Shadowable chain transitive sets of $C^1$-vector fields

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Motivations

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Conjecture

Abdenur and Díaz(2007)
Given a locally maximal transitive set $\Lambda$ of a generic diffeomorphisms $f$, then either,

(a) $\Lambda$ is hyperbolic or

(b) there are a neighborhood $U(f)$ of $f$ and a small locally maximal neighborhood $U$ of $\Lambda$ such that every $g \in U(f)$ is non-shadowable in the neighborhood $U$. 

Abdenur and Díaz(2007)
There is a residual set $G \subset \text{Diff}(M)$ such that $f \in G$ is shadowable if and only if it is hyperbolic.
Previous results

Lee and Wen (2012)
A locally maximal chain transitive set of a $C^1$-generic diffeomorphism is hyperbolic if and only if it is shadowable.

Main Theorem
For $C^1$ generic vector field $X$, a locally maximal chain transitive set is shadowable if and only if the chain transitive set is a hyperbolic basic set.

Basic notions

- $M$: a compact smooth Riemannian Manifold.
- $\mathcal{X}(M)$: the set of all $C^1$-vector fields of $M$ endowed with the $C^1$-topology.
- $d$: the distance induced from the Riemannian structure.

Shadowing

Pseudo orbit
For $\delta > 0$, a sequence
$$\{(x_i, t_i) : x_i \in M, t_i \geq 1\}_{i=a}^{b} (-\infty \leq a < b \leq \infty)$$
in $M$ is called a $\delta$-pseudo orbit of $X$ if $d(X_{t_i}(x_i), x_{i+1}) < \delta$ for all $a \leq i \leq b - 1$. 
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Basic notions

Shadowing

Let $\Lambda$ be a closed $X_t$-invariant set. We say that $X_t$ has the shadowing property on $\Lambda$ (or $\Lambda$ is shadowable) if for every $\epsilon > 0$ there is $\delta > 0$ such that for any $\delta$-pseudo-orbit
\[
((x_i, t_i))_{i=a}^{b} \subset \Lambda (-\infty \leq a < b \leq \infty), \text{let } T_i = t_0 + \cdots + t_i \text{ for any } 0 \leq i < b, \text{ and } T_i = -t_{-1} - t_{-2} - \cdots - t_i \text{ for any } a < i \leq 0,
\]
there exists a point $y \in M$ and an increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$ such that $d(X_{h(t)}(y), X_{t-T_i}(x_i)) < \epsilon$ for all $a \leq i \leq b - 1$, and $T_i < t < T_{i+1}$.

Star condition

- $P(X)$ : the set of the periodic orbits.

- $Sing(X)$ : the set of singularities.

- $Crit(X) = P(X) \cup Sing(X)$.

- $\mathcal{F}(M)$ : the set of $C^1$ vector fields in $M$ for which there is a $C^1$-neighborhood $\mathcal{U}(X)$ such that every critical orbit of every vector field in $\mathcal{U}(X)$ is hyperbolic.
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Basic notions

Star condition

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- $\text{Sing}(X)$: the set of singularities.
- $\text{Crit}(X) = P(X) \cup \text{Sing}(X)$.
- $\mathcal{F}(M)$: the set of $C^1$ vector fields in $M$ for which there is a $C^1$-neighborhood $U(X)$ such that every critical orbit of every vector field in $U(X)$ is hyperbolic.

We say that $X$ is star flow if $X \in \mathcal{F}(M)$.

If $X \in \mathcal{F}(M)$ and has no singularities, then $X$ is Axiom A and no-cycle condition (Gan and Wen(2006)).

Star condition

- We say that $X$ is star flow if $X \in \mathcal{F}(M)$.
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Chain transitive set

- We say that $\Lambda$ is transitive if there is a point $x \in \Lambda$ such that the closure of $O_{X_t}(x)(t \geq 0)$ is $\Lambda$.
- For given $x, y \in \Lambda$, we write $x \sim_{\Lambda} y$ if for any $\delta > 0$ there is a $\delta$-pseudo-orbit $\{(x_i, t_i)\}_{i=0}^{n}(n \geq 1, t_i \geq 1)$ of $X_t$ in $\Lambda$ such that $x_0 = x$ and $x_n = y$. 
We say that $\Lambda$ is **transitive** if there is a point $x \in \Lambda$ such that the closure of $O_{X_t}(x)(t \geq 0)$ is $\Lambda$.

For given $x, y \in \Lambda$, we write $x \sim_\Lambda y$ if for any $\delta > 0$ there is a $\delta$-pseudo-orbit $\{(x_i, t_i)\}_{i=0}^n (n \geq 1, t_i \geq 1)$ of $X_t$ in $\Lambda$ such that $x_0 = x$ and $x_n = y$.

We say that $C(X)$ is **chain transitive** if $x \sim_{C(X)} y$ for any $x, y \in C(X)$.

Note that every transitive set is chain transitive, but the converse is not true in general.
Basic set

- We say that $\Lambda$ is **locally maximal** if there is a neighborhood $U$ of $\Lambda$ such that

  $$\bigcap_{t \in \mathbb{R}} X_t(U) = \Lambda.$$ 

- We say that $\Lambda$ is **basic set** if it is locally maximal and transitive set.

Hyperbolic

We say that $\Lambda$ is **hyperbolic** for $X_t$ if the tangent bundle $T_{\Lambda}M$ has a $DX_t$-invariant splitting $E^s \oplus X > \oplus E^u$ and there exist constants $C > 0$ and $\lambda > 0$ such that

$$\|DX_t|E_x^s\| \leq Ce^{-\lambda t} \quad \text{and} \quad \|DX_{-t}|E_x^u\| \leq Ce^{-\lambda t}$$

for all $x \in \Lambda$ and $t > 0$.

Generic

- We say that a subset $G \subset \mathcal{X}(M)$ is **residual** if $G$ contains the intersection of a countable family of open and dense subsets of $\mathcal{X}(M)$.

- We say that a property holds $(C^1)$ **generically** if there exists a residual subset $G \subset \mathcal{X}(M)$ such that for any $X \in G$ has that property.
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Outline of the Proof

Step 1  If a locally maximal chain transitive set $C(X)$ is shadowable then $C(X)$ is transitive.

Step 2  For $C^1$-generic $X$, if $X$ has the shadowing property on $C(X)$, then for any hyperbolic periodic orbits $\gamma_1, \gamma_2 \in C(X)$,

$$\text{index}(\gamma_1) = \text{index}(\gamma_2),$$

where $\text{index}(\gamma) = \dim W^s(\gamma)$.

Main Theorem

For $C^1$ generic vector field $X$, a locally maximal chain transitive set $C(X)$ is shadowable if and only if $C(X)$ is a hyperbolic basic set.
Outline of the Proof

Step 3 For $C^1$-generic $X$, if $X$ has the shadowing property on a locally maximal chain transitive set $C(X)$, then $X \in \mathcal{F}(M)$.

Step 4 For $C^1$-generic $X$, if $X$ has the shadowing property on a locally maximal chain transitive set $C(X)$, then $C(X)$ is a hyperbolic basic set.

Sketch of Proof of Step 1

- If $X$ has the shadowing property on a locally maximal chain transitive set $C(X)$, then the shadowing point can be taken from $C(X)$.

- If $X$ has the shadowing property on a locally maximal chain transitive set $C(X)$, then $C(X)$ is transitive.

Sketch of Proof of Step 2

Crovisier(2006)

A compact $X_t$-invariant set $C(X)$ is chain transitive if and only if $C(X)$ is the Hausdorff limit of a sequence of periodic orbits of $X_t$. 
Sketch of Proof of Step 2

- Let $\gamma_1, \gamma_2 \in \mathcal{C}(X)$ be hyperbolic periodic orbits. If $X$ has the shadowing property on $\mathcal{C}(X)$, then

$$W^s(\gamma_1) \cap W^u(\gamma_2) \neq \emptyset, \text{ and } W^u(\gamma_1) \cap W^s(\gamma_2) \neq \emptyset.$$  

- Let $X \in \mathcal{X}(M)$. We say that $X$ is **Kupka-Smale** if all its critical points are hyperbolic, and the invariant manifolds of such elements intersect transversally.

Note that the set of Kupka-Smale vector fields is a residual subset of $\mathcal{X}(M)$.

**Lemma 1**

There is a residual set $G_1 \subset \mathcal{X}(M)$ such that for any $C^1$-neighborhood $\mathcal{U}(X)$ of $X$, if there is $Y \in \mathcal{U}(X)$ such that $Y$ has two distinct hyperbolic periodic orbits $\gamma_Y, \eta_Y$ with different indices, then $X$ has two different hyperbolic periodic orbits $\gamma, \eta$ with different indices.
Sketch of Proof of Step 3

Let \( p \in X \in \mathcal{X}(\mathcal{M}) \), \( x \in M \) and \( T_x M(r) = \{ v \in T_x M : ||v|| \leq r \} \). For every regular point \( x \in M(X(x) \neq 0) \), let \( N_x = \{ X(x) \}^{\perp} \subset T_x M \) and \( N_x(r) \) be the \( r \) ball in \( N_x \). Let \( \mathcal{N}_x,r = \exp_x(N_x(r)) \).

- Given a regular point \( x \in M \) and \( t \in \mathbb{R} \), there are \( r > 0 \) and a \( C^1 \) map \( \tau : \mathcal{N}_x,r \to \mathbb{R} \) with \( \tau(x) = t \) such that \( X_{\tau(y)}(y) \in \mathcal{N}_{X_{\tau(x)},1} \), for any \( y \in \mathcal{N}_x,r \).

Step 3

Lemma 2

There is a residual set \( G_2 \subset \mathcal{X}(\mathcal{M}) \) such that for any \( C^1 \)-neighborhood \( \mathcal{U}(X) \) of \( X \) if there is a \( Y \in \mathcal{U}(X) \) such that there exists at least one point in \( P_h(Y) \) with \( \delta \)-weak eigenvalue, then there exists a point in \( P_h(X) \) with \( 2 \delta \)-weak eigenvalue, where \( P_h(X) \) is the set of hyperbolic periodic orbits.

Poincaré map

Let \( X \in \mathcal{X}(\mathcal{M}) \), \( x \in M \) and \( T_x M(r) = \{ v \in T_x M : ||v|| \leq r \} \). For every regular point \( x \in M(X(x) \neq 0) \), let \( N_x = \{ X(x) \}^{\perp} \subset T_x M \) and \( N_x(r) \) be the \( r \) ball in \( N_x \). Let \( \mathcal{N}_x,r = \exp_x(N_x(r)) \).

- Given a regular point \( x \in M \) and \( t \in \mathbb{R} \), there are \( r > 0 \) and a \( C^1 \) map \( \tau : \mathcal{N}_x,r \to \mathbb{R} \) with \( \tau(x) = t \) such that \( X_{\tau(y)}(y) \in \mathcal{N}_{X_{\tau(x)},1} \), for any \( y \in \mathcal{N}_x,r \).

- We define the Poincaré map \( f_{x,t} : \mathcal{N}_x,r \to \mathcal{N}_{X_{\tau(x)},1} \) by \( f_{x,t}(y) = X_{\tau(y)}(y) \).
We define a flow $\Phi$.

**Poincaré map**

Let $X \in \mathfrak{X}(M), x \in M$ and $T_x M(r) = \{ v \in T_x M : \| v \| \leq r \}$. For every regular point $x \in M(X(x) \neq 0)$, let $N_x = \{ X(x) \} \subset T_x M$ and $N_x(r)$ be the $r$ ball in $N_x$. Let $N_{x,r} = \exp_x(N_x(r))$.

- Given a regular point $x \in M$ and $t \in \mathbb{R}$, there are $r > 0$ and a $C^1$ map $\tau : N_{x,r} \to \mathbb{R}$ with $\tau(x) = t$ such that $X_\tau(y) \in N_{X_\tau(x),1}$, for any $y \in N_{x,r}$.

- We define the **Poincaré map** $f_{x,t} : N_{x,r} \to N_{X_\tau(x),1}$ by $f_{x,t}(y) = X_\tau(y)$.

**Linear Poincaré flow**

Let $M_X = \{ x \in M : X(x) \neq 0 \}$, and let $N = \bigcup_{x \in M_X} N_x$ be the normal bundle based on $M_X$.

- We define a flow $\Phi_t : N \to N$ by $\Phi_t|_{N_x} = \pi_{N_x} \circ D_x X_t|_{N_x}$, where $\pi_{N_x} : T_x M \to N_x$ is the projection and $D_x X_t : T_x M \to T_{X_t(x)} M$ is the derivative map of $X_t$.

- $\Phi_t|_{N_x} = D_x f_{x,t}$.
Lemma 3
Let \( X \in \mathcal{X}(M) \) has no singularities, and let \( U(X) \) be a \( C^1 \)-neighborhood of \( X \) and \( \Lambda \) be locally maximal in \( U \). If \( \gamma \in \Lambda \cap P(Y) \) is not hyperbolic, then there is \( Y \in U(X) \) such that two distinct hyperbolic periodic orbits \( \gamma_1, \gamma_2 \in \Lambda_Y(U) \) with different indices, where \( \Lambda_Y(U) = \bigcap_{t \in \mathbb{R}} Y_t(U) \).

Lemma 4
Let \( C(X) \) be a locally maximal chain transitive set. There is a residual set \( G_4 \subset \mathcal{X}(M) \) such that if \( X \) has no singularities and \( X \) has the shadowing property on a locally maximal chain transitive set \( C(X) \), then there exist constants \( T > 0 \) and \( \lambda > 0 \) such that for any \( p \in \gamma \in P(X) \),

(a) \( \| \Phi_{X_t} |_{E^s(p)} \| \cdot \| \Phi_{X_{t-1}} |_{E^u(X_t(p))} \| \leq e^{-2\lambda t} \) for any \( t \geq T \),

(b) If \( \tau \) is the period of \( p \), \( m \) is any positive integer, and \( 0 = t_0 < t_1 < \cdots < t_k = m\tau \) is any partition of the time interval \([0, m\tau]\) with \( t_{i+1} - t_i \geq T \), then

\[
\frac{1}{m\tau} \sum_{i=0}^{k-1} \log \| \Phi_{X_{t_i+1-t_i}} |_{E^s(X_{t_i}(p))} \| < -\lambda, \quad \text{and}
\]

\[
\frac{1}{m\tau} \sum_{i=0}^{k-1} \log \| \Phi_{X_{-t_i+1-t_i}} |_{E^s(X_{t_i+1}(p))} \| < -\lambda.
\]
Sketch of Proof of Step 4

Let \( x \in M \setminus \text{Sing}(X) \) is called **strongly closable** if for any \( C^1 \)-neighborhood \( \mathcal{U}(X) \) of \( X \), for any \( \delta > 0 \), there are \( Y \in \mathcal{U}(X), p \in \gamma \in P(Y) \) and \( T > 0 \) such that

- (a) \( Y_T(p) = p \),
- (b) \( X(y) = Y(y) \) for any \( y \in M \setminus \bigcup_{t \in [0, T]} B(X_t(x), \delta) \),
- (c) \( d(X_t(x), Y_t(p)) < \delta \) for each \( t \in [0, T] \).

Let \( \Sigma(X) \) be the set of strongly closable points of \( X \).

### References

Wen (1996)

Let \( X \in \mathcal{X}(M) \). \( \mu(\Sigma(X) \cup \text{Sing}(X)) = 1 \), for every \( T > 0 \) and every \( X_T \)-invariant probability Borel measure \( \mu \).

Lee and Wen (2012)

There is a residual set \( \mathcal{G}_5 \subset \mathcal{X}(M) \) such that every \( X \in \mathcal{G}_5 \) satisfies the following property: Any ergodic measure \( \mu \) of \( X \) is the limit of sequence of ergodic invariant measures supported by periodic orbits \( \gamma_n \) of \( X \) in the weak* topology. Moreover, the orbits \( \gamma_n \) converges to the support of \( \mu \) in the Hausdorff topology.
Let $X \in \mathfrak{X}(M)$ without singularities, and let $X \in G_4 \cap G_5$. Then we prove the Main Theorem.

Thanks for your attention.