Maximally transitive semigroups of $n \times n$ matrices

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Definition

Let $G$ be a semigroup acting on a topological space $X$ by continuous maps. The action of $G$ on $X$ is called

- **hypercyclic**, if there exists $x \in G$ such that the $G$-orbit of $x$ defined by $\{f(x) : f \in G\}$ is dense in $X$. 
- **topologically transitive**, if for every pair of nonempty open sets $U$ and $V$ in $X$, there exists a map $f \in G$ so that $f(U) \cap V \neq \emptyset$.
- **topologically $k$-transitive**, if the induced action of $G$ on $X^k$ is topologically transitive.
Introduction

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Theorem

Let $G$ be a semigroup acting by continuous maps on a separable complete metric space $X$ without isolated points. If the action of $G$ is topologically transitive, then there exists a $G_\delta$ set $W \subseteq X$ so that the $G$-orbit of every $x \in W$ is dense in $X$. 

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Birkhoff transitivity theorem for semigroup actions

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For the action of $GL(n, \mathbb{K})$ on $\mathbb{K}^n$, the action of a subsemigroup is $n$-transitive if and only if the subsemigroup is dense in $GL(n, \mathbb{K})$. 
The one-dimensional case

- **The real case:** If \( \frac{\ln(-a)}{\ln b} \) is a negative irrational number, then \( \langle a, b \rangle \) is dense in \( \mathbb{R} \).
The one-dimensional case

- **The real case:** If $\ln(-a) / \ln b$ is a negative irrational number, then $\langle a, b \rangle$ is dense in $\mathbb{R}$.

- **The complex case:** If $\ln(-a) / \ln(b) < 0$ and the numbers $1, \frac{\ln(-a)}{\ln b}, \frac{\arg(c)}{2\pi}$, are rationally independent, then $\langle a, b \rangle$ is dense in $\mathbb{C}$.
The commutative case

- Feldman: The minimum number of generators for the semigroup of diagonal matrices is $n + 1$.
- Ayadi, Costakis, and Abels-Manoussos: minimum number of generators of an abelian semigroup of matrices with a dense orbit:
  - Real case: $\lfloor (n + 3)/2 \rfloor$
  - Complex case: $n + 1$
  - Real case triangular non-diagonalizable: $n + 1$
  - Complex case triangular non-diagonalizable: $n + 2$
The non-commutative case

- Does there exist a pair of matrices in $\text{GL}(n, \mathbb{K})$ that generates a dense subsemigroup of $\text{GL}(n, \mathbb{K})$?
- Does there exist a pair of matrices in $\text{SL}(n, \mathbb{K})$ that generates a dense subsemigroup of $\text{SL}(n, \mathbb{K})$?
- What is the minimum number of generators of a dense semigroup of lower-triangular matrices?
A 2-dimensional explicit example

- The semigroup of matrices generated by

\[
A = \begin{pmatrix} 1 & 1/2 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -8/3 \end{pmatrix}
\]

is dense in the set of $2 \times 2$ real matrices.
A 2-dimensional explicit example

▶ The semigroup of matrices generated by

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▶ The matrices

\[
A = \begin{pmatrix} 1/\sqrt{2} & -\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \sqrt{2}/3 & 0 \\ 0 & \sqrt{3}/2 \end{pmatrix}
\]

generate a dense subsemigroup of \( SL(2, \mathbb{R}) \).
n-transitive subsemigroups of matrices

Theorem

There exists a 2-generator semigroup of matrices whose action on the set of $\mathbb{K}^n$ is topologically $n$-transitive. Equivalently, this semigroup is dense in the set of $n \times n$ matrices with entries in $\mathbb{K}$. 
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This is an optimal result because the action of a singly generated subsemigroup is not even hypercyclic, while the action of a subsemigroup of $GL(n, \mathbb{K})$ can never be $(n + 1)$-transitive.
Dense subsemigroups of Lie groups

Abels-Vinberg: Connected Lie groups with finite center have 2-generator dense sub(semi)groups.

Sketch of the proof:

▶ Given a non-central element $g$, there exists elliptic $h$ so that $\langle g, h \rangle$ is dense.

In $\text{SL}(n, \mathbb{C})$, choose $g$ and $h$ (of finite order $p$) so that $\langle g, h \rangle$ is dense in $\text{SL}(n, \mathbb{C})$.

▶ Choose $a, b \in \mathbb{C}$ so that $\langle a^p, b^p \rangle$ is dense in $\mathbb{C}$. Then $\langle ag, bh \rangle$ is dense in $\text{GL}(n, \mathbb{C})$.

▶ In $\mathbb{R}$, further care is required.
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- Choose $a, b \in \mathbb{C}$ so that $\langle a^p, b^p \rangle$ is dense in $\mathbb{C}$. Then $\langle ag, bh \rangle$ is dense in $GL(n, \mathbb{C})$. 

In $R^2$, further care is required.
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Sketch of the proof:

- Given a non-central element $g$, there exists elliptic $h$ so that $\langle g, h \rangle$ is dense.
- In $SL(n, \mathbb{C})$, choose $g$ and $h$ (of finite order $p$) so that $\langle g, h \rangle$ is dense in $SL(n, \mathbb{C})$.
- Choose $a, b \in \mathbb{C}$ so that $\langle a^p, b^p \rangle$ is dense in $\mathbb{C}$. Then $\langle ag, bh \rangle$ is dense in $GL(n, \mathbb{C})$.
- In $\mathbb{R}$, further care is required.
An alternative approach

Lemma
Let $\Lambda$ be a closed subsemigroup of $(n + 1) \times (n + 1)$ matrices with entries in $\mathbb{K}$ such that

$$\forall F \in GL(n, \mathbb{K}) : \begin{pmatrix} F & 0 \\ 0 & 1 \end{pmatrix} \in \Lambda.$$ 

Suppose that there exists

$$K = \begin{pmatrix} F & X \\ Y & \eta \end{pmatrix} \in \Lambda,$$

such that

$$YF^{-1}X \neq 0, \eta.$$

Then $\Lambda$ contains all $(n + 1) \times (n + 1)$ matrices with entries in $\mathbb{K}$.
Inductive construction

Theorem
For any $n \geq 1$, there exists a pair of matrices in $\mathcal{M}_{n\times n}(\mathbb{C})$ that generates a dense subsemigroup of $\mathcal{M}_{n\times n}(\mathbb{C})$. Moreover, for $n \geq 2$, we can arrange for one of the matrices to be of the form

$$A = \begin{pmatrix}
Z_1 & 0 & \ldots & 0 \\
0 & Z_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & Z_n
\end{pmatrix},$$

where $Z_n = 1$, $Z_1 \neq 0$, and each $Z_i, 1 < i < n$, is a root of unity.
Proof

Induction: Given $A$ and $E$ generating a dense subsemigroup of $GL(n, \mathbb{C})$, let

$$C = \begin{pmatrix} Z_1' & 0 & \ldots & 0 \\ 0 & Z_2' & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & Z_n' \end{pmatrix}, \quad D = \begin{pmatrix} E & 0 \\ 0 & 1 \end{pmatrix},$$

where $Z_i' = \sqrt{Z_i}$ for $1 \leq i < n$, and

$$Z_n' = \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{pmatrix}.$$
Thank You!

Any Questions?