(0,T;\mathrm{BMO})}\le C(T,b)\|u_0\|_{\mathrm{BMO}}.$$

The proof of existence is based on the fact that (5) can be found a unique solution ϕ_t consisting of quasiconformal mappings, which preserve BMO by composition. Indeed, it is precisely the assumption $S_Ab\in L^1(0,T;L^\infty)$ what allows for a classical compactness argument in $\mathcal{Q}$. Uniqueness follows as a consequence of renormalization properties of solutions to the transport equations; see [DPL89, Am04].

It is a classical fact for harmonic analysts that BMO can be identified with the homogeneous Triebel-Lizorkin space $\dot F^0_{\infty,2}$. As it was proven by [KYZ11] (see also [RR75]), the homogeneous spaces $\dot F^\theta_{p,q}$ are quasiconformally invariant provided that $\theta p=n$ and $q>\frac{n}{n+\theta}$. As a consequence, we obtain well-posedness of (3) also in these spaces (see Theorem 9). Moreover, well-posedness also holds in the homogeneous Sobolev spaces $\dot W^{1,n}$ (see Theorem 9).

The paper is organized as follows. In Section 2 we show that for vector fields b satisfying the requirements from Theorem 1 the corresponding flow ϕ_t from (5) is a quasiconformal mapping for each t. The argument is based on Reimann's approach from [Re76], but we also relax the condition $S_Ab\in L^\infty(0,T;L^\infty)$ from [Re76] to $S_Ab\in L^1(0,T;L^\infty)$. In Section 3, we prove Theorem 1, and in the last section we address the Cauchy problem for the transport equation in some Triebel-Lizorkin spaces.

2 Flows of quasiconformal mappings

In this section, we deal with the flows of quasiconformal maps. The idea of this section is similar to Reimann [Re76].

Lemma 2. *If $b : \mathbb{R}^n \mapsto \mathbb{R}^n$ is differentiable at x and $|S_A b(x)| < \infty$, then*

$$\limsup_{y,z\to 0,\, 0<|z|,|y|} \left| \frac{\langle y, (b(x+y)-b(x))\rangle}{|y|^2} - \frac{\langle z, (b(x+z)-b(x))\rangle}{|z|^2} \right| \le 2|S_A b(x)|.$$

Proof. By [Re76, Proposition 13], it holds that

$$\limsup_{y\to 0} \left| \frac{\langle y, (b(x+y)-b(x))\rangle}{|y|^2} - \frac{\langle \frac{|y|}{|z|} z, (b(x+\frac{|y|}{|z|}z)-b(x))\rangle}{|y|^2} \right| \le 2|S_A b(x)|.$$

On the other hand, by the differentiability of b, we can further deduce that

$$\begin{aligned} &\lim_{y,z\to 0,\, 0<|z|,|y|} \left| \frac{\langle z, (b(x+z)-b(x))\rangle}{|z|^2} - \frac{\langle \frac{|y|}{|z|} z, (b(x+\frac{|y|}{|z|}z)-b(x))\rangle}{|y|^2} \right| \\ &= \lim_{y,z\to 0,\, 0<|z|,|y|} \left| \frac{\langle z, Db(x)z + o(|z|)\rangle}{|z|^2} - \frac{\langle \frac{|y|}{|z|} z, Db(x)\frac{|y|}{|z|}z + o(|y|)\rangle}{|y|^2} \right| \\ &= 0. \end{aligned}$$

The above two estimates give the desired conclusion. □

Definition 3 (Distortion)**.** *Let $\phi : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a homeomorphism. For each $x \in \mathbb{R}^n$ and each $r > 0$, define*

$$L_\phi(x,r) = \sup_{y:\, |y-x|=r} |\phi(y)-\phi(x)|,$$

and

$$\ell_\phi(x,r) = \inf_{y:\, |y-x|=r} |\phi(y)-\phi(x)|.$$

We then define the linear distortion function as

$$H_\phi(x) := \limsup_{r\to 0} \frac{L_\phi(x,r)}{\ell_\phi(x,r)}.$$

A homeomorphism $\phi : \mathbb{R}^n \mapsto \mathbb{R}^n$ is called a quasiconformal mapping, if there exists $H > 0$ such that the distortion $H_\phi(x) \le H$ for all $x \in \mathbb{R}^n$. Notice that this (metric) definition coincides with the usual (analytic) definition of K-quasiconformal mapping. Recall that a homeomorphism $\phi : \mathbb{R}^n \mapsto \mathbb{R}^n$ is called a K-quasiconformal mapping, if $\phi \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ with $|D\phi(x)|^n \le K_\phi J_\phi(x)$ for a.e. $x \in \mathbb{R}^n$. Then for any K-quasiconformal mapping ϕ, it holds $K_\phi^{\frac{1}{n-1}} \le H_\phi(x) \le K_\phi$ almost everywhere. See the book [IM01] for more information on quasiconformal mappings in $\mathbb{R}^n$.

Theorem 4. *Let $b(t,x) : [0,T]\times\mathbb{R}^n \mapsto \mathbb{R}^n$ be a vector field in $L^1(0,T;W^{1,1}_{\text{loc}})$ and $b(t,\cdot)\in C^2(\mathbb{R}^n)$ for each $t\in[0,T]$. Assume that b satisfies (6) and $S_A b\in L^1(0,T;L^\infty)$. Then there exists a unique flow of quasiconformal mappings $\phi_t(x)$ satisfying*

$$\frac{d}{dt}\,\phi_t(x) = b(t,\phi_t(x)), \qquad a.e.\ t\in[0,T], \qquad \phi_0(x)=x.$$

Moreover, for each $x\in\mathbb{R}^n$ and each $t\in[0,T]$, it holds that

$$H_{\phi_t}(x) \le \exp\left(\int_0^t 2|S_A b(s,\phi_s(x))|\,ds\right).$$

Proof. The existence and uniqueness of a flow $\phi_t(x)$ satisfying

$$\frac{d}{dt}\,\phi_t(x) = b(t,\phi_t(x)) \qquad \text{a.e. } t\in[0,T],$$

is a classical result; see [Ha80]. Moreover, for each $t\in[0,T]$, the flow

$$\phi_t(x) = x + \int_0^t b(s,\phi_s(x))\,ds$$

is a locally Lipschitz homeomorphism of $\mathbb{R}^n$ and preserves the class of sets of measure zero. By (6) one has

$$|\phi_t(x)| \le |x| + \int_0^t \left\|\frac{b(s,\cdot)}{1+|\cdot|\log^+|\cdot|}\right\|_\infty (1+|\phi_s(x)||\log^+|\phi_s(x)|)\,ds.$$

Then, using a Gronwall type inequality due to I. Bihari (see [Dr03, p. 3]) one gets

$$|\phi_t(x)| \le C(R,b) \quad \text{for all } (t,x)\in[0,T]\times B(0,R), \tag{7}$$

where $C(R,b)$ is a constant depending on the radius R and $\displaystyle\int_0^T \left\|\frac{b(s,\cdot)}{1+|\cdot|\log^+|\cdot|}\right\|_\infty ds$, that is, ϕ_t maps bounded sets into bounded sets in finite time.
Let $x\in\mathbb{R}^n$ and $t\in[0,T]$ be fixed. For each $y,z\in B(0,1)$, $|y|=|z|\neq 0$, define

$$A(t,x) = \phi_t(x+y) - \phi_t(x)$$

$$B(t,x) = \phi_t(x+z) - \phi_t(x),$$

$$D(t,x) = b(t,\phi_t(x+y)) - b(t,\phi_t(x)),$$

$$E(t,x) = b(t,\phi_t(x+z)) - b(t,\phi_t(x)).$$

and set

$$H_{y,z}(t,x) = \frac{|A(t,x)|}{|B(t,x)|}.$$

Because, for each $t \in [0,T]$, ϕ_t is a homeomorphism of $\mathbb{R}^n$, the quantity $H_{y,z}(t,x)$ is well defined. It is clear from the definiton that $\log H_{y,z}(t,x)$ as function of t is absolutely continuous on $[0,T]$. For $|s|$ small enough such that $t+s \in [0,T]$, one has

$$\begin{aligned} H_{y,z}(t+s,x) - H_{y,z}(t,x) &= \frac{|A(t+s,x)|}{|B(t+s,x)|} - \frac{|A(t,x)|}{|B(t,x)|} \\ &= \frac{|A(t,x)|}{|B(t+s,x)|}\left\{\frac{|A(t+s,x)|}{|A(t,x)|} - \frac{|B(t+s,x)|}{|B(t,x)|}\right\} \\ &= \frac{|A(t,x)|}{|B(t+s,x)|}\frac{\left\{\frac{|A(t+s,x)|^2}{|A(t,x)|^2} - \frac{|B(t+s,x)|^2}{|B(t,x)|^2}\right\}}{\left\{\frac{|A(t+s,x)|}{|A(t,x)|} + \frac{|B(t+s,x)|}{|B(t,x)|}\right\}}, \end{aligned}$$

and therefore,

$$\frac{H_{y,z}(t+s,x)}{H_{y,z}(t,x)} = \frac{|B(t,x)|}{|B(t+s,x)|}\frac{\left\{\frac{|A(t+s,x)|^2}{|A(t,x)|^2} - \frac{|B(t+s,x)|^2}{|B(t,x)|^2}\right\}}{\left\{\frac{|A(t+s,x)|}{|A(t,x)|} + \frac{|B(t+s,x)|}{|B(t,x)|}\right\}} + 1.$$

Using that,

$$A(t+s,x) = A(t,x) + \int_t^{t+s} D(r,x)\,dr$$

$$B(t+s,x) = B(t,x) + \int_t^{t+s} E(r,x)\,dr,$$

we can conclude that for a.e. $t \in [0,T]$ it holds

$$\begin{aligned} \frac{d\log H_{y,z}(t,x)}{dt} &= \lim_{s\to 0}\frac{1}{s}\log\left(\frac{H_{y,z}(t+s,x)}{H_{y,z}(t,x)}\right) \\ &= \lim_{s\to 0}\frac{1}{s}\log\left(\frac{|B(t,x)|}{|B(t+s,x)|}\frac{\left\{\frac{|A(t+s,x)|^2}{|A(t,x)|^2} - \frac{|B(t+s,x)|^2}{|B(t,x)|^2}\right\}}{\left\{\frac{|A(t+s,x)|}{|A(t,x)|} + \frac{|B(t+s,x)|}{|B(t,x)|}\right\}} + 1\right) \\ &= \frac{\langle A(t,x), D(t,x)\rangle}{|A(t,x)|^2} - \frac{\langle B(t,x), E(t,x)\rangle}{|B(t,x)|^2}. \end{aligned} \tag{8}$$

By the estimate (8), we see that

$$H_{y,z}(t,x) \le \exp\left\{\int_0^t \left|\frac{\langle A(s,x), D(s,x)\rangle}{|A(s,x)|^2} - \frac{\langle B(s,x), E(s,x)\rangle}{|B(s,x)|^2}\right| ds\right\}.$$

Now, since ϕ_t is locally Lipschitz continuous ($b(t,\cdot) \in C^2(\mathbb{R}^n)$ for each $t \in [0,T]$), we can apply Lemma 2 to obtain

$$\limsup_{|y|=|z|\to 0} H_{y,z}(t,x) \le \limsup_{|y|=|z|\to 0} \exp\left\{\int_0^t \left|\frac{\langle A(s,x), D(s,x)\rangle}{|A(s,x)|^2} - \frac{\langle B(s,x), E(s,x)\rangle}{|B(s,x)|^2}\right| ds\right\}$$

$$\leq \exp\left\{\int_0^t \limsup_{|y|=|z|\to 0}\left|\frac{\langle A(s,x),D(s,x)\rangle}{|A(s,x)|^2}-\frac{\langle B(s,x),E(s,x)\rangle}{|B(s,x)|^2}\right| ds\right\}$$
$$\leq \exp\left\{\int_0^t |2S_Ab(s,\phi_s(x))|\, ds\right\},$$

for all $x\in\mathbb{R}^n$ and all $t\in[0,T]$. The proof is completed. $\square$

Theorem 5. *Let $b(t,x):\,[0,T]\times\mathbb{R}^n\mapsto\mathbb{R}^n$ be a vector field in $L^1(0,T;W^{1,1}_{\mathrm{loc}})$. Assume (6) and $S_Ab\in L^1(0,T;L^\infty)$. Then there exists a unique flow of quasiconformal mappings $\phi_t(x)$ satisfying*

$$\frac{d}{dt}\,\phi_t(x)=b(t,\phi_t(x)),\qquad a.e.\ t\in[0,T],\qquad \phi_0(x)=x.$$

Moreover, for a.e. $x\in\mathbb{R}^n$ and each $t\in[0,T]$, it holds that

$$K_{\phi_t}\leq \exp\left((n-1)\int_0^t 2\|S_Ab(s,\cdot)\|_{L^\infty}\,ds\right).$$

Proof. Let $\rho\in C_c^\infty(B(0,1))$ be a non-negative smooth function that satisfies $\int_{\mathbb{R}^n}\rho(x)\,dx=1$. For each $\epsilon>0$ let $\rho_\epsilon(x)=\rho(x/\epsilon)/\epsilon^n$ and

$$b_\epsilon(t,x):=\int_{\mathbb{R}^n} b(t,x-y)\rho_\epsilon(y)\,dy.$$

Then for each $\epsilon>0$, b_ϵ satisfies the requirements from Theorem 4, and therefore there exists a unique flow of quasiconformal maps $\phi_{t,\epsilon}$ that satisfies

$$\frac{d}{dt}\,\phi_{t,\epsilon}(x)=b_\epsilon(t,\phi_{t,\epsilon}(x)),\qquad \text{a.e. } t\in[0,T],\qquad \phi_{0,\epsilon}(x)=x.$$

Moreover, since

$$|S_Ab_\epsilon(s,\phi_{s,\epsilon}(x))|\leq\int_{\mathbb{R}^n}|S_Ab(s,\phi_{s,\epsilon}(x)-y)|\,\rho_\epsilon(y)\,dy\leq\|S_Ab(s,\cdot)\|_{L^\infty},$$

we have that the linear distortion function $H_{\phi_{t,\epsilon}}$ of $\phi_{t,\epsilon}$ satisfies

$$H_{\phi_{t,\epsilon}}(x)\leq\exp\left(\int_0^t 2|S_Ab_\epsilon(s,\phi_{s,\epsilon}(x))|\,ds\right)\leq\exp\left(\int_0^t 2\|S_Ab(s,\cdot)\|_{L^\infty}\,ds\right). \tag{9}$$

Notice that from (6) and the argument used to obtain (7) we have that for any bounded set $U\subset\mathbb{R}^n$, $\phi_{t,\epsilon}(U)$ is uniformly bounded in $\mathbb{R}^n$ for any $t\in[0,T]$ and any $\epsilon<1$. That is, $\phi_{t,\epsilon}(U)\subset B(0,R)$ where the radius R depends on U and b. Moreover, by (9), for all $t\in[0,T]$ and $\epsilon<1$, the map $\phi_{t,\epsilon}$ is K-quasiconformal with

$$K_{\phi_{t,\epsilon}}\leq(H_{\phi_{t,\epsilon}}(x))^{n-1}\leq\exp\left((n-1)\int_0^t 2\|S_Ab(s,\cdot)\|_{L^\infty}\,ds\right).$$

So, the family $\{\phi_{t,\epsilon}\}_{0<\epsilon<1}$ is locally equicontinuous in the spatial direction. On the other hand, if $x \in U$

$$|\phi_{t,\epsilon}(x) - \phi_{s,\epsilon}(x)| \leq C(U,T) \int_s^t \left\| \frac{b(r,\cdot)}{1+|\cdot|\log^+|\cdot|} \right\|_\infty dr.$$

Therefore, we can conclude that $\phi_{t,\epsilon}(x)$ is locally uniformly bounded and equicontinuous in $[0,T]\times\mathbb{R}^n$. Applying the Arzelà-Ascoli theorem, we achieve that $\phi_{t,\epsilon}$ converges to some ϕ_t locally uniformly up to a subsequence. To get

$$\phi_t(x) = x + \int_0^t b(s,\phi_s(x))\,ds \tag{10}$$

from

$$\phi_{t,\epsilon}(x) = x + \int_0^t b_\epsilon(s,\phi_{s,\epsilon}(x))\,ds,$$

it is enough to prove that for each $x \in \mathbb{R}^n$

$$\int_0^t b_\epsilon(s,\phi_{s,\epsilon}(x))\,ds \longrightarrow \int_0^t b(s,\phi_s(x))\,ds \qquad \text{as } \epsilon \to 0.$$

Thus, we split

$$\begin{aligned}\int_0^t |b_\epsilon(s,\phi_{s,\epsilon}(x)) - b(s,\phi_s(x))|\,ds &\leq \int_0^t |b_\epsilon(s,\phi_{s,\epsilon}(x)) - b_\epsilon(s,\phi_s(x))|\,ds \\ &\quad + \int_0^t |b_\epsilon(s,\phi(x)) - b(s,\phi_s(x))|\,ds := I + II.\end{aligned}$$

Recall that $\|S_A b_\epsilon(s,\cdot)\|_\infty \leq \|S_A b(s,\cdot)\|_\infty < \infty$ a.e.. So, as in [Re76] , for a.e. $s \in [0,1]$ $b_\epsilon(s,\cdot)$ belongs to the Zygmund class. Then

$$I \leq C \int_0^t \|S_A b(s,\cdot)\|_\infty \, |\phi_{s,\epsilon}(x) - \phi_s(x)| \, |\log|\phi_{s,\epsilon}(x) - \phi_s(x)|| \; ds$$

which tends to 0 because $\sup_{s\in[0,T]} |\phi_{s,\epsilon}(x) - \phi_s(x)| \to 0$ as $\epsilon \to 0$. The dominated convergence theorem gives $II \to 0$ as $\epsilon \to 0$, because $b_\epsilon \longrightarrow b$ a.e. in $[0,T]\times\mathbb{R}^n$ and

$$|b_\epsilon(s,\phi(x)| \leq C(x,T) \left\| \frac{b(s,\cdot)}{1+|\cdot|\log^+|\cdot|} \right\|_\infty \in L^1([0,T]).$$

Equivalently to (10) we obtained

$$\frac{d}{dt}\,\phi_t(x) = b(t,\phi_t(x)), \qquad \text{a.e. } t \in [0,T], \qquad \phi_0(x) = x.$$

The uniqueness of the flow follows as a consequence that for a.e. t the vector field $b(t,\cdot)$ is in the Zygmund class and so it satisfies a quasi-Lipschitz condition (see [AL93, Theorem 1.5.1]).

By the fact that a uniform limit of K-quasiconformal mappings is a K-quasiconformal mapping or a constant (which can not happen here since ϕ_t satisfies the above ODE), we know that ϕ_t is a quasiconformal mapping with

$$K_{\phi_t} \le \exp\left((n-1)\int_0^t 2\|S_A b(s,\cdot)\|_{L^\infty}\, ds\right).$$

$\square$

3 Transport equation in BMO

In this section, we apply the theory of flows of quasiconformal mappings to the transport equation (3) with initial value in BMO. Recall that, a locally integrable function f is in the space BMO, if

$$\|f\|_{\mathrm{BMO}} := \sup_B \frac{1}{|B|}\int_B |f - f_B|\, dx < \infty,$$

where f_B denotes $\frac{1}{|B|}\int_B f\, dx$ and the supremum is taken over all open balls. In [Re74], Reimann proved that

Theorem 6. *The* BMO *space is invariant under quasiconformal mappings of* $\mathbb{R}^n$*. Precisely, for any* K*-quasiconformal mapping* ϕ*, there exists* $C = C(K,n)$ *such that for any* f *in* BMO*, it holds*

$$\|f\circ\phi\|_{\mathrm{BMO}} \le C\|f\|_{\mathrm{BMO}}.$$

A function $u \in L^1(0,T; L^1_{\mathrm{loc}})$ is called a *weak solution* to (3) if for each $\varphi \in C_c^\infty([0,T]\times\mathbb{R}^n)$ with compact support in $[0,T)\times\mathbb{R}^n$ it holds that

$$\int_0^T\int_{\mathbb{R}^n} u\,\frac{d\varphi}{dt}\, dx\, dt + \int_{\mathbb{R}^n} u_0\,\varphi(0,\cdot)\, dx - \int_0^T\int_{\mathbb{R}^n} u \operatorname{div}(b\,\varphi)\, dx\, dt = 0.$$

Proof of Theorem 1. Let us first prove the existence. By Theorem 5, we know that there exists a unique flow of quasiconformal mappings $\phi_t(x)$ satisfying

$$\frac{d}{dt}\phi_t(x) = b(t,\phi_t(x)), \qquad \text{a.e. } t\in[0,T], \qquad \phi_0(x) = x$$

and its inverse $\phi_t^{-1}(x) = \phi_{-t}(x)$ also satisfies

$$\frac{d}{dt}\phi_{-t}(x) = -b(t,\phi_{-t}(x)), \qquad \text{a.e. } t\in[0,T], \qquad \phi_0(x) = x.$$

Let $u_0 \in \mathrm{BMO}$ and $u(x,t) := u_0(\phi_t(x))$ for each $t\in[0,T]$. Since for each t, ϕ_t preserves zeros sets of $\mathbb{R}^n$, $u(x,t)$ is well defined. We deduce from Theorem 5 that for each fixed $t\in[0,T]$, $\phi_t(x)$ is a K-quasiconformal mapping with

$$K_{\phi_t} \le \exp\left((n-1)\int_0^t 2\|S_A b(s,\cdot)\|_{L^\infty}\, ds\right).$$

This, together with Theorem 6, implies that $u(x,t) \in L^\infty(0,T;\mathrm{BMO})$ with

$$\|u\|_{L^\infty(0,T;\mathrm{BMO})} \le C(T,b)\|u_0\|_{\mathrm{BMO}}.$$

Let us next show that u is a weak solution to (3). Choose an arbitrary $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $\psi \in C_c^\infty([0,T))$. Then we have

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^n} u \,\frac{d(\varphi\psi)}{dt}\, dx\, dt \\
&\quad = \int_0^T \int_{\mathbb{R}^n} u_0(\phi_t(x))\, \frac{d\psi(t)}{dt} \varphi(x)\, dx\, dt \\
&\quad = \int_0^T \int_{\mathbb{R}^n} u_0(x)\, \frac{d\psi(t)}{dt} \varphi(\phi_{-t}(x)) J_{\phi_{-t}}(x)\, dx\, dt \\
&\quad = -\int_{\mathbb{R}^n} u_0(x)\, \psi(0)\varphi(x)\, dx - \int_0^T \int_{\mathbb{R}^n} u_0(x)\psi(t)\, \frac{d(\varphi(\phi_{-t}(x))J_{\phi_{-t}}(x))}{dt}\, dx\, dt.
\end{aligned}$$

On the other hand, it holds

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^n} u \,\mathrm{div}(b\,\varphi\psi)\, dx\, dt \\
&\quad = \int_0^T \int_{\mathbb{R}^n} u_0(\phi_t(x))\psi(t)\,\mathrm{div}(b(t,x)\,\varphi(x))\, dx\, dt \\
&\quad = \int_0^T \int_{\mathbb{R}^n} u_0(x)\psi(t)\,\mathrm{div}(b(t,\phi_{-t}(x))\,\varphi(\phi_{-t}(x)))J_{\phi_{-t}}(x)\, dx\, dt.
\end{aligned}$$

By noticing that $\dfrac{d}{dt} J_{\phi_{-t}}(x) = -\mathrm{div}(b(t,\phi_{-t}(x))\, J_{\phi_{-t}}(x)$ (see [Am04, Re76]), one has

$$\begin{aligned}
&\frac{d}{dt}\,(\varphi(\phi_{-t}(x))J_{\phi_{-t}}(x)) \\
&\quad = -\nabla\varphi(\phi_{-t}(x)) \cdot b(t,\phi_{-t}(x))J_{\phi_{-t}}(x) - \varphi(\phi_{-t}(x))\mathrm{div}\, b(t,\phi_{-t}(x))J_{\phi_{-t}}(x)) \\
&\quad = -\,\mathrm{div}(b(t,\phi_{-t}(x))\,\varphi(\phi_{-t}(x)))J_{\phi_{-t}}(x).
\end{aligned}$$

We can conclude that

$$\int_0^T \int_{\mathbb{R}^n} u\, \frac{d(\varphi\phi)}{dt}\, dx\, dt + \int_{\mathbb{R}^n} u_0(x)\, \psi(0)\varphi(x)\, dx = \int_0^T \int_{\mathbb{R}^n} u\,\mathrm{div}(b\,\varphi\psi)\, dx\, dt,$$

which implies that $u(x,t) = u_0(\phi_t(x))$ is a weak solution to (3).
Let us prove the uniqueness. Let $u \in L^\infty(0,T;\mathrm{BMO})$ be a solution of the transport equation with initial value $u_0 = 0$. Notice that for b with $S_A b \in L^1(0,T;L^\infty)$, we have $Db \in L^\infty(0,T;L^q_{\mathrm{loc}})$ for each finite q; see [Re76, p.262]. Letting $0 \le \rho \in C_c^\infty(\mathbb{R}^n)$, and $\rho_\epsilon = \epsilon^{-n}\rho(\cdot/\epsilon)$ for each $\epsilon > 0$, we conclude by [DPL89, Theorem 2.1, Lemma 2.1] that

$$\frac{du_\epsilon}{dt} + b \cdot \nabla u_\epsilon = r_\epsilon,$$

where $u_\epsilon = u * \rho_\epsilon$, and $r_\epsilon \to 0$ in $L^1(0,T;L^1_{\rm loc})$ as $\epsilon \to 0$. Therefore, using the renormalization property of transport equation, one has that for each $\beta \in C^1(\mathbb{R})$ with $\beta(0) = 0$ and $\beta' \in L^\infty$, it holds

$$\frac{d\beta(u)}{dt} + b \cdot \nabla \beta(u) = 0,$$

i.e., $\beta(u)$ is a solution of the transport equation with initial value $\beta(u_0) = 0$. For each $M > 0$, let $\beta_M(t) = |t| \wedge M$ be a Lipschitz function on $\mathbb{R}$. A further approximation argument would give us that, $\beta_M(u) = |u| \wedge M$ is a solution of the transport equation with initial value $\beta(u_0) = 0$.
At this point, applying the well-posedness of the transport equation in L^∞ (see e.g. [CJMO, Theorem 2.2]) gives us that $\beta_M(u) = 0$ for each $M > 0$. Letting $M \to \infty$, we conclude that $u = 0$. $\square$

Remark 7. By the Banach-Alaoglu theorem, one has that the solution u found in Theorem 1 is continuous in time with respect to the weak-$*$ topology of BMO.
If instead of BMO one considers the vanishing mean oscillation space (VMO), which is defined as the closure of compactly supported smooth functions with the BMO norm, then under the assumptions of Theorem 1, for each $u_0 \in \mathrm{VMO}$, one can find a unique solution u in $L^\infty(0,T;\mathrm{VMO})$. Moreover, the solution u is continuous in time with respect to the norm topology of VMO, that is, $u \in C(0,T;\mathrm{VMO})$. Indeed, since the solution u is given as $u_0(\phi_t)$, where ϕ_t is as in Theorem 5, for each compactly supported smooth function u_0, it is easy to see that $u_0(\phi_s) \to u_0(\phi_t)$ uniformly as $s \to t$ and therefore,

$$\|u(s,\cdot) - u(t,\cdot)\|_{\mathrm{BMO}} \le 2\|u(s,\cdot) - u(t,\cdot)\|_{L^\infty} \to 0, \quad \text{as } s \to t.$$

An density argument gives the desired conclusion for any initial value in VMO.

4 Transport equation in Triebel-Lizorkin spaces

In this section, we show that the same conclusion of Theorem 1 holds with BMO replaced by certain Triebel-Lizorkin spaces.
The following result was proved in Koskela-Yang-Zhou [KYZ11]. We refer the reader to [KYZ11] for precise definitions of the Triebel-Lizorkin spaces.

Theorem 8. *Let $n \ge 2$, $s \in (0,1)$ and $q \in (n/(n+s), \infty]$ Then $\dot{F}^s_{n/s,q}(\mathbb{R}^n)$ is invariant under quasiconformal mappings of $\mathbb{R}^n$.*

Applying the above theorem and the well-known fact that $\dot{W}^{1,n}(\mathbb{R}^n)$ is also quasiconformally invariant (cf. [KYZ11]), similar to the proof of Theorem 1 we can conclude the following result, whose proof will be omitted.

Theorem 9. *Let $b(t,x) : [0,T] \times \mathbb{R}^n \mapsto \mathbb{R}^n$ be a vector field in $L^1(0,T;W^{1,1}_{\rm loc})$. Assume that b satisfies (6) and $S_A b \in L^1(0,T;L^\infty)$. Then*

(i) for each $u_0 \in \dot{F}^s_{n/s,q}(\mathbb{R}^n)$, $s \in (0,1)$ and $q \in (n/(n+s), \infty]$, there exists a unique solution $u \in L^\infty(0,T;\dot{F}^s_{n/s,q}(\mathbb{R}^n))$ of (3).

(ii) for each $u_0 \in \dot{W}^{1,n}(\mathbb{R}^n)$*, there exists a unique solution* $u \in L^\infty(0,T;\dot{W}^{1,n}(\mathbb{R}^n))$ *of* (3).

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