Periods of periodic orbits for vertex maps on graphs

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Introduction

One of the basic starting points for one-dimension combinatorial dynamics is Sharkovsky’s Theorem.

Theorem

Let \( f : \mathbb{R} \to \mathbb{R} \) be continuous. If \( f \) has a periodic point of least period \( v \) then \( f \) also has a periodic point of least period \( m \) for any \( m \triangleleft v \), where

\[
1 \triangleleft 2 \triangleleft 4 \triangleleft \ldots \ldots 28 \triangleleft 20 \triangleleft 12 \triangleleft \ldots \ldots 14 \triangleleft 10 \triangleleft 6 \ldots 7 \triangleleft 5 \triangleleft 3.
\]
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\[
\begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 1 \\
0 & 1 & 1
\end{bmatrix}
\]
Theorem

Let $M$ be the Markov matrix associated to a directed graph that has vertices labeled $E_1, \ldots, E_n$, then the $ij$th entry of $M^k$ gives the number of walks of length $k$ from $E_j$ to $E_i$. 
a basic result

Theorem

Let $M$ be the Markov matrix associated to a directed graph that has vertices labeled $E_1, \ldots, E_n$, then the $ij$th entry of $M^k$ gives the number of walks of length $k$ from $E_j$ to $E_i$.

Corollary

The trace of $M^k$ gives the total number of closed walks of length $k$.
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\[ \theta \]

\[
\begin{align*}
E_1 & \quad E_2 \\
E_3 & \quad E_4
\end{align*}
\]

\[
\begin{align*}
1 & \quad 2 \\
3 & \quad 4
\end{align*}
\]
an example – with orientation

\[ M_1(\theta) = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \]
an example – with orientation

\[ M_0(\theta) = \begin{pmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix},
M_1(\theta) = \begin{pmatrix}
0 & 0 & -1 \\
1 & 0 & -1 \\
0 & 1 & -1
\end{pmatrix} \]
Theorem

The $ij$th entry of $(M_1(\theta))^k$ gives the number of positively oriented walks of length $k$ from $E_j$ to $E_i$ minus the number negatively oriented walks from $E_j$ to $E_i$. 
basic properties

Theorem

The $ij$th entry of $(M_1(\theta))^k$ gives the number of positively oriented walks of length $k$ from $E_j$ to $E_i$ minus the number negatively oriented walks from $E_j$ to $E_i$.

Corollary

The trace of $(M_1(\theta))^k$ gives the number of positively oriented closed walks of length $k$ minus the number of negatively oriented closed walks of length $k$. 
basic properties

Theorem

1. \((M_0(\theta))^k = M_0(\theta^k)\)
2. \((M_1(\theta))^k = M_1(\theta^k)\)
basic properties

**Theorem**

1. \((M_0(\theta))^k = M_0(\theta^k)\)
2. \((M_1(\theta))^k = M_1(\theta^k)\)

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M_0(\theta) = \begin{pmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix},
M_1(\theta) = \begin{pmatrix}
0 & 0 & -1 \\
1 & 0 & -1 \\
0 & 1 & -1
\end{pmatrix}
\]
basic properties

Theorem

\[ \text{Trace } (M_0(\theta)) - \text{Trace } (M_1(\theta)) = 1. \]
first lemma

Lemma

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that $f$ has a periodic point of period 17. Then $f$ has a periodic point of period $2^k$ for any non-negative integer $k$. 
first lemma

Proof.

Since 17 is not a divisor of $2^k$ we know that $\theta^{2^k}$ does not fix any of the integers in $\{1, 2, \ldots, 17\}$. So $\text{Trace} (M_0(\theta^{2^k})) = 0$. 
first lemma

Proof.

Since 17 is not a divisor of $2^k$ we know that $\theta^{2^k}$ does not fix any of the integers in $\{1, 2, \ldots, 17\}$. So $\text{Trace} \left( M_0(\theta^{2^k}) \right) = 0$. So $\text{Trace} \left( M_1(\theta^{2^k}) \right) = -1$. 

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first lemma

Proof.

Since 17 is not a divisor of $2^k$ we know that $\theta^{2^k}$ does not fix any of the integers in $\{1, 2, \ldots, 17\}$. So $\operatorname{Trace}(M_0(\theta^{2^k})) = 0$. So $\operatorname{Trace}(M_1(\theta^{2^k})) = -1$. So the oriented Markov graph has a vertex $E_j$ with a closed walk from $E_j$ to itself of length $2^k$ with negative orientation.
First lemma

Definition.

Since 17 is not a divisor of $2^k$ we know that $\theta^{2^k}$ does not fix any of the integers in $\{1, 2, \ldots, 17\}$. So $\text{Trace} (M_0(\theta^{2^k})) = 0$. So $\text{Trace} (M_1(\theta^{2^k})) = -1$. So the oriented Markov graph has a vertex $E_j$ with a closed walk from $E_j$ to itself of length $2^k$ with negative orientation. Since the orientation is negative it cannot be the repetition of a shorter closed walk, as any shorter closed walk would have to be repeated an even number of times.
first lemma

Proof.

Since 17 is not a divisor of $2^k$ we know that $\theta^{2^k}$ does not fix any of the integers in $\{1, 2, \ldots, 17\}$. So $\text{Trace } (M_0(\theta^{2^k})) = 0$. So $\text{Trace } (M_1(\theta^{2^k})) = -1$. So the oriented Markov graph has a vertex $E_j$ with a closed walk from $E_j$ to itself of length $2^k$ with negative orientation. Since the orientation is negative it cannot be the repetition of a shorter closed walk, as any shorter closed walk would have to be repeated an even number of times. So there is a periodic point in $E_j$ with minimum period $2^k$. 


second lemma

**Lemma**

Let \( f : \mathbb{R} \to \mathbb{R} \) be continuous. Suppose that \( f \) has a periodic point of period 17. Then \( f \) has a periodic point of period \( m \) for any non-negative integer \( m > 17 \).
second lemma

**Proof.**

\[ \text{Trace} \left( M_1(\theta) \right) = -1. \] So there vertex \( E_j \) in the Markov graph with a closed walk of length one with negative orientation.
second lemma

**Proof.**

Trace \( (M_1(\theta)) = -1 \). So there vertex \( E_j \) in the Markov graph with a closed walk of length one with negative orientation. \( M_1(\theta)^{17} \) is the identity matrix. So there is a closed walk from \( E_j \) to itself with length 17 and with positive orientation.
Proof.

Trace \((M_1(\theta)) = -1\). So there vertex \(E_j\) in the Markov graph with a closed walk of length one with negative orientation. \(M_1(\theta)^{17}\) is the identity matrix. So there is a closed walk from \(E_j\) to itself with length 17 and with positive orientation. The closed walk of length 17 is not a repetition of the walk of length 1.
second lemma

Proof.

Trace \((M_1(\theta)) = -1.\) So there vertex \(E_j\) in the Markov graph with a closed walk of length one with negative orientation. \(M_1(\theta)^{17}\) is the identity matrix. So there is a closed walk from \(E_j\) to itself with length 17 and with positive orientation. The closed walk of length 17 is not a repetition of the walk of length 1. We can construct a non-repetitive closed walk of length \(m\) by going once around the walk of length 17 and then \(m - 17\) times around the walk of length 1.
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$M_0(\theta) = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}$
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\[
M_1(\theta) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \\
1 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 \\
0 & 0 & -1 & 0 & -1 & -1 & 0 & 1 \\
0 & 1 & 0 & -1 & 0 & 1 & 0 & 1
\end{pmatrix}
\]
basic properties

Theorem

1. \((M_0(\theta))^k = M_0(\theta^k)\)
2. \((M_1(\theta))^k = M_1(\theta^k)\)
basic properties

Theorem

1. \((M_0(\theta))^k = M_0(\theta^k)\)
2. \((M_1(\theta))^k = M_1(\theta^k)\)
3. \(\text{Trace} (M_0(\theta)) – \text{Trace} (M_1(\theta)) = L_f\)
basic properties

Theorem

1. $(M_0(\theta))^k = M_0(\theta^k)$
2. $(M_1(\theta))^k = M_1(\theta^k)$
3. $\text{Trace } (M_0(\theta)) - \text{Trace } (M_1(\theta)) = L_f$

Corollary

If the underlying map is homotopic to the identity, then
$\text{Trace } (M_0(\theta)) - \text{Trace } (M_1(\theta)) = v - e$
Lemma

Let $G$ be a graph and $f$ a vertex map from $G$ to itself that is homotopic to the identity. Suppose that the vertices form one periodic orbit. Suppose $f$ flips an edge. If $v$ is not a divisor of $2^k$, then $f$ has a periodic point with period $2^k$. 
first lemma – redux

Proof.

Since $f$ flips an edge, there must be at least one loop in the Markov graph that has length 1 and has negative orientation.
first lemma – redux

**Proof.**

Since \( f \) flips an edge, there must be at least one loop in the Markov graph that has length 1 and has negative orientation. Since \( \text{Trace}(M_1(f)) = e - v \), there must be at least \( e - v + 1 \) loops in Markov graph of length 1 that have positive orientation.
**Proof.**

Since $f$ flips an edge, there must be at least one loop in the Markov graph that has length 1 and has negative orientation. Since $\text{Trace}(M_1(f)) = e - v$, there must be at least $e - v + 1$ loops in Markov graph of length 1 that have positive orientation. By going around each of these loops in the Markov graph twice we can see that there must be at least $e - v + 2$ loops of length 2 that have positive orientation.
Proof.

Since $f$ flips an edge, there must be at least one loop in the Markov graph that has length 1 and has negative orientation. Since $\text{Trace}(M_1(f)) = e - v$, there must be at least $e - v + 1$ loops in Markov graph of length 1 that have positive orientation. By going around each of these loops in the Markov graph twice we can see that there must be at least $e - v + 2$ loops of length 2 that have positive orientation. Since $\text{Trace}(M_1(f)^2) = e - v$, there must be at least one loop of length 2 with negative orientation. Since it has negative orientation, it cannot be the repetition of a shorter loop.
first lemma – redux

Proof.

Since $f$ flips an edge, there must be at least one loop in the Markov graph that has length 1 and has negative orientation. Since $\text{Trace}(M_1(f)) = e - v$, there must be at least $e - v + 1$ loops in Markov graph of length 1 that have positive orientation. By going around each of these loops in the Markov graph twice we can see that there must be at least $e - v + 2$ loops of length 2 that have positive orientation.

Since $\text{Trace}(M_1(f)^2) = e - v$, there must be at least one loop of length 2 with negative orientation. Since it has negative orientation, it cannot be the repetition of a shorter loop. So the Markov graph of $f$ has a non-repetitive loop of length 2 with negative orientation.
first lemma – redux

Proof.

Since $f$ flips an edge, there must be at least one loop in the Markov graph that has length 1 and has negative orientation. Since $\text{Trace}(M_1(f)) = e - \nu$, there must be at least $e - \nu + 1$ loops in Markov graph of length 1 that have positive orientation. By going around each of these loops in the Markov graph twice we can see that there must be at least $e - \nu + 2$ loops of length 2 that have positive orientation.

Since $\text{Trace}(M_1(f)^2) = e - \nu$, there must be at least one loop of length 2 with negative orientation. Since it has negative orientation, it cannot be the repetition of a shorter loop. So the Markov graph of $f$ has a non-repetitive loop of length 2 with negative orientation.

e tc – use induction
Lemma

Let $G$ be a graph and $f$ a map from $G$ to itself that is homotopic to the identity. Suppose that the vertices form one periodic orbit. Suppose $f$ flips an edge. If $v = 2^p q$, where $q > 1$ is odd and $p \geq 0$, then $f$ has a periodic point with period $2^p r$ for any $r \geq q$. 

Proof. Similar trace argument.
Lemma

Let $G$ be a graph and $f$ a map from $G$ to itself that is homotopic to the identity. Suppose that the vertices form one periodic orbit. Suppose $f$ flips an edge. If $v = 2^p q$, where $q > 1$ is odd and $p \geq 0$, then $f$ has a periodic point with period $2^p r$ for any $r \geq q$.

Proof.

Similar trace argument.
The Sharkovsky ordering can be defined as follows:
(what positive integers does $v$ force?)

1. $2^l \triangleleft 2^k = v$ if $l \leq k$. 

Sharkovsky ordering
Sharkovsky ordering

The Sharkovsky ordering can be defined as follows: (what positive integers does $v$ force?)

1. $2^l < 2^k = v$ if $l \leq k$.
2. If $v = 2^k s$, where $s > 1$ is odd, then
The Sharkovsky ordering can be defined as follows: (what positive integers does \( v \) force?)

1. \( 2^l \triangleleft 2^k = v \) if \( l \leq k \).
2. If \( v = 2^k s \), where \( s > 1 \) is odd, then \( 2^l \triangleleft v \), for all positive integers \( l \).
The Sharkovsky ordering can be defined as follows: (what positive integers does $v$ force?)

1. $2^l \triangleleft 2^k = v$ if $l \leq k$.
2. If $v = 2^k s$, where $s > 1$ is odd, then
   1. $2^l \triangleleft v$, for all positive integers $l$.
   2. $2^k r \triangleleft v$, where $r \geq s$. 

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The Sharkovsky ordering can be defined as follows:
(what positive integers does \( v \) force?)

1. \( 2^l \prec 2^k = v \) if \( l \leq k \).
2. If \( v = 2^k s \), where \( s > 1 \) is odd, then
   1. \( 2^l \prec v \), for all positive integers \( l \).
   2. \( 2^k r \prec v \), where \( r \geq s \).
   3. \( 2^l r \prec v \), where \( l > k \) and such that \( 2^l r < v \).
This is a way of generalizing from maps of the interval and circle to maps on graphs.
1. This is a way of generalizing from maps of the interval and circle to maps on graphs.

2. This is not the most general method of generalizing, but it leads to interesting results, and is very accessible.
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More info at: *Sharkovsky's theorem and one-dimensional combinatorial dynamics* arxiv.org/abs/1201.3583
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