Topological and algebraic reducibility for patterns on trees

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Outline

Motivation

The notion of reducibility plays a fundamental role in the study of the combinatorial and topological dynamics of discrete dynamical systems.

Reducible systems are those such that the space can be decomposed in connected pieces with pairwise disjoint interiors which are permuted by the map.

In this situation the behavior of the original map can be related with the dynamics of an iterate of the map on the reduced pieces.

This approach plays a crucial role, for instance in the study of both the surface homeomorphisms and the interval dynamics related to periodic orbits (where reducibility is formalized through the notion of *block structure*).

Aims

The aim of this talk is to clarify the notions of reducibility and irreducibility for periodic orbits of tree maps and study the dynamical implications of these notions at a topological and algebraic level.

Thanks to this study, we obtain some interesting properties of the topological entropy of reducible systems and we clarify its relation with the decomposition of the space and the Markov matrix of the map.

This study is done at a combinatorial level. So we need the notion of combinatorial type or *pattern* for finite invariant sets of trees.

The definition of a pattern

Let T be a tree and let $P \subset T$ be a finite subset of T. The pair (T, P) will be called a *pointed tree*. A set $Q \subset P$ is said to be a *discrete component* of (T, P) if either $|Q| > 1$ and there is a connected component C of $T \setminus P$ such that $Q = \mathrm{Cl}(C) \cap P$, or $|Q| = 1$ and $Q = P$.

We say that two pointed trees (T, P) and (T', P') are *equivalent* if there exists a bijection $\phi\colon P \longrightarrow P'$ which preserves discrete components. In this case, two discrete components C of (T, P) and C' of (T', P') will be called *equivalent* if $C' = \phi(C)$.

The equivalence class of a pointed tree (T, P) will be denoted by $[T, P]$, and the equivalence class of a discrete component of (T, P) will be called a *discrete component of* $[T, P]$.

The definition of a pattern

Let (T, P) and (T', P') be equivalent pointed trees, and let $\theta\colon P \longrightarrow P$ and $\theta'\colon P' \longrightarrow P'$ be maps. We will say that θ and θ' are *equivalent* if $\theta' = \varphi \circ \theta \circ \varphi^{-1}$ for a bijection $\varphi\colon P \longrightarrow P'$ which preserves discrete components. The equivalence class of θ by this relation will be denoted by $[\theta]$.

Definition

If $[T, P]$ is an equivalence class of pointed trees and $[\theta]$ is an equivalence class of maps then the pair $([T, P], [\theta])$ will be called a *pattern*.

Any discrete component of $[T, P]$ will be also called a *discrete component of the pattern* $([T, P], [\theta])$.

We say that a model (T, P, f) *exhibits* a pattern $(\mathcal{T}, \Theta)$ if $\mathcal{T} = [\langle P \rangle_T, P]$ and $\Theta = [f\big|_P]$. Alternatively, we will say that the model (T, P, f) is a *representative* of the pattern $(\mathcal{T}, \Theta)$.

Entropy of a pattern

The topological entropy of a map $f\colon T \longrightarrow T$ will be denoted by $h(f)$.

Given a pattern $\mathcal{P}$, the topological entropy of $\mathcal{P}$ is defined to be

$$\boxed{h(\mathcal{P}) := \inf\{h(f) : (T, P, f) \text{ is a model exhibiting } \mathcal{P}\}.}$$

Monotone models

The simplest models exhibiting a given pattern are the monotone ones, according to the following definition. Let S and T be trees and let $f\colon T \longrightarrow S$ be a map. Given $a, b \in T$ we say that $f\big|_{[a,b]}$ is *monotone* if $f([a, b])$ is either an interval or a point and $f\big|_{[a,b]}$ is monotone as an interval map.

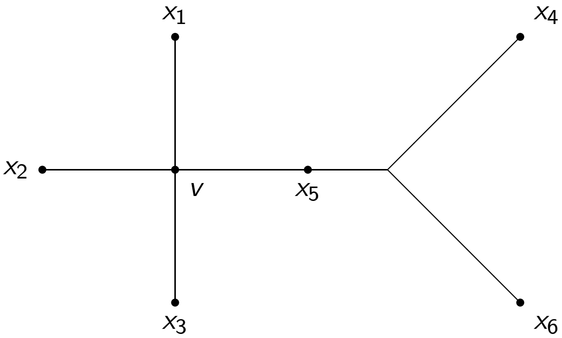
Let (T, P, f) be a model. A pair $\{a, b\} \subset P$ will be called a *basic path of (T, P)* if it is contained in a single discrete component of (T, P).

Definition

We will say that f is *P-monotone* if $\operatorname{En}(T) \subset P$ and $f\big|_{[a,b]}$ is monotone for any basic path $\{a, b\}$. The model (T, P, f) will be called *monotone*.

In such case, one can see that the set $P \cup V(T)$ is f-invariant and the map f (which is P-monotone, is also $(P \cup V(T))$-monotone).

Monotone models may not exist. Hence, the space cannot be fixed!!



Monotone models

Theorem (Theorem A of [aglmm])

Let $\mathcal{P}$ be a pattern. Then the following statements hold.

1. *There exists a monotone model of $\mathcal{P}$.*
2. *Every monotone model (T, P, f) of $\mathcal{P}$ satisfies $h(f) = h(\mathcal{P})$.*

[aglmm] Ll. Alsedà, J. Los, F. Mañosas, and P. Mumbrú, *Canonical representatives for patterns of tree maps*, Topology **36** (1997), no. 5, 1123–1153. MR 1445556 (99f:58062)

Canonical models

The monotone models from the above are essentially unique in the following sense. Let (T, P, f) be a monotone model and let S be a non-empty union of edges disjoint from P. We will say that S is an *invariant forest of (T, P, f)* if either $f^i(S) \cap P = \emptyset$ for every $i \geq 0$ or there exists $n > 0$ such that $f^i(S) \cap P = \emptyset$ for every $i = 0, 1, \ldots, n-1$ and $f^n(S)$ degenerates to a point of P.

(T, P, f) is a *canonical model* of the pattern $[T, P, f]$ if it has no invariant forests. From [aglmm, Theorem B] it follows that every pattern has a canonical model. Moreover, given two canonical models (T, P, f) and (T', P', f') of the same pattern there exists a homeomorphism $\varphi \colon T \longrightarrow T'$ such that $\varphi(P) = P'$, and $f' \circ \varphi\big|_P = \varphi \circ f\big|_P$. Hence, the canonical model of a pattern is essentially unique.

On the reducibility of patterns

We need to introduce the notions of *trivial pattern*, *collapsing interval*, *Markov matrix*, *block structure* and *rotational structure*, which depend only on the combinatorial data of the pattern.

Trivial Pattern

An n-periodic pattern $\mathcal{P}$ will be called *trivial* if it has only one discrete component.

In this case, for $n \geq 2$, let (T, P) be a pointed tree such that T is an n-star with $\mathrm{En}(T) = P = \{x_1, x_2, \ldots, x_n\}$ and let y be its central point. Consider a rigid rotation on T, that is, a model (T, P, f) such that $f(y) = y$ and f maps bijectively $[y, x_i]$ onto $[y, x_{i+1}]$ for $1 \leq i < n$ and $[y, x_n]$ onto $[y, x_1]$. Clearly, (T, P, f) is a monotone model with no invariant forests. In consequence, (T, P, f) is the canonical model of $\mathcal{P}$. Therefore, it easily follows that every trivial pattern has entropy 0.

Collapsing interval

Let $\mathcal{P}$ be a periodic pattern and let (T, P, f) be the canonical model of $\mathcal{P}$. Any $(P \cup V(T))$-basic interval $[a, b]$ such that $f([a, b])$ reduces to a point will be called a *collapsing interval* of $\mathcal{P}$.

Note that, in this case, since $\mathcal{P}$ is periodic, $\{a, b\} \not\subset P$. On the other hand, since (T, P, f) has no invariant forests, $\{a, b\} \not\subset V(T) \setminus P$. Therefore, each collapsing interval has the form $[a, b]$ with $a \in P$ and $b \in V(T) \setminus P$ (the interval $[c, 8]$ in the canonical model of the pattern $\mathcal{P}$ shown in Example 1 is an example of a collapsing interval, since $f(c) = f(8) = 9$).

There is a purely combinatorial criterion to decide whether a pattern $\mathcal{P}$ has collapsing intervals (without constructing its canonical model). Indeed, a pattern $([T, P], [f])$ has collapsing intervals if and only if there is a discrete component C of (T, P) such that $|\operatorname{En}(\langle f(C) \rangle)| < |C|$. This definition is independent from the particular model (T, P, f) realizing the pattern.

Example 1

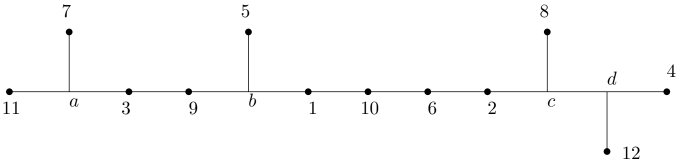


Figure: The canonical model (T, P, f) of a 12-periodic pattern $\mathcal{P}$, which satisfies $f(a) = d$, $f(b) = 6$, $f(c) = 9$, $f(d) = b$. The pattern $\mathcal{P}$ has a separated 4-block structure given by the partition $\{1, 5, 9\} \cup \{2, 6, 10\} \cup \{3, 7, 11\} \cup \{4, 8, 12\}$ and a separated 2-block structure given by the partition $\{1, 3, 5, 7, 9, 11\} \cup \{2, 4, 6, 8, 10, 12\}$.

Recall that $[c, 8]$ is a collapsing interval for $\mathcal{P}$ and observe that the discrete component $C = \{2, 4, 8, 12\}$ verifies $\langle f(C)\rangle = \langle\{3, 5, 9, 1\}\rangle$, a tree with 3 endpoints.

Markov Matrix

Let (T, Q, f) be a monotone model such that $Q \supset V(T)$. In this case, any connected component of $T \setminus Q$ is an open interval.

An interval of T will be called *Q-basic* if it is the closure of a connected component of $T \setminus Q$. Observe that two different Q-basic intervals have pairwise disjoint interiors.

Given $K, L \subset T$, we will say that *K f-covers L* if $f(K) \supset L$.

Consider a labeling $I_1, I_2, \ldots I_k$ of all Q-basic intervals. The *Markov graph of (T, Q, f)* associated to this labeling is a combinatorial directed graph whose vertices are the Q-basic intervals and there is an arrow from I_i to I_j if and only if I_i f-covers I_j.

Markov Matrix

The *Markov matrix of* (T, Q, f) associated to this labeling is a $k \times k$ matrix $(m_{i,j})_{i,j=1}^{k}$ such that $m_{i,j} = 1$ if and only if I_i f-covers I_j, and $m_{i,j} = 0$ otherwise.

Given two different labellings of the set of Q-basic intervals and their associated Markov matrices M and N, there exists a permutation matrix A such that $M = A^T NA$ (where A^T denotes the transpose of A), and the corresponding Markov graphs are isomorphic.

Recall that if (T, P, f) is the canonical model of a pattern $\mathcal{P}$ then the model $(T, P \cup V(T), f)$ is monotone. Thus, we can consider their Markov graph and matrix. Since both objects depend only on the canonical model of $\mathcal{P}$, which is uniquely determined by the combinatorial data of the pattern $\mathcal{P}$, they will be respectively called *Markov graph of* $\mathcal{P}$ and *Markov matrix of* $\mathcal{P}$.

Algebraic reducibility

We recall that a square matrix with non-negative entries is called *reducible* if there exists a permutation matrix A such that

$$A^T M A = \begin{pmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{pmatrix} \tag{1}$$

where M_{11} and M_{22} are square matrices of sizes $l \times l$ and $m \times m$ ($l, m \geq 1$) respectively and 0 stands for the $l \times m$ matrix whose entries are all 0. If there does not exist such A then the matrix M is called *irreducible*.

Primitivity

An irreducible matrix M is called *primitive* if all powers M^n are irreducible for $n \geq 2$. Otherwise M is called *imprimitive*. It is well known [gant, Theorem 8] that an irreducible matrix M is primitive if and only if there exists $n \geq 1$ such that all the entries of M^n are positive.

[gant] F.R. Gantmacher, *The theory of matrices*, AMS Chelsea Publishing Company, New York 1989–90 (2nd edition).

Ciclicity

A square matrix with non-negative entries M will be called *cyclic* if there exist $p \geq 2$ and a permutation matrix A such that

$$A^T M A = \begin{pmatrix} 0 & M_1 & 0 & \dots & 0 \\ 0 & 0 & M_2 & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & M_{p-1} \\ M_p & 0 & \dots & 0 & 0 \end{pmatrix} \tag{2}$$

where the diagonal 0 blocks are square (possibly with pairwise different sizes). Of course, the matrix of a cyclic permutation is cyclic.

Remarks

Remark

Recall that if (T, P, f) is the canonical model of a trivial n-periodic pattern $\mathcal{P}$ with $n \geq 3$ then T is an n-star with $\text{En}(T) = P$ and $f(y) = y$, where y is the central point of T. It is straightforward to check that the Markov matrix M of $\mathcal{P}$ is the permutation matrix corresponding to the cyclic permutation $(2, 3, \ldots, n, 1)$. In consequence, M is cyclic.

Remark

Let M be an irreducible matrix. It is well known [gant] that M is imprimitive if and only if M is cyclic.

Block Structure

Let $\mathcal{P} = ([T, P], [f])$ be an n-periodic pattern with $n \geq 3$. For $n > p \geq 2$, we will say that $\mathcal{P}$ *has a p-block structure* (or simply a *block structure*) if there exists a partition $P = P_1 \cup P_2 \cup \ldots \cup P_p$ such that $f(P_i) = P_{i+1}$ for $1 \leq i < p$, $f(P_p) = P_1$, and $\langle P_i \rangle_T \cap P_j = \emptyset$ whenever $i \neq j$.

In this case, p is a strict divisor of n and $|P_i| = n/p$ for $1 \leq i \leq p$.

The trees $\langle P_i \rangle_T$ (which do depend on the particular model (T, P, f) realizing the pattern) will be called *blocks*. See the pattern $\mathcal{P}$ in Example 2: the partition $P = P_1 \cup P_2 = \{1, 3, 5, 7, 9, 11, 13, 15\} \cup \{2, 4, 6, 8, 10, 12, 14, 16\}$ defines a 2-block structure for $\mathcal{P}$, since $\langle P_1 \rangle_T \cap P_2 = \langle P_2 \rangle_T \cap P_1 = \emptyset$ no matter what particular model (T, P, f) represents $\mathcal{P}$.

Example 2

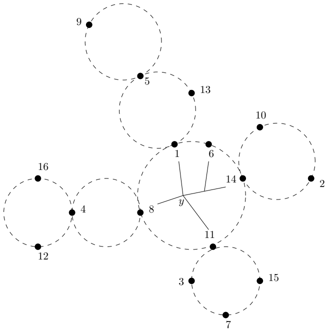


Figure: A 16-periodic pattern $\mathcal{P} = ([T, P], [f])$. The dashed circles stand for the discrete components of $\mathcal{P}$. The points of P are labeled with natural numbers, $f(i) = i + 1$ for $1 \leq i < 16$ and $f(16) = 1$. The pointed tree corresponding to the discrete component $\{1, 6, 8, 11, 14\}$ in the canonical model is shown. The partition $P = P_1 \cup P_2 = \{1, 3, 5, 7, 9, 11, 13, 15\} \cup \{2, 4, 6, 8, 10, 12, 14, 16\}$ defines a 2-block structure for $\mathcal{P}$. There is also a 4-block structure given by the partition $P = Q_1 \cup Q_2 \cup Q_3 \cup Q_4 = \{1, 5, 9, 13\} \cup \{2, 6, 10, 14\} \cup \{3, 7, 11, 15\} \cup \{4, 8, 12, 16\}$.

The notion of block structure is pattern intrinsic

Observe that from the equivalence relation which defines the class of models belonging to the pattern $\mathcal{P}$ it easily follows that this notion does not depend on the particular model (T, P, f) representing $\mathcal{P}$.

We note that if a pattern has a p-block structure, this p-block structure is essentially unique up to relabeling of blocks. Observe also that a pattern can have several different block structures: see again Example 2.

More block structures

The existence of a block structure for a periodic pattern $\mathcal{P}$ is essentially equivalent to the fact that, for some $k \geq 1$, the k-th power M^k of the Markov matrix M of $\mathcal{P}$ is reducible. To look closer at the algebraic properties of M (more precisely, to discriminate whether M is reducible itself and to decide whether M is cyclic) we need to define a couple of particular block structures, which we will respectively call *separated structure* and *rotational structure*.

Separated Block Structure

Let $\mathcal{P}$ be an n-periodic pattern with $n \geq 3$ and let (T, P, f) be the canonical model of $\mathcal{P}$. Assume that $\mathcal{P}$ has a p-block structure defined by a partition $P = P_1 \cup P_2 \cup \ldots \cup P_p$. We say that this p-block structure is *separated* if $\langle P_i \rangle_T \cap \langle P_j \rangle_T = \emptyset$ whenever $i \neq j$.

For instance, the 4-block structure $Q_1 \cup Q_2 \cup Q_3 \cup Q_4$ for the pattern $\mathcal{P}$ in Example 2 is separated, since the blocks have pairwise disjoint intersections in any model representing $\mathcal{P}$ (in particular, in the canonical model). On the other hand, a part of the tree T corresponding to the canonical model is shown in Example 2. Observe that $\langle P_1 \rangle_T \cap \langle P_2 \rangle_T = \{y\}$, where y is a branching point of valence 4. Therefore, the 2-block structure $P_1 \cup P_2$ is not separated.

Rotational Block Structure

The cyclicity of the Markov matrix of a periodic pattern is related to the existence of another particular case of block structure.

Let $\mathcal{P}$ be a non-trivial n-periodic pattern with $n \geq 3$ and let (T, P, f) be the canonical model of $\mathcal{P}$. Assume that there exists a branching point $y \in T$ such that $f(y) = y$. For $n > p \geq 2$, we will say that $\mathcal{P}$ *has a p-rotational structure* (or simply a *rotational structure*) if there exist subtrees $Y_1, Y_2, \ldots, Y_p$ such that each Y_i is the closure of a union of connected components of $T \setminus \{y\}$, $f(Y_i) = Y_{i+1}$ for $1 \leq i < p$ and $f(Y_p) = Y_1$. Note that the sets $Y_i \setminus \{y\}$ form a partition of $T \setminus \{y\}$.

A rotational block structure is a block structure

In this situation, the partition $P = P_1 \cup P_2 \cup \ldots \cup P_p$, where $P_i := P \cap Y_i$ for $1 \leq i \leq p$, defines a p-block structure for $\mathcal{P}$ (that is, a rotational structure is a particular case of block structure). Moreover,

1. Either all blocks $\langle P_i \rangle$ are pairwise disjoint,
2. Or $\langle P_i \rangle \cap \langle P_j \rangle = \{y\}$ whenever $i \neq j$.

Hence, a rotational structure is either separated or every pair of blocks in the canonical model intersect at a fixed branching point.

Examples

A pattern can have several different rotational structures. For an example, consider the 8-periodic pattern $\mathcal{P}$ whose canonical model (T, P, f) is depicted in Example 3. In this case, y is the only branching point in T and $f(y) = y$. The connected components of $T \setminus \{y\}$ are the intervals $(y, 5]$, $(y, 6]$, $(y, 7]$ and $(y, 8]$, whose closures are mapped cyclically by f. Hence, $\{1, 5\} \cup \{2, 6\} \cup \{3, 7\} \cup \{4, 8\}$ defines a 4-rotational structure for $\mathcal{P}$. Since the blocks $\langle 1, 5\rangle$, $\langle 2, 6\rangle$, $\langle 3, 7\rangle$, $\langle 4, 8\rangle$ are pairwise disjoint, (a) holds and this rotational structure is separated. On the other hand, since f also maps cyclically the sets $[y, 5] \cup [y, 7]$ and $[y, 6] \cup [y, 8]$, the partition $\{1, 3, 5, 7\} \cup \{2, 4, 6, 8\}$ defines a 2-rotational structure for $\mathcal{P}$, which is not separated because the blocks $\langle\{1, 3, 5, 7\}\rangle$ and $\langle\{2, 4, 6, 8\}\rangle$ intersect at y. Observe that the points of P rotate around the discrete component $\{1, 2, 3, 4\}$ under the action of f. This fact justifies the name *rotational structure*.

Example 3

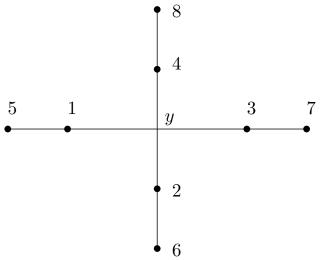


Figure: The canonical model (T, P, f) of a 8-periodic pattern $\mathcal{P}$, which satisfies $f(y) = y$. The pattern $\mathcal{P}$ has a separated and rotational 4-block structure given by the partition $\{1, 5\} \cup \{2, 6\} \cup \{3, 7\} \cup \{4, 8\}$ and also a rotational 2-block structure given by the partition $\{1, 3, 5, 7\} \cup \{2, 4, 6, 8\}$, which is not separated.

Remarks on block structures

There exist block structures that are not rotational neither separated (see below).The 8-periodic pattern $\mathcal{P}$ has no rotational structures, since the only fixed point in the canonical model lies on the open interval $(1, 2)$ and, in consequence, is not a branching point. On the other hand, the 4-block structure given by the partition $P_1 \cup P_2 \cup P_3 \cup P_4 = \{1, 5\} \cup \{2, 6\} \cup \{3, 7\} \cup \{4, 8\}$ is not separated. However, the pattern has also a separated 2-block structure given by the partition $\{1, 3, 5, 7\} \cup \{2, 4, 6, 8\}$, obtained by grouping together some sets P_i.

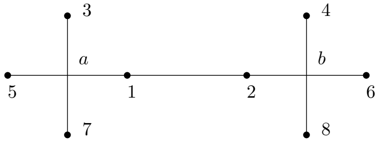


Figure: The canonical model (T, P, f) of a 8-periodic pattern $\mathcal{P}$, which satisfies $f(a) = b$, $f(b) = a$.

Remarks on block structures

The notion of *block structure* is purely combinatorial, since it depends only on the discrete components of $\mathcal{P}$. In contrast, a block structure will be *separated* or *rotational* depending on some topological properties of the blocks in the canonical model of $\mathcal{P}$. However, since the canonical model is unique and it is constructed by means of a well defined algorithm uniquely determined by the combinatorial data of $\mathcal{P}$, in fact both notions are also intrinsic (in the sense that depend only on the combinatorial data of $\mathcal{P}$).

In the literature one can find several kinds of block structures and related notions for periodic orbits. In the interval case, the Sharkovskii's *square root construction* is an early example of a block structure. Also the notion of *extension* gives rise to some particular cases of block structures for interval periodic orbits. Finally, the notion of *division*, introduced for interval periodic orbits and generalized by A.–Ye in order to study the topological entropy and the set of periods for tree maps, has a strong connection with the notion of rotational structure.

Characterization of the topological, algebraic and combinatorial reducibility

Theorem

Let $\mathcal{P}$ be an n-periodic pattern with $n \geq 3$ and let M be the Markov matrix of $\mathcal{P}$. The following statements hold:

1. *M is reducible if and only if $\mathcal{P}$ has separated block structures or collapsing intervals.*
2. *M is cyclic if and only if either $\mathcal{P}$ is trivial or has rotational structures.*

Characterization of the topological, algebraic and combinatorial reducibility

Corollary

Let $\mathcal{P}$ be a non-trivial n-periodic pattern with $n \geq 3$ and let M be the Markov matrix of $\mathcal{P}$. Then, M^k is reducible for some $k \geq 1$ if and only if $\mathcal{P}$ has collapsing intervals or block structures.
Equivalently, M is primitive if and only if $\mathcal{P}$ has no collapsing intervals and no block structures.

Characterization of the topological, algebraic and combinatorial reducibility

The above theorem generalizes some well known results for interval patterns. It is folk knowledge that a periodic interval pattern has a block structure if and only if its Markov matrix is reducible. In fact, this is true not just for interval patterns but for a broader class of patterns, which we call *simplicial*.

A pattern $([T, P], [\theta])$ is called *simplicial* if each discrete component of (T, P) has two points. Observe that, in this case, for each pointed tree $(S, Q) \in [T, P]$ we have that $V(S) \subset Q$ and, for each discrete component C of (S, Q), $\langle C \rangle_S$ is an interval.

Corollary

Let $\mathcal{P}$ be a simplicial n-periodic pattern with $n \geq 3$ and let M be the Markov matrix of $\mathcal{P}$. Then, M is reducible if and only if $\mathcal{P}$ has a block structure. Moreover, if M is irreducible then M is primitive.

Reducibility and entropy

Now we study the topological entropy of patterns with a block structure.

It is a generalization of the following classical result for interval patterns which gives a formula for the entropy of *extensions*.

In order to state it we need to introduce the notion of *skeleton*.

Let $\mathcal{P}$ be an n-periodic pattern and let (T, P, f) be the canonical model of $\mathcal{P}$. Let $P = P_1 \cup P_2 \cup \ldots \cup P_p$ be a partition of P which defines a separated p-block structure or a p-rotational structure for $\mathcal{P}$. It follows that, in both cases, $f(\langle P_i \rangle) = \langle P_{i+1} \rangle$ for $1 \leq i < p$ and $f(\langle P_p \rangle) = \langle P_1 \rangle$.

The skeleton of a pattern

The *skeleton of $\mathcal{P}$* (associated to this partition of P) is a p-periodic pattern $\mathcal{Q}$ defined as follows:

1. If $P_1 \cup P_2 \cup \ldots \cup P_p$ defines a p-rotational structure for $\mathcal{P}$, then $\mathcal{Q}$ is defined to be a trivial p-periodic pattern.
2. If $P_1 \cup P_2 \cup \ldots \cup P_p$ defines a separated p-block structure for $\mathcal{P}$ which is not a p-rotational structure, consider the tree S obtained from T by collapsing each block $\langle P_i \rangle$ to a point x_i. Let $\kappa\colon T \longrightarrow S$ be the standard projection, which is bijective on $T \setminus \cup_i \langle P_i \rangle$ and satisfies $\kappa(\langle P_i \rangle) = x_i$. Set $Q = \kappa(P) = \{x_1, x_2, \ldots, x_p\}$ and define $\theta\colon Q \longrightarrow Q$ by $\theta(x_i) = x_{i+1}$ for $1 \leq i < p$ and $\theta(x_p) = x_1$. Then the skeleton $\mathcal{Q}$ of $\mathcal{P}$ is defined to be the p-periodic pattern $([S, Q], [\theta])$. Observe that $\theta \circ \kappa\big|_P = \kappa \circ f\big|_P$.

Reducibility and entropy

Theorem

Let $\mathcal{P}$ be an n-periodic pattern and let (T, P, f) be the canonical model of $\mathcal{P}$. Assume that there is a partition $P = P_1 \cup P_2 \cup \ldots \cup P_p$ which defines either a p-rotational structure or a separated p-block structure for $\mathcal{P}$. Let $\mathcal{Q}$ be the associated skeleton of $\mathcal{P}$. Then, all the entropies $h(f^p\big|_{\langle P_i \rangle})$ are equal and

$$h(\mathcal{P}) = \max\left\{h(\mathcal{Q}), (1/p)h(f^p\big|_{\langle P_i \rangle})\right\} \text{ for any } 1 \leq i \leq p.$$

Entropy 0

Now we describe the zero entropy periodic patterns (i.e. periodic patterns $\mathcal{P}$ such that $h(\mathcal{P}) = 0$) in terms of the existence of a very particular class of block structures.

Let $\mathcal{P} = ([T, P], [f])$ be a periodic pattern with a p-block structure defined by a partition $P = P_1 \cup P_2 \cup \ldots \cup P_p$. We will say that this p-block structure *has trivial blocks* if the patterns $([\langle P_i \rangle, P_i], [f^p\big|_{P_i}])$ are trivial for $1 \leq i \leq p$. Equivalently, P_i is contained in a discrete component of (T, P) for $1 \leq i \leq p$. Observe that this notion is independent from the particular model (T, P, f) representing the pattern $\mathcal{P}$.

Starry patterns

An n-periodic pattern $\mathcal{P}$ will be called *1-starry* if $\mathcal{P}$ is trivial. For $k \geq 2$, $\mathcal{P}$ will be called *k-starry* if $\mathcal{P}$ has a separated p-block structure with trivial blocks whose associated skeleton is (k-1)-starry.

For an example, consider the 12-periodic pattern $\mathcal{P}_3$ of Example 4. By constructing the canonical model of $\mathcal{P}_3$ one checks that the block structure $\{1,7\} \cup \{2,8\} \cup \{3,9\} \cup \{4,10\} \cup \{5,11\} \cup \{6,12\}$, with trivial blocks, is separated. The associated skeleton $\mathcal{P}_2$ has also a separated 3-block structure $\{1,4\} \cup \{2,5\} \cup \{3,6\}$, again with trivial blocks. Finally, its associated skeleton $\mathcal{P}_1$ is a trivial pattern. Hence, the patterns $\mathcal{P}_k$ are k-starry for $k = 1, 2, 3$.

Example 4

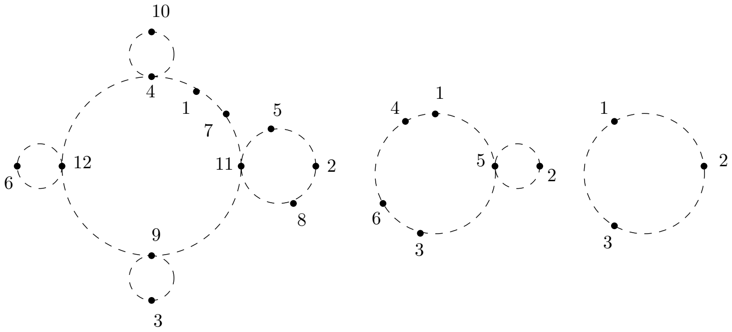


Figure: On the left, a 12-periodic pattern $\mathcal{P}_3$ with a separated 6-block structure $\{1,7\} \cup \{2,8\} \cup \{3,9\} \cup \{4,10\} \cup \{5,11\} \cup \{6,12\}$ with trivial blocks. The dashed circles stand for the discrete components of $\mathcal{P}$. In the center, the corresponding skeleton $\mathcal{P}_2$, with a 3-block structure $\{1,4\} \cup \{2,5\} \cup \{3,6\}$ with trivial blocks. On the right, the corresponding skeleton $\mathcal{P}_1$, a trivial pattern.

Entropy 0; first characterization

Theorem

A periodic pattern $\mathcal{P}$ has entropy zero if and only if $\mathcal{P}$ is k-starry for some $k \geq 1$.

Observe the recursive nature of Theorem 10: the fact that an n-periodic pattern has entropy 0 is translated to the fact that a collection of periodic patterns (the skeleton and those associated to the blocks), with periods strictly smaller than n, have entropy 0. It is well known that the same happens for interval periodic patterns. However, we emphasize that in order for $\mathcal{P}$ to have entropy 0 it is not enough that the patterns exhibited by f^p on each block have entropy 0. In addition, they must be trivial.

Entropy 0; second characterization

Theorem

Let $\mathcal{P} = ([T, P], [f])$ be an n-periodic pattern. Then:

1. *$\mathcal{P}$ has zero entropy if and only if all patterns $([T, P], [f^k])$, for each $k \in \mathbb{N}$ such that k and n are relatively prime, have zero entropy.*
2. *$\mathcal{P}$ has positive entropy if and only if all patterns $([T, P], [f^k])$, for each $k \in \mathbb{N}$ such that k and n are relatively prime, have positive entropy.*

As far as we know, this result was not explicitly stated in the literature, even for interval patterns. We also remark that in general the entropies of the patterns $([T, P], [f^k])$ in the statement (b) of the Theorem need not be equal.