Shadowable chain transitive sets of C^1-vector fields

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Outline

Conjecture

Abdenur and Díaz(2007)

There is a residual set $\mathcal{G} \subset \mathrm{Diff}(M)$ such that $f \in \mathcal{G}$ is shadowable if and only if it is hyperbolic.

Previous results

Abdenur and Díaz(2007)

Given a locally maximal transitive set Λ of a generic diffeomorphisms f, then either,

(a) Λ is hyperbolic or

(b) there are a neighborhood $\mathcal{U}(f)$ of f and a small locally maximal neighborhood U of Λ such that every $g \in \mathcal{U}(f)$ is non-shadowable in the neighborhood U.

Previous results

Lee and Wen(2012)

A locally maximal chain transitive set of a C^1-generic diffeomorphism is hyperbolic if and only if it is shadowable.

Main Theorem

For C^1 generic vector field X, a locally maximal chain transitive set is shadowable if and only if the chain transitive set is a hyperbolic basic set.

- M : a compact smooth Riemannian Manifold.
- $\mathfrak{X}(M)$: the set of all C^1-vector fields of M endowed with the C^1-topology.
- d : the distance induced from the Riemannian structure.

Shadowing

Pseudo orbit

For $\delta > 0$, a sequence $\{(x_i, t_i) : x_i \in M, t_i \geq 1\}_{i=a}^{b}(-\infty \leq a < b \leq \infty)$ in M is called a δ-pseudo orbit of X if $d(X_{t_i}(x_i), x_{i+1}) < \delta$ for all $a \leq i \leq b-1$.

Shadowing

Let Λ be a closed X_t-invariant set. We say that X_t has the shadowing property on Λ (or Λ is shadowable) if for every $\epsilon > 0$ there is $\delta > 0$ such that for any δ-pseudo-orbit $\{(x_i, t_i)\}_{i=a}^{b} \subset \Lambda(-\infty \leq a < b \leq \infty)$, let $T_i = t_0 + \cdots + t_i$ for any $0 \leq i < b$, and $T_i = -t_{-1} - t_{-2} - \cdots - t_i$ for any $a < i \leq 0$, there exists a point $y \in M$ and an increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$ such that $d(X_{h(t)}(y), X_{t-T_i}(x_i)) < \epsilon$ for all $a \leq i \leq b-1$, and $T_i < t < T_{i+1}$.

Star condition

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- $\mathfrak{F}(M)$: the set of C^1 vector fields in M for which there is a C^1-neighborhood $\mathcal{U}(X)$ such that every critical orbit of every vector field in $\mathcal{U}(X)$ is hyperbolic.

Star condition

- We say that X is star flow if $X \in \mathfrak{F}(M)$.
- If $X \in \mathfrak{F}(M)$ and has no singularities, then X is Axiom A and no-cycle condition (Gan and Wen(2006)).

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Chain transitive set

- We say that Λ is transitive if there is a point $x \in \Lambda$ such that the closure of $\mathcal{O}_{X_t}(x)(t \geq 0)$ is Λ.
- For given $x, y \in \Lambda$, we write $x \rightsquigarrow_\Lambda y$ if for any $\delta > 0$ there is a δ-pseudo-orbit $\{(x_i, t_i)\}_{i=0}^{n}(n \geq 1, t_i \geq 1)$ of X_t in Λ such that $x_0 = x$ and $x_n = y$.

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- We say that $\mathcal{C}(X)$ is chain transitive if $x \rightsquigarrow_{\mathcal{C}(X)} y$ for any $x, y \in \mathcal{C}(X)$.

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Basic set

- We say that Λ is locally maximal if there is a neighborhood U of Λ such that

$$\bigcap_{t\in\mathbb{R}} X_t(U) = \Lambda.$$

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Hyperbolic

We say that Λ is hyperbolic for X_t if the tangent bundle $T_\Lambda M$ has a DX_t-invariant splitting $E^s \oplus < X > \oplus E^u$ and there exist constants $C > 0$ and $\lambda > 0$ such that

$$\|DX_t|_{E^s_x}\| \leq Ce^{-\lambda t} \text{ and } \|DX_{-t}|_{E^u_x}\| \leq Ce^{-\lambda t}$$

for all $x \in \Lambda$ and $t > 0$.

Generic

- We say that a subset $\mathcal{G} \subset \mathfrak{X}(M)$ is residual if $\mathcal{G}$ contains the intersection of a countable family of open and dense subsets of $\mathfrak{X}(M)$
- We say that a property holds (C^1) generically if there exists a residual subset $\mathcal{G} \subset \mathfrak{X}(M)$ such that for any $X \in \mathcal{G}$ has that property.

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Main Theorem

For C^1 generic vector field X, a locally maximal chain transitive set $\mathcal{C}(X)$ is shadowable if and only if $\mathcal{C}(X)$ is a hyperbolic basic set.

Outline of the Proof

Step 1 If a locally maximal chain transitive set $\mathcal{C}(X)$ is shadowable then $\mathcal{C}(X)$ is transitive.

Step 2 For C^1-generic X, if X has the shadowing property on $\mathcal{C}(X)$, then for any hyperbolic periodic orbits $\gamma_1, \gamma_2 \in \mathcal{C}(X)$,

$$\mathrm{index}(\gamma_1) = \mathrm{index}(\gamma_2),$$

where $\mathrm{index}(\gamma) = \dim W^s(\gamma)$.

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Step 3 For C^1-generic X, if X has the shadowing property on a locally maximal chain transitive set $\mathcal{C}(X)$, then $X \in \mathfrak{F}(M)$.

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Sketch of Proof of Step 1

- If X has the shadowing property on a locally maximal chain transitive set $\mathcal{C}(X)$, then the shadowing point can be taken from $\mathcal{C}(X)$.
- If X has the shadowing property on a locally maximal chain transitive set $\mathcal{C}(X)$, then $\mathcal{C}(X)$ is transitive.

Sketch of Proof of Step 2

Crovisier(2006)

A compact X_t-invariant set $\mathcal{C}(X)$ is chain transitive if and only if $\mathcal{C}(X)$ is the Hausdorff limit of a sequence of periodic orbits of X_t.

Sketch of Proof of Step 2

- Let $\gamma_1, \gamma_2 \in \mathcal{C}(X)$ be hyperbolic periodic orbits. If X has the shadowing property on $\mathcal{C}(X)$, then

$$W^s(\gamma_1) \cap W^u(\gamma_2) \neq \emptyset, \text{ and } W^u(\gamma_1) \cap W^s(\gamma_2) \neq \emptyset.$$

- Let $X \in \mathfrak{X}(M)$. We say that X is Kupka-Smale if all its critical points are hyperbolic, and the invariant manifolds of such elements intersect transversaly.

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Sketch of Proof of Step 3

Lemma 1

There is a residual set $\mathcal{G}_1 \subset \mathfrak{X}(M)$ such that for any C^1-neighborhood $\mathcal{U}(X)$ of X, if there is $Y \in \mathcal{U}(X)$ such that Y has two distinct hyperbolic periodic orbits γ_Y, η_Y with different indices, then X has two different hyperbolic periodic orbits γ, η with different indices.

Sketch of Proof of Step 3

Let $p \in \gamma \in P(X)$ be hyperbolic. For any $\delta > 0$, We say that a point p has a δ-weak eigenvalue if there is an eigenvalue λ of $DX_T(p)$ such that $(1-\delta) < |\lambda| < (1+\delta)$.

Step 3

Lemma 2

There is a residual set $\mathcal{G}_2 \subset \mathfrak{X}(M)$ such that for any C^1-neighborhood $\mathcal{U}(X)$ of X if there is a $Y \in \mathcal{U}(X)$ such that there exists at least one point in $P_h(Y)$ with δ-weak eigenvalue, then there exists a point in $P_h(X)$ with 2δ-weak eigenvalue, where $P_h(X)$ is the set of hyperbolic periodic orbits.

Poincaré map

Let $X \in \mathfrak{X}(M), x \in M$ and $T_xM(r) = \{v \in T_xM : \|v\| \leq r\}$. For every regular point $x \in M(X(x) \neq 0)$, let $N_x =< X(x) >^{\perp} \subset T_xM$ and $N_x(r)$ be the r ball in N_x. Let $\mathcal{N}_{x,r} = \exp_x(N_x(r))$.

- Given a regular point $x \in M$ and $t \in \mathbb{R}$, there are $r > 0$ and a C^1 map $\tau : \mathcal{N}_{x,r} \to \mathbb{R}$ with $\tau(x) = t$ such that $X_{\tau(y)}(y) \in \mathcal{N}_{X_{T(x)},1}$, for any $y \in \mathcal{N}_{x,r}$.

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- We define the Poincaré map $f_{x,t} : \mathcal{N}_{x,r} \to \mathcal{N}_{X_T(x),1}$ by $f_{x,t}(y) = X_{\tau(y)}(y)$.

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Linear Poincaré flow

Let $M_X = \{x \in M : X(x) \neq 0\}$, and let $N = \bigcup_{x \in M_X} N_x$ be the normal bundle based on M_X.

- We define a flow $\Phi_t : N \to N$ by $\Phi_t|_{N_x} = \pi_{N_x} \circ D_x X_t|_{N_x}$, where $\pi_{N_x} : T_x M \to N_x$ is the projection and $D_x X_t : T_x M \to T_{X_t(x)} M$ is the derivative map of X_t.

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- $\Phi_t|_{N_x} = D_x f_{x,t}$.

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Sketch of Proof of Step 3

Lemma 3

Let $X \in \mathfrak{X}(M)$ has no singularities, and let $\mathcal{U}(X)$ be a C^1-neighborhood of X and Λ be locally maximal in U. If $\gamma \in \Lambda \cap P(Y)$ is not hyperbolic, then there is $Y \in \mathcal{U}(X)$ such that two distinct hyperbolic periodic orbits $\gamma_1, \gamma_2 \in \Lambda_Y(U)$ with different indices, where $\Lambda_Y(U) = \bigcap_{t \in \mathbb{R}} Y_t(U)$.

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Sketch of Proof of Step 3

Lemma 4

Let $\mathcal{C}(X)$ be a locally maximal chain transitive set. There is a residual set $\mathcal{G}_3 \subset \mathfrak{X}(M)$ such that for any $X \in \mathcal{G}_3$, if X has the shadowing property on $\mathcal{C}(X)$, then there is $\delta > 0$ such that every hyperbolic periodic orbit in $\mathcal{C}(X)$ has no δ-weak eigenvalue.

Proposition

There is a residual set $\mathcal{G}_4 \subset \mathfrak{X}(M)$ such that if X has no singularities and X has the shadowing property on a locally maximal chain transitive set $\mathcal{C}(X)$, then there exist constants $T > 0$ and $\lambda > 0$ such that for any $p \in \gamma \in P(X)$,

(a) $\|\Phi_{X_t}|_{E^s(p)}\| \cdot \|\Phi_{X_{-t}}|_{E^u(X_t(p))}\| \leq e^{-2\lambda t}$ for any $t \geq T$,

(b) If τ is the period of p, m is any positive integer, and $0 = t_0 < t_1 < \cdots < t_k = m\tau$ is any partition of the time interval $[0, m\tau]$ with $t_{i+1} - t_i \geq T$, then

$$\frac{1}{m\tau}\sum_{i=0}^{k-1} \log \|\Phi_{X_{t_{i+1}-t_i}}|_{E^s(X_{t_i}(p))}\| < -\lambda, \text{ and}$$

$$\frac{1}{m\tau}\sum_{i=0}^{k-1} \log \|\Phi_{X_{-(t_{i+1}-t_i)}}|_{E^s(X_{t_{i+1}}(p))}\| < -\lambda.$$

Sketch of Proof of Step 4

- Let $x \in M \setminus Sing(X)$ is called strongly closable if for any C^1-neighborhood $\mathcal{U}(X)$ of X, for any $\delta > 0$, there are $Y \in \mathcal{U}(X), p \in \gamma \in P(Y)$and $T > 0$ such that
  (a) $Y_T(p) = p$,
  (b) $X(y) = Y(y)$ for any $y \in M \setminus \bigcup_{t \in [0,T]} B(X_t(x), \delta)$,
  (c) $d(X_t(x), Y_t(p)) < \delta$ for each $t \in [0, T]$.

- Let $\Sigma(X)$ be the set of strongly closable points of X.

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Sketch of Proof of Step 4

Let $\mathcal{M}$ be the space of all Borel measures μ on M endowed with the weak* topology. Then for any ergodic measure $\mu \in \mathcal{M}$ of X, μ is supported on a periodic point $p \in \gamma$ of $X(X_T(p) = p, T > 0)$ if and only if

$$\int f d\mu = \frac{1}{T}\int_0^T f(X_t(p))dt,$$

where $f : C^0(M) \to \mathbb{R}$.

Sketch of Proof of Step 4

Wen(1996)

Let $X \in \mathfrak{X}(M)$. $\mu(\Sigma(X) \cup Sing(X)) = 1$, for every $T > 0$ and every X_T-invariant probability Borel measure μ.

Lee and Wen(2012)

There is a residual set $\mathcal{G}_5 \subset \mathfrak{X}(M)$ such that every $X \in \mathcal{G}_5$ satisfies the following property: Any ergodic measure μ of X is the limit of sequence of ergodic invariant measures supported by periodic orbits γ_n of X in the weak* topology. Moreover, the orbits γ_n converges to the support of μ in the Hausdorff topology.

Let $X \in \mathfrak{X}(M)$ without singularities, and let $X \in \mathcal{G}_4 \cap \mathcal{G}_5$. Then we prove the Main Theorem.

Thanks for your attention.