Shadowable chain transitive sets of $C^1$-vector fields

Manseob Lee
Joint work with Prof. K. Lee

Mokwon University, Daejeon, Korea.

July 2, 2012
Outline

Motivations
   Conjecture
   Previous results

Main Theorem

Basic notions
   Shadowing
   Chain transitive set

Proof of Main Theorem
   Outline of the Proof
   End of the Proof of Main Theorem
Conjecture

Abdenur and Díaz (2007)
There is a residual set $\mathcal{G} \subset \text{Diff}(M)$ such that $f \in \mathcal{G}$ is shadowable if and only if it is hyperbolic.
Abdenur and Díaz(2007)

Given a locally maximal transitive set $\Lambda$ of a generic diffeomorphisms $f$, then either,

(a) $\Lambda$ is hyperbolic or

(b) there are a neighborhood $U(f)$ of $f$ and a small locally maximal neighborhood $U$ of $\Lambda$ such that every $g \in U(f)$ is non-shadowable in the neighborhood $U$. 
Previous results

Lee and Wen (2012)

A locally maximal chain transitive set of a $C^1$-generic diffeomorphism is hyperbolic if and only if it is shadowable.
Main Theorem
For $C^1$ generic vector field $X$, a locally maximal chain transitive set is shadowable if and only if the chain transitive set is a hyperbolic basic set.
Basic notions

- $M$: a compact smooth Riemannian Manifold.

- $\mathcal{X}(M)$: the set of all $C^1$-vector fields of $M$ endowed with the $C^1$-topology.

- $d$: the distance induced from the Riemannian structure.
Shadowable chain transitive sets of $C^1$-vector fields

---

Basic notions

---

Shadowing

---

Shadowing

Pseudo orbit

For $\delta > 0$, a sequence

$\{(x_i, t_i) : x_i \in \mathcal{M}, t_i \geq 1\}_{i=a}^{b} (-\infty \leq a < b \leq \infty)$ in $\mathcal{M}$ is called a $\delta$-pseudo orbit of $X$ if $d(X_{t_i}(x_i), x_{i+1}) < \delta$ for all $a \leq i \leq b - 1$. 
Shadowable chain transitive sets of $C^1$-vector fields

### Basic notions

#### Shadowing

Let $\Lambda$ be a closed $X_t$-invariant set. We say that $X_t$ has the shadowing property on $\Lambda$ (or $\Lambda$ is shadowable) if for every $\epsilon > 0$ there is $\delta > 0$ such that for any $\delta$-pseudo-orbit $\{(x_i, t_i)\}_{i=a}^{b} \subset \Lambda(-\infty \leq a < b \leq \infty)$, let $T_i = t_0 + \cdots + t_i$ for any $0 \leq i < b$, and $T_i = -t_{i-1} - t_{i-2} - \cdots - t_i$ for any $a < i \leq 0$, there exists a point $y \in M$ and an increasing homeomorphism $h : \mathbb{R} \rightarrow \mathbb{R}$ with $h(0) = 0$ such that $d(X_{h(t)}(y), X_{t-T_i}(x_i)) < \epsilon$ for all $a \leq i \leq b - 1$, and $T_i < t < T_{i+1}$. 
Star condition

- $P(X)$: the set of the periodic orbits.
- $\text{Sing}(X)$: the set of singularities.
Shadowable chain transitive sets of $C^1$-vector fields

Basic notions

Star condition

- $P(X)$ : the set of the periodic orbits.
- $Sing(X)$ : the set of singularities.
- $Crit(X) = P(X) \cup Sing(X)$. 
Star condition

- $P(X)$: the set of the periodic orbits.
- $Sing(X)$: the set of singularities.
- $Crit(X) = P(X) \cup Sing(X)$.
- $\mathcal{F}(M)$: the set of $C^1$ vector fields in $M$ for which there is a $C^1$-neighborhood $\mathcal{U}(X)$ such that every critical orbit of every vector field in $\mathcal{U}(X)$ is hyperbolic.
Shadowable chain transitive sets of $C^1$-vector fields

### Basic notions

#### Shadowing

- $P(X)$: the set of the periodic orbits.

- $Sing(X)$: the set of singularities.

- $Crit(X) = P(X) \cup Sing(X)$.

- $\mathfrak{F}(M)$: the set of $C^1$ vector fields in $M$ for which there is a $C^1$-neighborhood $\mathcal{U}(X)$ such that every critical orbit of every vector field in $\mathcal{U}(X)$ is hyperbolic.

### Star condition

- $\mathfrak{F}(M)$: the set of $C^1$ vector fields in $M$ for which there is a $C^1$-neighborhood $\mathcal{U}(X)$ such that every critical orbit of every vector field in $\mathcal{U}(X)$ is hyperbolic.
Star condition

- We say that $X$ is star flow if $X \in \mathcal{F}(M)$.

- If $X \in \mathcal{F}(M)$ and has no singularities, then $X$ is Axiom A and no-cycle condition (Gan and Wen(2006)).
Star condition

- We say that $X$ is star flow if $X \in \mathcal{F}(M)$.

- If $X \in \mathcal{F}(M)$ and has no singularities, then $X$ is Axiom A and no-cycle condition (Gan and Wen(2006)).
Chain transitive set

- We say that $\Lambda$ is **transitive** if there is a point $x \in \Lambda$ such that the closure of $\mathcal{O}_{X_t}(x)(t \geq 0)$ is $\Lambda$.

- For given $x, y \in \Lambda$, we write $x \rightsquigarrow_{\Lambda} y$ if for any $\delta > 0$ there is a $\delta$-pseudo-orbit $\{(x_i, t_i)\}_{i=0}^{n}(n \geq 1, t_i \geq 1)$ of $X_t$ in $\Lambda$ such that $x_0 = x$ and $x_n = y$. 

Note that every transitive set is chain transitive, but the converse is not true in general.
Basic notions

Chain transitive set

- We say that $\Lambda$ is **transitive** if there is a point $x \in \Lambda$ such that the closure of $O_{X_t}(x)(t \geq 0)$ is $\Lambda$.

- For given $x, y \in \Lambda$, we write $x \leadsto^\Lambda y$ if for any $\delta > 0$ there is a $\delta$-pseudo-orbit $\{(x_i, t_i)\}_{i=0}^n (n \geq 1, t_i \geq 1)$ of $X_t$ in $\Lambda$ such that $x_0 = x$ and $x_n = y$.

- We say that $C(X)$ is **chain transitive** if $x \leadsto^C(X) y$ for any $x, y \in C(X)$.
Chain transitive set

- We say that $\Lambda$ is **transitive** if there is a point $x \in \Lambda$ such that the closure of $\mathcal{O}_{X_t}(x)(t \geq 0)$ is $\Lambda$.

- For given $x, y \in \Lambda$, we write $x \rightsquigarrow^\Lambda y$ if for any $\delta > 0$ there is a $\delta$-pseudo-orbit $\{(x_i, t_i)\}_{i=0}^n (n \geq 1, t_i \geq 1)$ of $X_t$ in $\Lambda$ such that $x_0 = x$ and $x_n = y$.

- We say that $C(X)$ is **chain transitive** if $x \rightsquigarrow^ {C(X)} y$ for any $x, y \in C(X)$.

- Note that every transitive set is chain transitive, but the converse is not true in general.
Chain transitive set

- We say that $\Lambda$ is transitive if there is a point $x \in \Lambda$ such that the closure of $O_{X_t}(x)(t \geq 0)$ is $\Lambda$.

- For given $x, y \in \Lambda$, we write $x \rightsquigarrow^\Lambda y$ if for any $\delta > 0$ there is a $\delta$-pseudo-orbit $\{(x_i, t_i)\}_{i=0}^n (n \geq 1, t_i \geq 1)$ of $X_t$ in $\Lambda$ such that $x_0 = x$ and $x_n = y$.

- We say that $C(X)$ is chain transitive if $x \rightsquigarrow^C(X) y$ for any $x, y \in C(X)$.

- Note that every transitive set is chain transitive, but the converse is not true in general.
Chain transitive set

- We say that $\Lambda$ is **transitive** if there is a point $x \in \Lambda$ such that the closure of $O_{X_t}(x)(t \geq 0)$ is $\Lambda$.

- For given $x, y \in \Lambda$, we write $x \leadsto_\Lambda y$ if for any $\delta > 0$ there is a $\delta$-pseudo-orbit $\{(x_i, t_i)\}_{i=0}^n (n \geq 1, t_i \geq 1)$ of $X_t$ in $\Lambda$ such that $x_0 = x$ and $x_n = y$.

- We say that $C(X)$ is **chain transitive** if $x \leadsto_{C(X)} y$ for any $x, y \in C(X)$.

- Note that every transitive set is chain transitive, but the converse is not true in general.
Basic set

We say that $\Lambda$ is **locally maximal** if there is a neighborhood $U$ of $\Lambda$ such that

$$\bigcap_{t \in \mathbb{R}} X_t(U) = \Lambda.$$ 

We say that $\Lambda$ is **basic set** if it is locally maximal and transitive set.
Basic set

- We say that $\Lambda$ is **locally maximal** if there is a neighborhood $U$ of $\Lambda$ such that
  \[ \bigcap_{t \in \mathbb{R}} X_t(U) = \Lambda. \]

- We say that $\Lambda$ is **basic set** if it is locally maximal and transitive set.
We say that $\Lambda$ is hyperbolic for $X_t$ if the tangent bundle $T\Lambda M$ has a $DX_t$-invariant splitting $E^s \oplus \langle X \rangle \oplus E^u$ and there exist constants $C > 0$ and $\lambda > 0$ such that

$$\| DX_t|_{E^s_x} \| \leq Ce^{-\lambda t} \quad \text{and} \quad \| DX_{-t}|_{E^u_x} \| \leq Ce^{-\lambda t}$$

for all $x \in \Lambda$ and $t > 0$. 
Generic

We say that a subset $\mathcal{G} \subset \mathcal{X}(M)$ is residual if $\mathcal{G}$ contains the intersection of a countable family of open and dense subsets of $\mathcal{X}(M)$.

We say that a property holds ($C^1$) generically if there exists a residual subset $\mathcal{G} \subset \mathcal{X}(M)$ such that for any $X \in \mathcal{G}$ has that property.
Generic

- We say that a subset $G \subset \mathcal{X}(M)$ is residual if $G$ contains the intersection of a countable family of open and dense subsets of $\mathcal{X}(M)$.

- We say that a property holds \((C^1)\) generically if there exists a residual subset $G \subset \mathcal{X}(M)$ such that for any $X \in G$ has that property.
Main Theorem

For $C^1$ generic vector field $X$, a locally maximal chain transitive set $\mathcal{C}(X)$ is shadowable if and only if $\mathcal{C}(X)$ is a hyperbolic basic set.
Outline of the Proof

Step 1 If a locally maximal chain transitive set \( \mathcal{C}(X) \) is shadowable then \( \mathcal{C}(X) \) is transitive.

Step 2 For \( C^1 \)-generic \( X \), if \( X \) has the shadowing property on \( \mathcal{C}(X) \), then for any hyperbolic periodic orbits \( \gamma_1, \gamma_2 \in \mathcal{C}(X) \),

\[
\text{index}(\gamma_1) = \text{index}(\gamma_2),
\]

where \( \text{index}(\gamma) = \dim W^s(\gamma) \).
Outline of the Proof

Step 1 If a locally maximal chain transitive set \( C(X) \) is shadowable then \( C(X) \) is transitive.

Step 2 For \( C^1 \)-generic \( X \), if \( X \) has the shadowing property on \( C(X) \), then for any hyperbolic periodic orbits \( \gamma_1, \gamma_2 \in C(X) \),

\[
\text{index}(\gamma_1) = \text{index}(\gamma_2),
\]

where \( \text{index}(\gamma) = \dim W^s(\gamma) \).
Outline of the Proof

Step 3 For $C^1$-generic $X$, if $X$ has the shadowing property on a locally maximal chain transitive set $C(X)$, then $X \in \mathcal{F}(M)$.

Step 4 For $C^1$-generic $X$, if $X$ has the shadowing property on a locally maximal chain transitive set $C(X)$, then $C(X)$ is a hyperbolic basic set.
Outline of the Proof

**Step 3** For $C^1$-generic $X$, if $X$ has the shadowing property on a locally maximal chain transitive set $C(X)$, then $X \in \mathcal{F}(M)$.

**Step 4** For $C^1$-generic $X$, if $X$ has the shadowing property on a locally maximal chain transitive set $C(X)$, then $C(X)$ is a hyperbolic basic set.
Sketch of Proof of Step 1

- If $X$ has the shadowing property on a locally maximal chain transitive set $C(X)$, then the shadowing point can be taken from $C(X)$.

- If $X$ has the shadowing property on a locally maximal chain transitive set $C(X)$, then $C(X)$ is transitive.
Sketch of Proof of Step 2

Crovisier (2006)

A compact $X_t$-invariant set $C(X)$ is chain transitive if and only if $C(X)$ is the Hausdorff limit of a sequence of periodic orbits of $X_t$. 
Sketch of Proof of Step 2

Let $\gamma_1, \gamma_2 \in \mathcal{C}(X)$ be hyperbolic periodic orbits. If $X$ has the shadowing property on $\mathcal{C}(X)$, then

$$W^s(\gamma_1) \cap W^u(\gamma_2) \neq \emptyset, \text{ and } W^u(\gamma_1) \cap W^s(\gamma_2) \neq \emptyset.$$ 

Let $X \in \mathcal{X}(M)$. We say that $X$ is Kupka-Smale if all its critical points are hyperbolic, and the invariant manifolds of such elements intersect transversally.
Sketch of Proof of Step 2

Let $\gamma_1, \gamma_2 \in \mathcal{C}(X)$ be hyperbolic periodic orbits. If $X$ has the shadowing property on $\mathcal{C}(X)$, then

$$W^s(\gamma_1) \cap W^u(\gamma_2) \neq \emptyset, \text{ and } W^u(\gamma_1) \cap W^s(\gamma_2) \neq \emptyset.$$ 

Let $X \in \mathfrak{X}(M)$. We say that $X$ is **Kupka-Smale** if all its critical points are hyperbolic, and the invariant manifolds of such elements intersect transversally.

Note that the set of Kupka-Smale vector fields is a residual subset of $\mathfrak{X}(M)$. 


Sketch of Proof of Step 2

- Let $\gamma_1, \gamma_2 \in \mathcal{C}(X)$ be hyperbolic periodic orbits. If $X$ has the shadowing property on $\mathcal{C}(X)$, then

$$W^s(\gamma_1) \cap W^u(\gamma_2) \neq \emptyset, \text{ and } W^u(\gamma_1) \cap W^s(\gamma_2) \neq \emptyset.$$  

- Let $X \in \mathcal{X}(M)$. We say that $X$ is Kupka-Smale if all its critical points are hyperbolic, and the invariant manifolds of such elements intersect transversally.

Note that the set of Kupka-Smale vector fields is a residual subset of $\mathcal{X}(M)$. 
Sketch of Proof of Step 3

**Lemma 1**
There is a residual set $G_1 \subset \mathcal{X}(M)$ such that for any $C^1$-neighborhood $\mathcal{U}(X)$ of $X$, if there is $Y \in \mathcal{U}(X)$ such that $Y$ has two distinct hyperbolic periodic orbits $\gamma_Y, \eta_Y$ with different indices, then $X$ has two different hyperbolic periodic orbits $\gamma, \eta$ with different indices.
Sketch of Proof of Step 3

Let \( p \in \gamma \in P(X) \) be hyperbolic. For any \( \delta > 0 \), We say that a point \( p \) has a \( \delta \)-weak eigenvalue if there is an eigenvalue \( \lambda \) of \( DX_T(p) \) such that \((1 - \delta) < |\lambda| < (1 + \delta)\).
Step 3

Lemma 2
There is a residual set $\mathcal{G}_2 \subset \mathcal{X}(M)$ such that for any $C^1$-neighborhood $\mathcal{U}(X)$ of $X$ if there is a $Y \in \mathcal{U}(X)$ such that there exists at least one point in $P_h(Y)$ with $\delta$-weak eigenvalue, then there exists a point in $P_h(X)$ with $2\delta$-weak eigenvalue, where $P_h(X)$ is the set of hyperbolic periodic orbits.
Poincaré map

Let $X \in \mathcal{X}(M), x \in M$ and $T_x M(r) = \{ v \in T_x M : \| v \| \leq r \}$. For every regular point $x \in M(X(x) \neq 0)$, let $N_x = \langle X(x) \rangle^\perp \subset T_x M$ and $N_x(r)$ be the $r$ ball in $N_x$. Let $\mathcal{N}_{x,r} = \exp_x(N_x(r))$.

- Given a regular point $x \in M$ and $t \in \mathbb{R}$, there are $r > 0$ and a $C^1$ map $\tau : \mathcal{N}_{x,r} \to \mathbb{R}$ with $\tau(x) = t$ such that $X_{\tau(y)}(y) \in \mathcal{N}_{X_{\tau}(x),1}$, for any $y \in \mathcal{N}_{x,r}$. 
Poincaré map

Let $X \in \mathfrak{X}(M)$, $x \in M$ and $T_x M(r) = \{v \in T_x M : \|v\| \leq r\}$. For every regular point $x \in M (X(x) \neq 0)$, let $N_x = \langle X(x) \rangle ^\perp \subset T_x M$ and $N_x(r)$ be the $r$ ball in $N_x$. Let $\mathcal{N}_{x,r} = \exp_x(N_x(r))$.

- Given a regular point $x \in M$ and $t \in \mathbb{R}$, there are $r > 0$ and a $C^1$ map $\tau : \mathcal{N}_{x,r} \to \mathbb{R}$ with $\tau(x) = t$ such that $X_{\tau(y)}(y) \in \mathcal{N}_{X_{\tau(x)},1}$, for any $y \in \mathcal{N}_{x,r}$.

- We define the Poincaré map $f_{x,t} : \mathcal{N}_{x,r} \to \mathcal{N}_{X_{\tau(x)},1}$ by $f_{x,t}(y) = X_{\tau(y)}(y)$.
Let $X \in \mathcal{X}(M), x \in M$ and $T_x M(r) = \{ v \in T_x M : \|v\| \leq r \}$. For every regular point $x \in M (X(x) \neq 0)$, let $N_x = \langle X(x) \rangle^\perp \subset T_x M$ and $N_x(r)$ be the $r$ ball in $N_x$. Let $\mathcal{N}_{x,r} = \exp_x(N_x(r))$.

- Given a regular point $x \in M$ and $t \in \mathbb{R}$, there are $r > 0$ and a $C^1$ map $\tau : \mathcal{N}_{x,r} \to \mathbb{R}$ with $\tau(x) = t$ such that $X_{\tau(y)}(y) \in \mathcal{N}_{X_\tau(x),1}$, for any $y \in \mathcal{N}_{x,r}$.

- We define the Poincaré map $f_{x,t} : \mathcal{N}_{x,r} \to \mathcal{N}_{X_\tau(x),1}$ by $f_{x,t}(y) = X_{\tau(y)}(y)$. 
Linear Poincaré flow

Let \( M_X = \{ x \in M : X(x) \neq 0 \} \), and let \( N = \bigcup_{x \in M_X} N_x \) be the normal bundle based on \( M_X \).

We define a flow \( \Phi_t : N \to N \) by \( \Phi_t|_{N_x} = \pi_{N_x} \circ D_x X_t|_{N_x} \), where \( \pi_{N_x} : T_x M \to N_x \) is the projection and \( D_x X_t : T_x M \to T_{X_t(x)} M \) is the derivative map of \( X_t \).
Linear Poincaré flow

Let $M_X = \{ x \in M : X(x) \neq 0 \}$, and let $N = \bigcup_{x \in M_X} N_x$ be the normal bundle based on $M_X$.

- We define a flow $\Phi_t : N \to N$ by $\Phi_t|_{N_x} = \pi_{N_x} \circ D_x X_t|_{N_x}$, where $\pi_{N_x} : T_x M \to N_x$ is the projection and $D_x X_t : T_x M \to T_{X_t(x)} M$ is the derivative map of $X_t$.

- $\Phi_t|_{N_x} = D_x f_{x,t}$. 
Linear Poincaré flow

Let $M_X = \{ x \in M : X(x) \neq 0 \}$, and let $N = \bigcup_{x \in M_X} N_x$ be the normal bundle based on $M_X$.

- We define a flow $\Phi_t : N \to N$ by $\Phi_t|_{N_x} = \pi_{N_x} \circ D_x X_t|_{N_x}$, where $\pi_{N_x} : T_x M \to N_x$ is the projection and $D_x X_t : T_x M \to T_{X_t(x)} M$ is the derivative map of $X_t$.

- $\Phi_t|_{N_x} = D_x f_{x,t}$. 
Sketch of Proof of Step 3

Lemma 3
Let $X \in \mathcal{X}(M)$ has no singularities, and let $U(X)$ be a $C^1$-neighborhood of $X$ and $\Lambda$ be locally maximal in $U$. If $\gamma \in \Lambda \cap P(Y)$ is not hyperbolic, then there is $Y \in U(X)$ such that two distinct hyperbolic periodic orbits $\gamma_1, \gamma_2 \in \Lambda_Y(U)$ with different indices, where $\Lambda_Y(U) = \bigcap_{t \in \mathbb{R}} Y_t(U)$. 
Lemma 3
Let $X \in \mathfrak{X}(M)$ has no singularities, and let $U(X)$ be a $C^1$-neighborhood of $X$ and $\Lambda$ be locally maximal in $U$. If $\gamma \in \Lambda \cap P(Y)$ is not hyperbolic, then there is $Y \in U(X)$ such that two distinct hyperbolic periodic orbits $\gamma_1, \gamma_2 \in \Lambda_Y(U)$ with different indices, where $\Lambda_Y(U) = \bigcap_{t \in \mathbb{R}} Y_t(U)$.
Sketch of Proof of Step 3

Lemma 4
Let $\mathcal{C}(X)$ be a locally maximal chain transitive set. There is a residual set $\mathcal{G}_3 \subset \mathcal{X}(M)$ such that for any $X \in \mathcal{G}_3$, if $X$ has the shadowing property on $\mathcal{C}(X)$, then there is $\delta > 0$ such that every hyperbolic periodic orbit in $\mathcal{C}(X)$ has no $\delta$-weak eigenvalue.
Proposition

There is a residual set $\mathcal{G}_4 \subset \mathcal{X}(M)$ such that if $X$ has no singularities and $X$ has the shadowing property on a locally maximal chain transitive set $C(X)$, then there exist constants $T > 0$ and $\lambda > 0$ such that for any $p \in \gamma \in P(X),$

(a) $\|\Phi_{X_t}|_{Es(p)}\| \cdot \|\Phi_{X_{-t}}|_{Eu(X_t(p))}\| \leq e^{-2\lambda t}$ for any $t \geq T$,

(b) If $\tau$ is the period of $p$, $m$ is any positive integer, and $0 = t_0 < t_1 < \cdots < t_k = m\tau$ is any partition of the time interval $[0, m\tau]$ with $t_{i+1} - t_i \geq T$, then

$$\frac{1}{m\tau} \sum_{i=0}^{k-1} \log \|\Phi_{X_{t_{i+1}-t_i}}|_{Es(X_{t_i}(p))}\| < -\lambda,$$

and

$$\frac{1}{m\tau} \sum_{i=0}^{k-1} \log \|\Phi_{X_{-(t_{i+1}-t_i)}}|_{Es(X_{t_i+1}(p))}\| < -\lambda.$$
Let $x \in M \setminus \text{Sing}(X)$ is called **strongly closable** if for any $C^1$-neighborhood $U(X)$ of $X$, for any $\delta > 0$, there are $Y \in U(X)$, $p \in \gamma \in P(Y)$ and $T > 0$ such that

(a) $Y_T(p) = p$,

(b) $X(y) = Y(y)$ for any $y \in M \setminus \bigcup_{t \in [0, T]} B(X_t(x), \delta)$,

(c) $d(X_t(x), Y_t(p)) < \delta$ for each $t \in [0, T]$.

Let $\Sigma(X)$ be the set of strongly closable points of $X$. 

---

**Sketch of Proof of Step 4**

- Let $x \in M \setminus \text{Sing}(X)$ is called **strongly closable** if for any $C^1$-neighborhood $U(X)$ of $X$, for any $\delta > 0$, there are $Y \in U(X)$, $p \in \gamma \in P(Y)$ and $T > 0$ such that

(a) $Y_T(p) = p$,

(b) $X(y) = Y(y)$ for any $y \in M \setminus \bigcup_{t \in [0, T]} B(X_t(x), \delta)$,

(c) $d(X_t(x), Y_t(p)) < \delta$ for each $t \in [0, T]$.

- Let $\Sigma(X)$ be the set of strongly closable points of $X$. 

Sketch of Proof of Step 4

Let $x \in M \setminus \text{Sing}(X)$ is called strongly closable if for any $C^1$-neighborhood $\mathcal{U}(X)$ of $X$, for any $\delta > 0$, there are $Y \in \mathcal{U}(X)$, $p \in \gamma \in P(Y)$ and $T > 0$ such that

(a) $Y_{T}(p) = p$,
(b) $X(y) = Y(y)$ for any $y \in M \setminus \bigcup_{t \in [0, T]} B(X_t(x), \delta)$,
(c) $d(X_t(x), Y_t(p)) < \delta$ for each $t \in [0, T]$.

Let $\Sigma(X)$ be the set of strongly closable points of $X$. 
Let $\mathcal{M}$ be the space of all Borel measures $\mu$ on $M$ endowed with the weak$^*$ topology. Then for any ergodic measure $\mu \in \mathcal{M}$ of $X$, $\mu$ is supported on a periodic point $p \in \gamma$ of $X(X_T(p) = p, T > 0)$ if and only if

$$
\int f d\mu = \frac{1}{T} \int_0^T f(X_t(p)) dt,
$$

where $f : C^0(M) \to \mathbb{R}$. 
Sketch of Proof of Step 4

Wen(1996)
Let $X \in \mathcal{X}(M)$. $\mu(\Sigma(X) \cup \text{Sing}(X)) = 1$, for every $T > 0$ and every $X_T$-invariant probability Borel measure $\mu$.

Lee and Wen(2012)
There is a residual set $\mathcal{G}_5 \subset \mathcal{X}(M)$ such that every $X \in \mathcal{G}_5$ satisfies the following property: Any ergodic measure $\mu$ of $X$ is the limit of sequence of ergodic invariant measures supported by periodic orbits $\gamma_n$ of $X$ in the weak* topology. Moreover, the orbits $\gamma_n$ converges to the support of $\mu$ in the Hausdorff topology.
Let $X \in \mathcal{X}(M)$ without singularities, and let $X \in G_4 \cap G_5$. Then we prove the Main Theorem.
Thanks for your attention.