Weak product recurrence and related properties

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Recurrence

1. $X$ - compact,
2. $f : X \to X$ - continuous
3. $x \in X$ is recurrent if $x \in \omega(x, f)$.
   - or in other words, $N(x, U, f) \neq \emptyset$ for any neighborhoods $U$ of $x$,
   - where $N(x, U, f) = \{ i > 0 : f^i(x) \in U \}$.

4. $x \in X$ is uniformly recurrent (or minimal) if it is recurrent and $\omega(x, f)$ is a minimal set.
   - or equivalently $N(x, U, f)$ is syndetic (has bounded gaps between its elements, i.e. any sufficiently long block of consecutive integers intersects it).
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4. \( x \in X \) is uniformly recurrent (or minimal) if it is recurrent and \( \omega(x, f) \) is a minimal set.
   - or equivalently \( N(x, U, f) \) is syndetic (has bounded gaps between its elements, i.e. any sufficiently long block of consecutive integers intersects it).
1. \( x \in X \) is **uniformly** recurrent if \( x \in \omega(x, f) \) (and it is a minimal set).

2. \( x \in X \) is **product recurrent** if
   - given any recurrent point \( y \) in any dynamical system \( g \)
   - and any neighborhoods \( U \) of \( x \) and \( V \) of \( y \),
   - \( N(x, U, f) \cap N(y, V, g) \neq \emptyset \).

   where \( N(x, U, f) = \{ i > 0 : f^i(x) \in U \} \).

3. \( x, z \in X \) are proximal if \( \lim \inf_{n \to \infty} d(f^n(x), f^n(z)) = 0 \)

4. \( x \) is distal if it is not proximal to any point in its orbit closure other than itself.

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**Theorem (Furstenberg)**

A point \( x \) is product recurrent if and only if it is (uniformly recurrent) distal point.
Product recurrence

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A point \( x \) is product recurrent if and only if it is (uniformly recurrent) distal point.
Weak product recurrence

1. $x \in X$ is **weakly** product recurrent if
   1. given any **uniformly** recurrent (= almost periodic) point $y$ in any dynamical system $g$
   2. and any neighborhoods $U$ of $x$ and $V$ of $y$,
   3. $N(x, U, f) \cap N(y, V, g) \neq \emptyset$.

**Question**

„Another question (even for $\mathbb{Z}$ or $\mathbb{N}$ actions): If $(x, y)$ is recurrent for all almost periodic points $y$, is $x$ necessarily a distal point?”


2. It was first by Haddad and Ott that product recurrence and weak product recurrence are not equivalent (Answer **NO** to the above).

Theorem

A point $x \in X$ is weakly product recurrent if it has the following property:

- for every neighborhood $V$ of $x$ there exists $n$ such that if $S \subset \mathbb{N}$ is any finite set satisfying $|s - t| > n$ for all distinct $s, t \in S$, then there exists $l \in \mathbb{N}$ such that $l + s \in N(x, V, f)$ for every $s \in S$.

1. the above conditions are satisfied by many points/systems (e.g. point with dense orbit in full shift on 2 symbols)
2. dynamical system satisfying above must be at least mixing
3. dynamical system satisfying above cannot be minimal
Disjointness

1. We a closed set \( \emptyset \neq J \subset X \times Y \) is a joining of \((X, f)\) and \((Y, g)\) if it is invariant (for the product map \( f \times g \)) and its projections on first and second coordinate are \( X \) and \( Y \) respectively.

2. If \( X \times Y \) is the only joining of \( f \) and \( g \) then we say that they are disjoint.

Question

How to characterize systems disjoint from any distal or minimal system?


Theorem (Petersen, 1970)

A system is disjoint with every distal system iff it is weakly mixing and minimal.
**Disjointness**

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**Theorem (Petersen, 1970)**

A system is disjoint with every distal system iff it is weakly mixing and minimal.
Only partial answers are known when a system is disjoint with all minimal systems.

**Theorem (Furstenberg, 1967)**

If $f$ is weakly mixing with dense periodic points then it is disjoint from every minimal systems.

**Theorem (Huang & Ye; Oprocha)**

If $(X, f)$ is disjoint from every minimal system then every transitive point in $(X, f)$ is weakly product recurrent.
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Remark

The class of weak product recurrent points is much wider than can be detected by disjointness theorems, e.g.

- If \([0, 1], f\) is mixing and \((S^1, R)\) is irrational rotation then for any \(z \in S^1\) there is a residual set in \([0, 1] \times \{z\} \subset (S^1, R)\) in dynamical system \(([0, 1] \times S^1, f \times R)\) consisting of weakly product recurrent points.

- But \(([0, 1] \times S^1, f \times R)\) is not disjoint with \((S^1, R)\).
Product recurrence in terms of Furstenberg families (Dong, Shao, Ye)

1. $\mathcal{F}$ - upward hereditary set of subsets of $\mathbb{N} = \text{Furstenberg family}$
2. $x \in X$ is $\mathcal{F}$-recurrent if $N(x, U, f) \in \mathcal{F}$ for any open neighborhood $U$ of $x$,
3. recurrence $= \mathcal{F}_{\text{inf}}$-recurrence ($\mathcal{F}_{\text{inf}} = \text{infinite subsets of } \mathbb{N}$)
4. $x \in X$ is $\mathcal{F}$-product recurrent ($\mathcal{F}$-PR for short) if for any dynamical system $(Y, g)$ and any $\mathcal{F}$-recurrent point $y \in Y$ the pair $(x, y)$ is recurrent for $(X \times Y, f \times g)$.
5. $\mathcal{F}$-PR$_0 = \mathcal{F}$-PR but only with $(Y, g)$ of topological entropy zero.
6. $\mathcal{F}_{\text{inf}}$-PR $= \text{product recurrence (as introduced by Furstenberg)}$
7. $\mathcal{F}_s$-PR $= \text{weak product recurrence (where } \mathcal{F}_s \text{ is the family of syndetic sets)}$
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Further results on product recurrence

\[ \mathcal{F}_{\text{inf}} - \text{PR} \xleftarrow{?} \mathcal{F}_{\text{pubd}} - \text{PR} \xleftarrow{?} \mathcal{F}_{\text{ps}} - \text{PR} \xleftarrow{\text{not}} \mathcal{F}_{\text{s}} - \text{PR} \]

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**Figure:** Product recurrence and product recurrence with zero entropy systems (Dong, Shao & Ye)

- \( \mathcal{F}_{ps} = \) piecewise syndetic, i.e. intersections of syndetic and thick set
- \( \mathcal{F}_{pubd} = \) sets with positive upper Banach densitity

\[
0 < D(A) = \lim_{n \to \infty} \sup_{n} \frac{1}{n} \sup_{i \geq 0} \#(A \cap [i, i + n])
\]
Results on PR obtained by results on disjointness

1. If \((X, f)\) is a minimal flow (i.e. homeomorphism) such that any of its invariant measures is a \(K\)-measure, then it is disjoint from any transitive zero entropy \(E\)-system.
   - If \((X, f)\) is a strictly ergodic flow with its unique invariant measure being a \(K\)-measure, then every point \(x \in X\) is \(\mathcal{F}_{\text{pubd}}\)-PR\(_0\).
   - But it has positive topological entropy, so also asymptotic pairs...
   - So there are points in \(X\) which are not recurrent in pair with minimal points.
   - Hence we have an example \(\mathcal{F}_{\text{pubd}} - \text{PR}_0 \not\iff \mathcal{F}_s - \text{PR}\).


2. If \(x\) is \(\mathcal{F}_{\text{ps}}\)-PR\(_0\) then it is a minimal point.
   - Hence we have an example \(\mathcal{F}_s - \text{PR} \not\iff \mathcal{F}_{ps} - \text{PR}_0\).

Further results on product recurrence (cont.)

\[ \mathcal{F}_{\text{inf}} \rightarrow \text{PR} \quad \mathcal{F}_{\text{pubd}} \rightarrow \text{PR} \quad \mathcal{F}_{\text{ps}} \rightarrow \text{PR} \quad \mathcal{F}_{s} \rightarrow \text{PR} \]

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- \( \mathcal{F}_{pubd} \) = sets with positive upper Banach density
Further results on product recurrence (cont.)

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Figure: Product recurrence and product recurrence with zero entropy systems (Dong, Shao & Ye) + work of Oprocha and G.H. Zhang

- \( \mathcal{F}_{\text{ps}} = \) piecewise syndetic, i.e. intersections of syndetic and thick set
- \( \mathcal{F}_{\text{pubd}} = \) sets with positive upper Banach density
Theorem

If \( x \) is \( \mathcal{F}_{ps}\)-PR then it is distal.

Theorem

The following statements are equivalent:

1. \( x \) is distal,
2. \( (x, y) \) is recurrent for any recurrent point \( y \) of any system \((Y, g)\),
3. \( (x, y) \) is \( \mathcal{F}_{pubd}\)-recurrent for any \( \mathcal{F}_{pubd}\)-recurrent point \( y \) of any system \((Y, g)\),
4. \( (x, y) \) is \( \mathcal{F}_{ps}\)-recurrent for any \( \mathcal{F}_{ps}\)-recurrent point \( y \) of any system \((Y, g)\),
5. \( (x, y) \) is minimal for any minimal point \( y \) of any system \((Y, g)\).
Open problems

1. $\mathcal{F}_{ps} - PR_0 \iff \mathcal{F}_{pubd} - PR_0$?
2. $\mathcal{F}_s - PR + \text{minimal} \implies \text{distal}$?