Snap-back repellers and Li-Yorke chaos in rational difference equations

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An order one difference equation (DE) is:

$$x_{k+1} = f(x_k) \qquad k \geq 0 \tag{1}$$

where $x_0 \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a (continuous) function.

The equation is rational (RDE) if f is a rational function.

A solution of the DE/RDE is a sequence $(x_k)_k \subseteq \mathbb{R}^n$ obtained from (1).

If the sequence $(x_k)_k$ has a finite number of elements, we say that x_0 is an element of the **forbidden set** of (1).

- The forbidden set is empty when $\mathrm{Dom} f = \mathbb{R}^n$.
- If f is rational, the forbidden set includes the poles of f.

General open problem in *RDE*

To determine the forbidden set.

Generalized Li-Yorke chaos or Marotto chaos

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a **continuous** function. $x_{k+1} = f(x_k)$ is Li-Yorke chaotic in a generalized sense or Marotto's chaotic if:

LY1 There is an $N \in \mathbb{N}$ such that $\forall p \geq N$ there are (prime) periodic sequences with period p.

LY2 There exist an **scrambled set of Li-Yorke pairs**, i.e., an uncountable set $S \subset \mathbb{R}^n$ holding $f(S) \subset S$ and non containing periodic points, such that for every $x \neq y \in S$,

$$\overline{\lim}_{k\to\infty} ||f^k(x) - f^k(y)|| > 0$$

and for any periodic p of f

$$\overline{\lim}_{k\to\infty} ||f^k(x) - f^k(p)|| > 0$$

LY3 there is an uncountable subset $S_0 \subset S$, such that for every $x_0 \neq y_0 \in S_0$ is

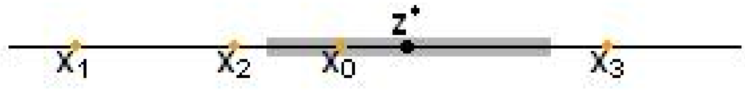
$$\underline{\lim}_{k\to\infty} ||f^k(x_0) - f^k(y_0)|| = 0$$

Notion of snap-back repeller (SBR)

Let $f \in C^1$ in a neighborhood of a fixed point $z*$ of f. We say that $z*$ is a SBR if:

SBR1 All the eigenvalues of $Df(z*)$ have modulus greater than 1 ($z*$ is a repeller)

SBR2 There exist a finite sequence $x_0, x_1, \ldots, x_M$ such that $x_{k+1} = f(x_k)$, $x_M = z*$ and $x_0 \neq z*$ belongs to a repelling neighborhood of $z*$. Moreover, $|Df(x_k)| \neq 0$ for $0 \leq k \leq M-1$.



Theorem (Marotto, 1978; improved by Shi and Chen in 2004)

The existence of a SBR implies generalized Li-Yorke or Marotto chaos.

A similar result, without derivatives and only continuity was proved by P.Kloeden in 1981. In the proof, he used the fact that the existence of continuous one-to-one mappings have continuous inverses on compact spaces and a simplified application of Brouwer fixed point theorem. In this case, saddle points can be considered as well as repellers and Marotto's result can not be applied.

It is apparent that the existence of a snap-back repeller for a one-dimensional map f is equivalent to the existence of a point of period 3 for the map f^n for some positive integer n

Questions:

- Does Marotto's theorem remain true for a RDE?
- How estimate in an easy way the forbidden set of a RDE

Motivation

We consider systems of difference equations

$$x_{n+1} = \frac{P_1(x_n, y_n)}{Q_1(x_n, y_n)}$$

$$y_{n+1} = \frac{P_2(x_n, y_n)}{Q_2(x_n, y_n)}$$

where P_1, Q_1, P_2, Q_2 are quadratic polynomials in x_n, y_n and look for condition for the existence of chaotic behavior, since in simulations we appreciate it. For some particular values of the parameters, Mazrooei and Sebdani have recently have given an example of the former system, where the origin $(0, 0)$ is a snap-back repeller and as a consequence there is a generalized Li - Yorke chaos in a neighborhood of it. Fortunately in such neighborhood, there is no point belonging to the forbidden set.

Motivation

In population dynamics we find many models of RDE even with several delays and also in application of the Newton's method to polynomial equations

Easy examples are

- inverse parabolas, $x_{k+1} = \frac{1}{x_k^2 - r}$
- inverse logistic equation, $x_{k+1} = \frac{1}{r x_k (1 - x_k)}$

In a RDE, we define SBR in the same way that in the continuous case, only adding that none of the elements of the finite sequence x_0, x_1,... , $x_M = z*$ can be in the forbidden set.

Theorem (LY1)

In a RDE, the existence of a SBR implies the existence of infinite periods (LY1).

Sketch of proof

The key idea is fix a special compact neighborhood $B_\epsilon(z*)$ and to apply Brouwer's fixed point theorem to each function of the form

$$f^{-k} \circ g$$

where

- g is the local inverse of f^M in a neighborhood of $z*$.
- f^{-k} is the k-times composition of a local inverse of f near $z*$.

Sketch of proof II

These concepts appear in Li-Yorke and Marotto works. In the RDE setting, they remain valid because of the local continuity and differentiability of the iteration function.

An important remark is that the repelling character of $z*$ assures that $f^{-1}(A) \subseteq A$ where A is a neighborhood of $z*$, and therefore $f^{-k}(x)$ is well defined for all $x \in A$ and $k > 0$.

Second part of Marotto's theorem

We wonder if it is true or not the existence of an uncountable set of Li-Yorke pairs when the *RDE* has a *SBR*
We claim that the answer is not, but we have not be able to find yet a counterexample.

Our claim is based on the necessity of an additional topological condition if we want to obtain similar results to those of Marotto. We have called it *the compact preimage property (CPC)*.

Definition (Itinerary)

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ *a function. A sequence of compact subsets of* $\mathbb{R}^n$, $\{I_k\}_{k=0}^{+\infty}$, *is an itinerary of* f *when* $f(I_k) \supseteq f(I_{k+1})$.

Itineraries are the tool that allow us to locate special solutions of a difference equation behaving in a particular way. Indeed, they are used several times in the proof of the second part of Marotto's theorem, where we find the result:

Lemma

If f *is continuous, and* $\{I_k\}_{k=0}^{+\infty}$ *is an itinerary, there exist* $x_0 \in \mathbb{R}^n$ *such that* $f^k(x_0) \in I_k$, $\forall k \geq 0$.

Itineraries' lemma in the real line is an application of the intermediate value property.

In the nth-dimensional case the lemma was proved by P. Diamond in 1976.

In the RDE setting the problem is more complicate because we have in addition the presence of the the forbidden set and the non continuity in the maps.

Definition (CPP property)

We say that a function f has the compact pre-image property (CPP) if for every compact subset K in the range of f, there exist a compact L such that $f(L) = K$.

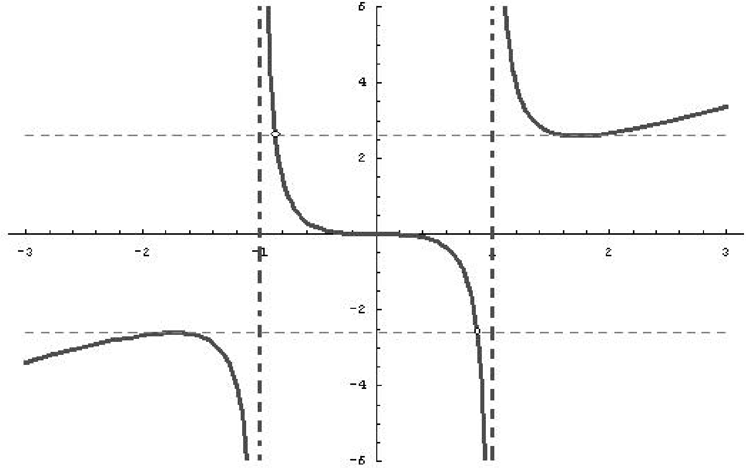
- If f has CPP, this is also the case for f^k.
- If $f : \mathbb{R} \to \mathbb{R}$ is continuous, the Intermediate Value Property proves that f is CPP.
- If $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, the CPP property is also true.

Lemma

If f is CPP, and $\{I_k\}_{k=0}^{+\infty}$ is an itinerary, there exist $x_0 \in \mathbb{R}^n$ such that $f^k(x_0) \in I_k$, $\forall k \geq 0$.

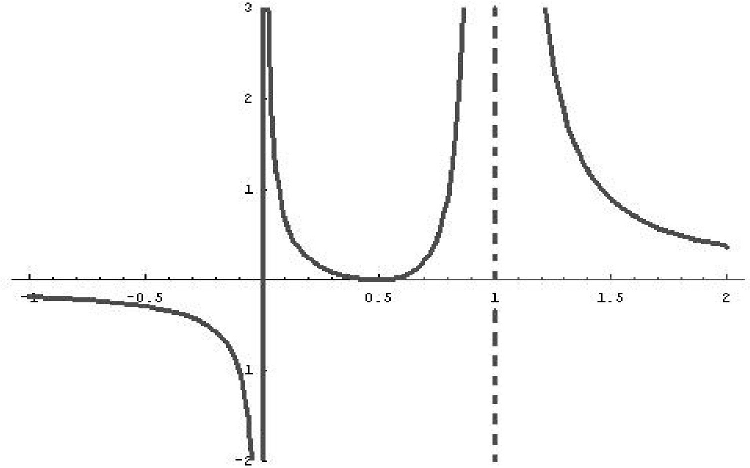
Example of non-CPP function: reducible case

$$f(x) = \frac{4x^5 - 3x^3}{4x^4 - 7x^2 + 3}$$



Another non-CPP function

$$f(x) = \frac{1}{3} \cdot \frac{(x - 1/2)^2}{x(x-1)^2}$$



Theorem

A RDE with the CPP property and with a $z*$ *is generalized Li-Yorke chaotic.*

LY2: Sketch of proof.

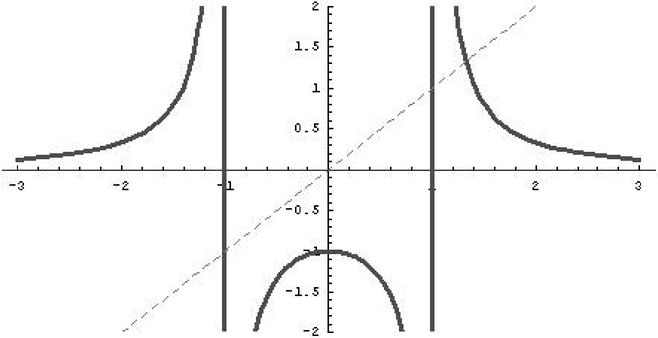
The definition of SBR allows to construct two compact sets U, V of $\mathbb{R}^n$ and $N \in \mathbb{N}$ such that:

- $U \cap V = \emptyset$
- $V \subset f^N(U)$
- $U, V \subset f^N(V)$

Therefore, any sequence $\{I_k\}_{k=0}^{+\infty}$ with $I_k = U, V$ and with the restriction that $I_k = U$ implies $I_{k+1} = V$, is an itinerary of f^N.
The set of all these itineraries define an uncountable subset of $\mathbb{R}^n$ which can be refined to obtain the scrambled set of Li-Yorke pairs.

$$x_{k+1} = \frac{1}{x_k^2 - 1} \tag{2}$$

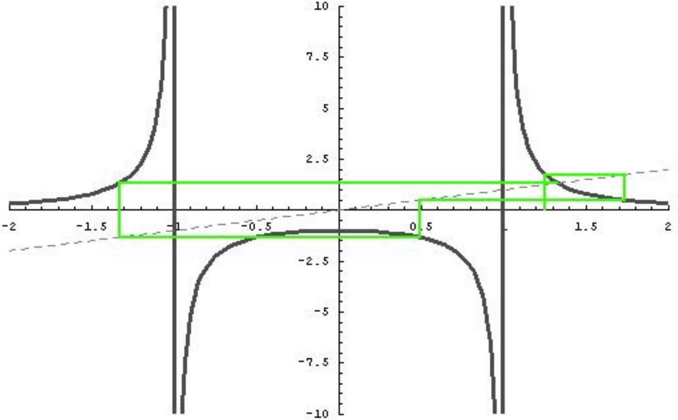
We apply Marotto's criterion to this particular RDE. This is possible because function $f(x) = \frac{1}{x^2-1}$ is CPP, for each compact interval in the image of f belongs to one of the three branches of the graph, and we have continuity over them.



Moreover, RDE (2) has a SBR in

$$z* = \frac{\sqrt[3]{9-\sqrt{69}} + \sqrt[3]{9+\sqrt{69}}}{\sqrt[3]{18}} \approx 1.32472$$

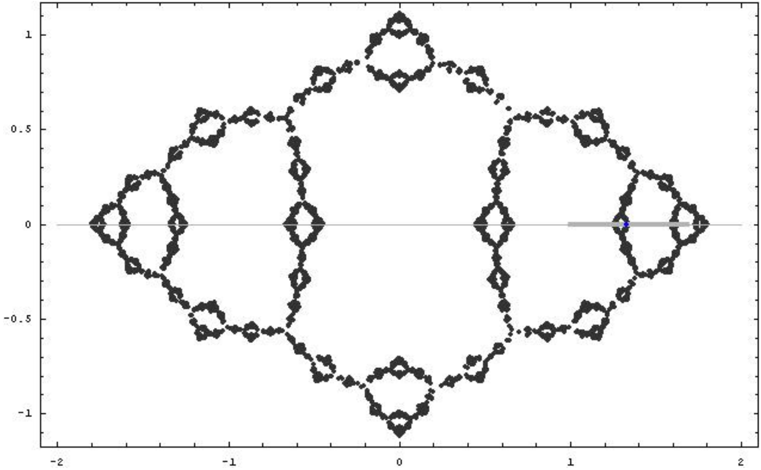
This is illustrated in the following cobweb diagram.



There is another way to locate the SBR of RDE (2). We consider the inverse difference equation:

$$x_{k+1} = \pm\sqrt{1 + \frac{1}{x_k}} \qquad (3)$$

Now, we compute the *solution* of (3) corresponding to $x_0 = z*$. So we must work in the complex domain, but it is easy to verify if some of the branches of this multi-valuate solution belongs to a repeller neighborhood of $z*$.

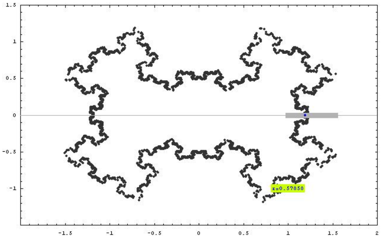


If we start the inverse iteration with the poles of (2), we obtain some type of fractal structure. Therefore, it is an estimation of the forbidden set.

Now, we consider the family

$$x_{k+1} = \frac{1}{x_k^2 - r} \qquad r \in \mathbb{R} \tag{4}$$

These RDE are CPP, so it will be possible to apply Marotto's theorem if we can locate SBRs. A fast way to do that is to draw the fractals corresponding to each RDE of the family.



We have estimated that there is a bifurcation point in $r_0 = \frac{1}{\sqrt[3]{2}}$.
We conjecture that if $r \geq r_0$, then RDE (4) has a SBR and therefore is chaotic.

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